

Mixed Models with R:

From Hierarchical to Mixed Models

*Two equivalent views of hierarchical data reveal different and important features*

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## Notes:

- A version of this document that includes the data use to generate graphs will be available through the course web page.
- Lab 1, also available through the course web page, presents many additional concepts that complement the material in this document.
- Many concepts relevant to Hierarchical and Mixed Models will be seen in the sections on Longitudinal, Non-Linear and Generalized Linear Mixed Models.

# The many hierarchies of statistics

## ***Hierarchical Data:***

refers to the structure of data with nested sampling levels: e.g. students sampled in schools and schools sampled from a population of schools or patients whose symptoms are measured on a number of visits.

## ***Hierarchical Models:***

is often used to refer to a set of models used where some models are 'nested' within each other, i.e. a simpler model is obtained by restricting the parameters of a more complex model. This is the usual basis for ANOVA.

## ***Hierarchical Model:***

(the sense in which we use it) a model with hierarchical components intended to analyze hierarchical data. Of course, nothing prevents us from considering hierarchies of hierarchical models in which case we are using both concepts in the same sentence – although they refer to entirely different hierarchies.

Handwritten mathematical models in blue ink, including  $Y \sim X$ ,  $Y \sim X + G$ ,  $Y \sim X * G$ , and  $Y \sim X + 2$ , with arrows indicating relationships between them.

## Statistical goals for estimation:

We need to keep our goals in mind as we consider various approaches to analyze data. When you want to estimate something, e.g. a treatment effect or a comparison between two groups, you want your procedure to be:

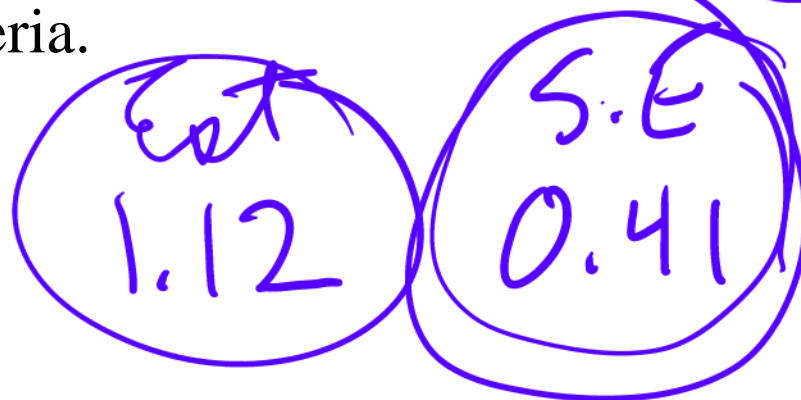
1) **consistent**: You want to know that you are estimating the right thing with little bias. i.e. you are aiming at the right target and, although your aim might be shaky, you won't be consistently off in any direction.

2) **efficient**: you want to shake as little as possible. You want to use the 'best' method available with this data and model to minimize the true standard error of estimation (what it really is, not what your procedure reports it to be)

3) **honest**: you would like to have an honest estimate of the true standard error. Otherwise, your CIs will have the wrong size and your p-values will be off. You may have more power than you think leading you to commit Type II errors unnecessarily or you may have less power than you think leading you to commit Type I errors too often.

4) **robust**: the more a good method remains good when assumptions are violated, the more robust it is. Robustness is more important if you are not confident of assumptions or if you know that the formal assumptions are not satisfied.

We don't necessarily need hierarchical models to analyze hierarchical data so we consider simpler approaches first and we will see how they measure up to our four criteria.



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p

# Hierarchical data

## High school example:

For multilevel modeling we will use a subset of a data set presented at length in Bryk and Raudenbush (1992). The major variables are math achievement (`mathach`) and socio-economic status (`ses`) measured for each student in a sample of high school students in each of 40 selected schools in the U.S. Each school belongs to one of two sectors: 21 are Public schools and 19 are Catholic schools. There are 1,977 students in the sample. The sample size from each school ranges from 29 students to 66 students. The data are available as the data frame '`hs`' in the package '`spida`'. The full data set is '`hsfull`' and two split halves are '`hs1`' and '`hs2`'. The following is a listing of the first 50 lines of the '`hs`' file:

2

```
> head(hs, 50)
```

	school	mathach	ses	Sex	Minority	Size	Sector	PRACAD	DISCLIM
1	1317	12.862	0.882	Female	No	455	Catholic	0.95	-1.694
2	1317	8.961	0.932	Female	Yes	455	Catholic	0.95	-1.694
3	1317	4.756	-0.158	Female	Yes	455	Catholic	0.95	-1.694
4	1317	21.405	0.362	Female	Yes	455	Catholic	0.95	-1.694
5	1317	20.748	1.372	Female	No	455	Catholic	0.95	-1.694
6	1317	18.362	0.132	Female	Yes	455	Catholic	0.95	-1.694
7	1317	14.752	0.132	Female	No	455	Catholic	0.95	-1.694
8	1317	11.290	-0.008	Female	Yes	455	Catholic	0.95	-1.694
9	1317	10.493	-0.108	Female	Yes	455	Catholic	0.95	-1.694
10	1317	10.956	0.612	Female	Yes	455	Catholic	0.95	-1.694
11	1317	21.405	0.482	Female	Yes	455	Catholic	0.95	-1.694
12	1317	23.355	0.502	Female	No	455	Catholic	0.95	-1.694
13	1317	12.283	0.482	Female	Yes	455	Catholic	0.95	-1.694
14	1317	9.257	0.472	Female	Yes	455	Catholic	0.95	-1.694
15	1317	11.502	-0.578	Female	No	455	Catholic	0.95	-1.694
16	1317	20.039	1.152	Female	Yes	455	Catholic	0.95	-1.694
17	1317	21.405	-0.288	Female	Yes	455	Catholic	0.95	-1.694
18	1317	23.736	0.942	Female	Yes	455	Catholic	0.95	-1.694
19	1317	11.027	0.722	Female	Yes	455	Catholic	0.95	-1.694
20	1317	17.203	-0.108	Female	Yes	455	Catholic	0.95	-1.694
21	1317	10.661	1.462	Female	Yes	455	Catholic	0.95	-1.694
22	1317	7.031	-0.028	Female	Yes	455	Catholic	0.95	-1.694
23	1317	13.677	0.702	Female	No	455	Catholic	0.95	-1.694
24	1317	13.373	0.082	Female	Yes	455	Catholic	0.95	-1.694
25	1317	10.121	-0.108	Female	Yes	455	Catholic	0.95	-1.694

26	1317	10.394	0.322	Female	Yes	455	Catholic	0.95	-1.694
27	1317	6.973	0.302	Female	Yes	455	Catholic	0.95	-1.694
28	1317	11.064	-0.098	Female	No	455	Catholic	0.95	-1.694
29	1317	11.531	-0.848	Female	Yes	455	Catholic	0.95	-1.694
30	1317	8.253	-1.248	Female	Yes	455	Catholic	0.95	-1.694
31	1317	7.142	0.122	Female	Yes	455	Catholic	0.95	-1.694
32	1317	3.220	0.272	Female	Yes	455	Catholic	0.95	-1.694
33	1317	15.784	0.582	Female	No	455	Catholic	0.95	-1.694
34	1317	17.246	0.642	Female	Yes	455	Catholic	0.95	-1.694
35	1317	9.337	0.952	Female	Yes	455	Catholic	0.95	-1.694
36	1317	15.555	-0.258	Female	Yes	455	Catholic	0.95	-1.694
37	1317	8.382	0.492	Female	Yes	455	Catholic	0.95	-1.694
38	1317	11.621	0.992	Female	No	455	Catholic	0.95	-1.694
39	1317	4.810	0.832	Female	Yes	455	Catholic	0.95	-1.694
40	1317	17.869	-0.068	Female	Yes	455	Catholic	0.95	-1.694
41	1317	8.057	-0.088	Female	Yes	455	Catholic	0.95	-1.694
42	1317	11.794	0.972	Female	Yes	455	Catholic	0.95	-1.694
43	1317	18.939	0.542	Female	No	455	Catholic	0.95	-1.694
44	1317	20.261	0.132	Female	Yes	455	Catholic	0.95	-1.694
45	1317	10.066	-0.008	Female	Yes	455	Catholic	0.95	-1.694
46	1317	20.236	0.812	Female	No	455	Catholic	0.95	-1.694
47	1317	4.508	1.122	Female	No	455	Catholic	0.95	-1.694
48	1317	18.827	0.062	Female	No	455	Catholic	0.95	-1.694
49	1906	14.449	0.132	Female	Yes	400	Catholic	0.87	-0.939
50	1906	20.455	0.382	Female	No	400	Catholic	0.87	-0.939

R

SAC

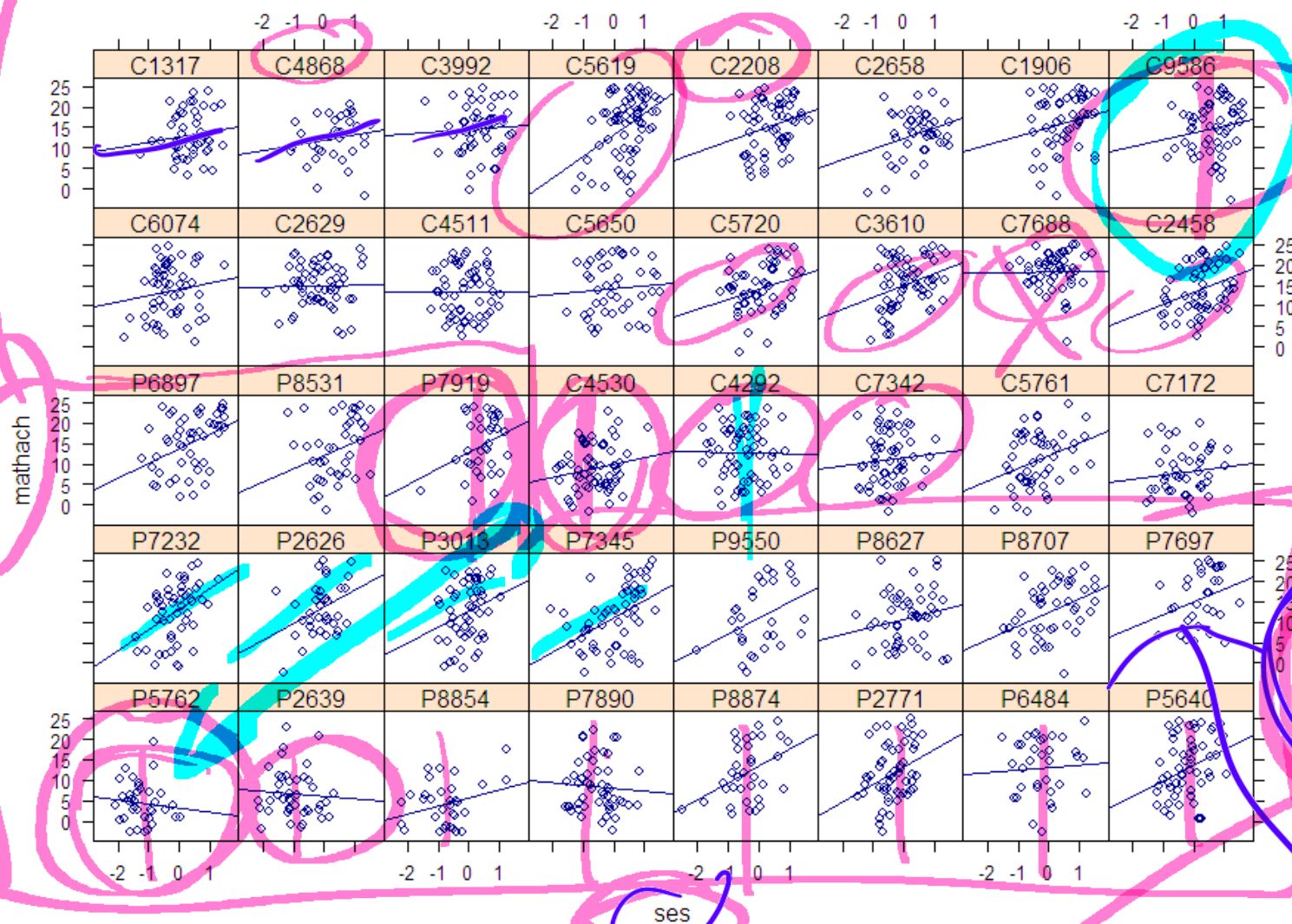
The first 48 lines are students belonging to school labelled 1317. The last 2 lines are the first cases of the second school in the sample labelled 1906.

The uniform quantile plot of each variable gives a good snapshot of the data. Think of lining up each variables from shortest to tallest and plotting the result:



**Figure 1: Uniform quantile plots of high school data**

*xqplot(hs)*



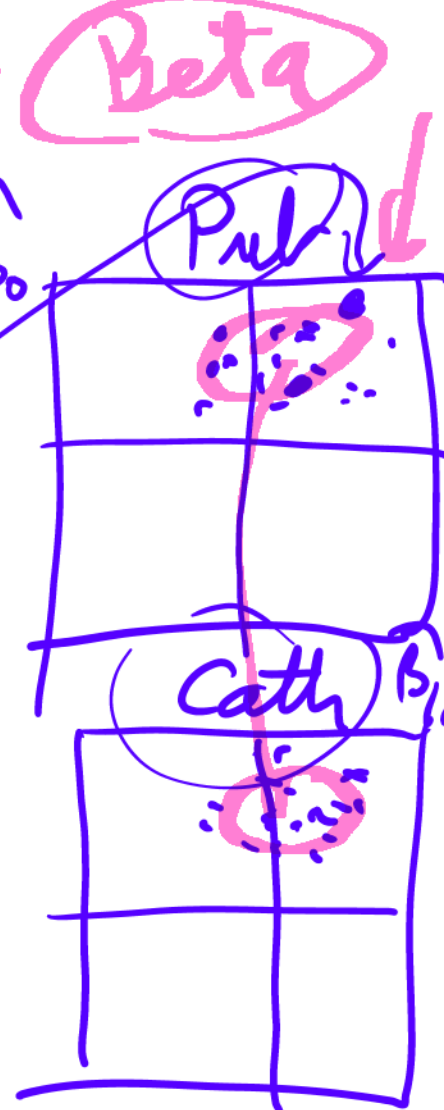
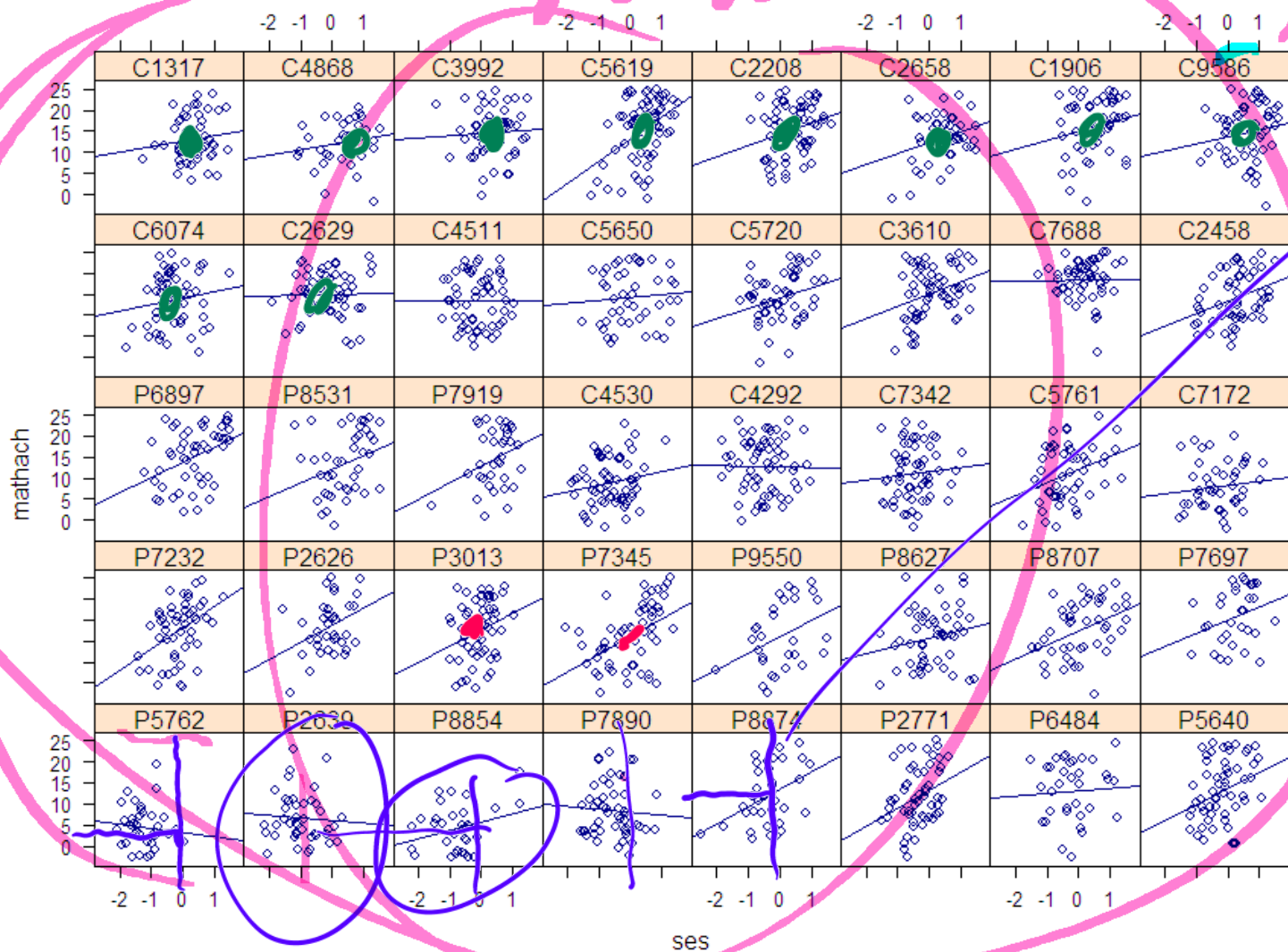
$I + S_c$   
is factor

$1 + X$   
 $+ 39$

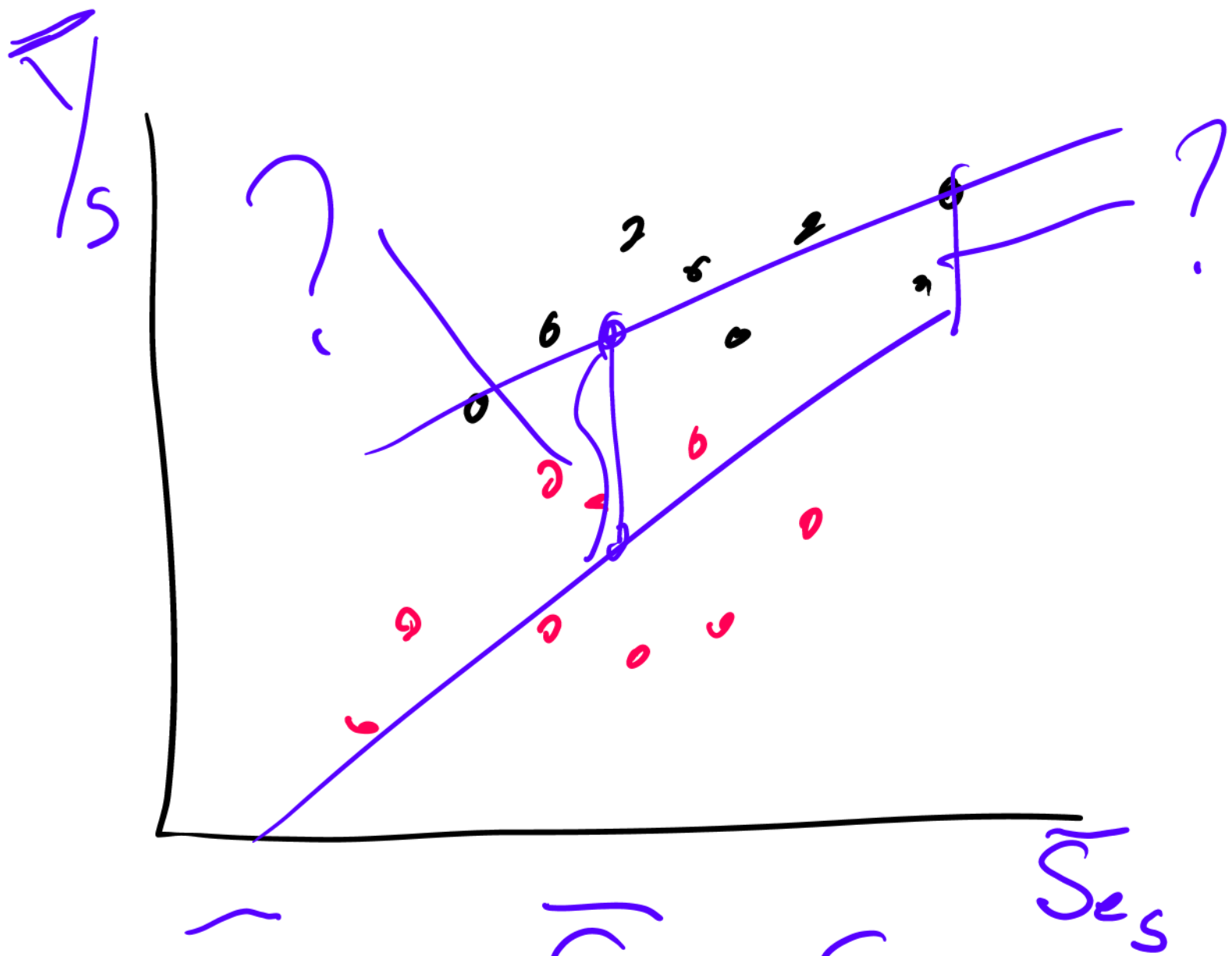
$Y \sim X * S_c$

Figure 2: Trellis plot with least-squares line in each school. Note that the LS line could vary because of randomness in the observations within a school – i.e. they would vary even if all schools had exactly the same relationship between mathach and ses – and because the underlying relationship might vary from school to school.

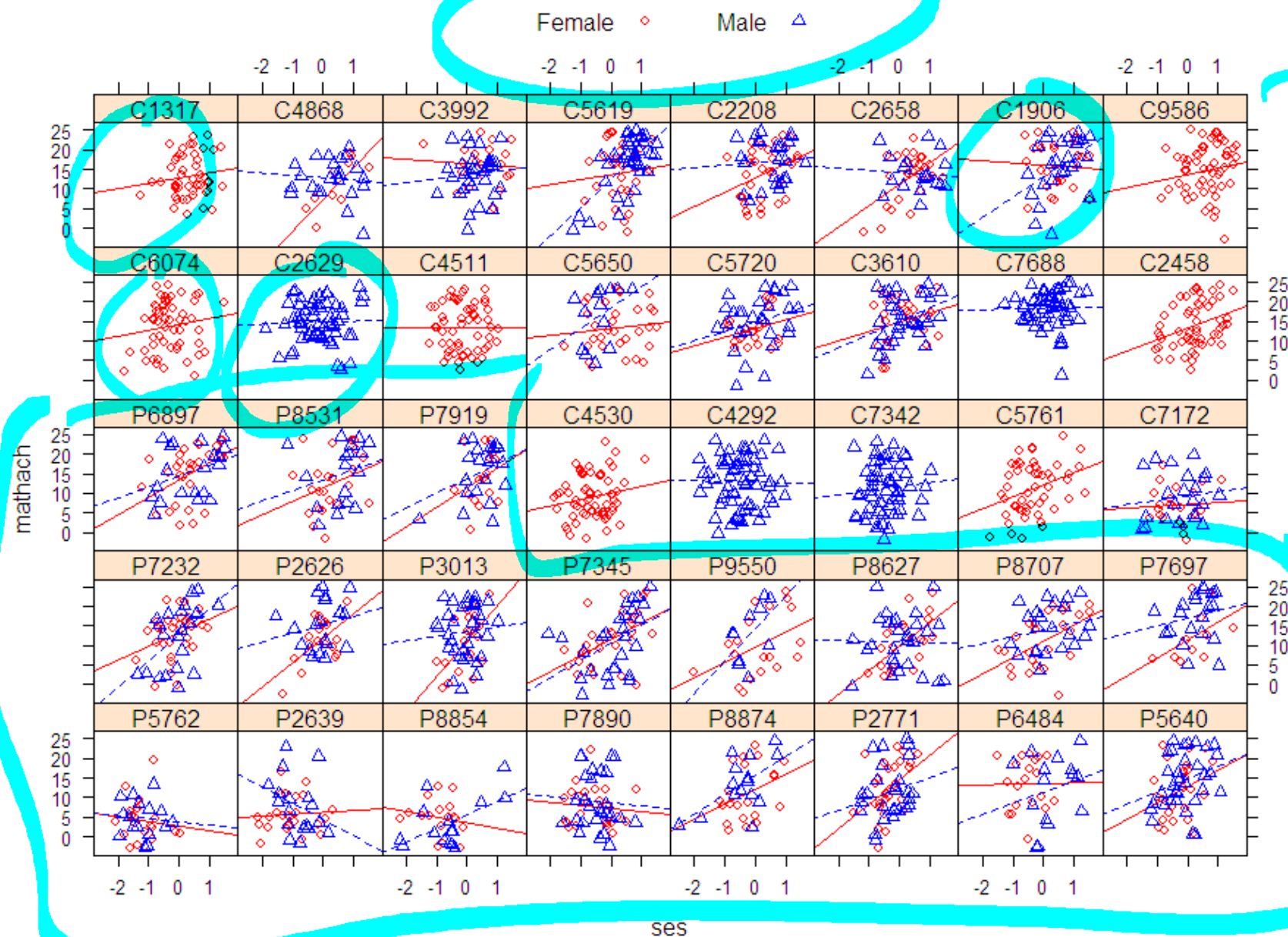
If  $S_c$  is an integer.  $1 + X + S_c + X * S_c$



**Figure 2: Trellis plot with least-squares line in each school. Note that the LS line could vary because of randomness in the observations within a school – i.e. they would vary even if all schools had exactly the same relationship between mathach and ses – and because the underlying relationship might vary from school to school.**



$$\hat{Y}_S \sim \hat{S}_{\text{Sec}} * S_{\text{Sec}}$$



**Figure 3: Trellis plot of high school data with sex of students.**

## *Comparing mathach and its relationship with ses in the two school sectors.*

### Some possible approaches:

1) Pool the data from the schools within each sector and analyze with OLS. i.e. completely ignore the individual schools and regress mathach on ses and sector alone.


2a) Use a fixed effects model (Allison, 2005) to estimate relationship in each school and then compare the mean level of each sector. Can we just fit a model on SES, School and Sector to estimate the effect of Sector?

2b) Use a fixed effects model with varying intercept from school to school but assume same slope within each Sector.

Handwritten notes:

$Y \sim X + S_c + X \# \text{Sector}$

3) Use a two step approach: fit a regression to each school and then estimate the mean intercept and slope of the schools in each sector with a multivariate analysis of the using the fitted intercepts and slopes as data.



4) Fit a 'between school' model: take the average ses and average mathach from each school and then perform a regression on the resulting means.

5) Use a hierarchical model

6) Use a hierarchical model with a contextual variable to see that we were really estimating two things to begin with.

## Looking at hierarchical data

id cluster (2)

```
> library( spida ) # also loads nlme, lattice, car, MASS
```

```
> data( hs )
```

```
> head( hs )
```

	X	school	mathach	ses	sector	female	Sex	Minority	Size	Sector
1	141	1317	12.862	0.882	1	1	Female	No	455	Catholic
2	142	1317	8.961	0.932	1	1	Female	Yes	455	Catholic
3	143	1317	4.756	-0.158	1	1	Female	Yes	455	Catholic
4	144	1317	21.405	0.362	1	1	Female	Yes	455	Catholic
5	145	1317	20.748	1.372	1	1	Female	No	455	Catholic
6	146	1317	18.362	0.132	1	1	Female	Yes	455	Catholic

	PRACAD	DISCLIM	HIMINTY
1	0.95	-1.694	1
2	0.95	-1.694	1
3	0.95	-1.694	1
4	0.95	-1.694	1
5	0.95	-1.694	1
6	0.95	-1.694	1

Level 1 variables: vary from student to student within schools

Level 2 variables: characteristics of schools

mathach ~ ses \* factor(school)

$$Y \sim \frac{X * 2}{X * 2}$$

$X * 2$   
not expand

$$\sim \frac{1 + \cancel{X} + 2 + \cancel{X * 2}}{\cancel{X * 2}}$$

Principle of Marginality

$$G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

X num

$$Y \sim 1 + X + G + X * G$$

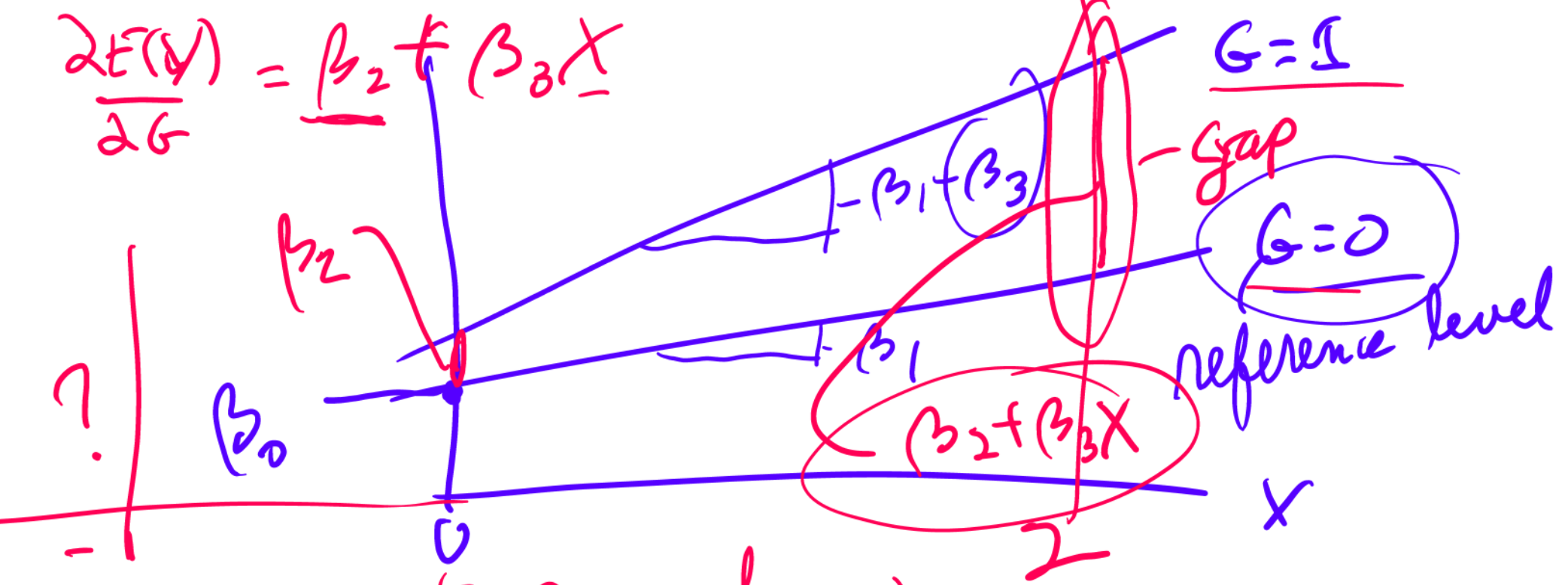
$$E(Y) = \beta_0 + \beta_1 X + \beta_2 G + \beta_3 X * G$$

$$\frac{\partial E(Y)}{\partial X} = \beta_1 + \beta_3 G$$

$$G = 0 : \frac{\partial E(Y)}{\partial X} = \beta_1$$

$$G = 1 : \frac{\partial E(Y)}{\partial X} = \beta_1 + \beta_3$$

$$\frac{\partial E(Y)}{\partial G} = \beta_2 + \beta_3 X$$



Estimate Gap when  $X = 1.5$

$\beta_2 + \beta_3 \times 1.5$

Coeff	Est	SE
$\beta_0$	1.2	1.1
$\beta_1$	0.5	1.2
$\beta_2$	1.3	1.3
$\beta_3$	0.1	1.4

$$y = \beta_2 + \beta_3 \times 1.5$$

$$\hat{y} = 1 \cdot \hat{\beta}_2 + \hat{\beta}_3 \times \underline{1.5}$$

$$SE(\hat{y}) =$$

$$1^2 \times 1.3^2 + 1.5^2 \times 1.4^2$$

$$+ 2 \times 1 \times 1.5 \text{Cov}(\hat{\beta}_2, \hat{\beta}_3)$$

GLM Wald test

Matrix L lc of  $\beta$ 's

$$\beta_2 + 1.5 \beta_3$$
$$L \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1.5 \end{bmatrix}_{1 \times 4} \beta =$$

$$L = \begin{bmatrix} 0 & 0 & 1 & 1.5 \end{bmatrix}$$

> L = rbind('gap at 1.5' = c(0, 0, 1, 1.5))

> wald(fit, L)

```
> supply( hs, class)
```

X	school	mathach	ses	sector	female	Sex
"integer"	"integer"	"numeric"	"numeric"	"integer"	"integer"	"factor"
Minority	Size	Sector	PRACAD	DISCLIM	HIMINTY	
"factor"	"integer"	"factor"	"numeric"	"numeric"	"integer"	

```
> tab( hs, ~ Sector + school )
```

	school												
Sector	1317	1906	2208	2458	2626	2629	2639	2658	2771	3013	3610	3992	4292
Catholic	48	53	60	57	0	57	0	45	0	0	64	53	65
Public	0	0	0	0	38	0	42	0	55	53	0	0	0
Total	48	53	60	57	38	57	42	45	55	53	64	53	65

	school												
Sector	4511	4530	4868	5619	5640	5650	5720	5761	5762	6074	6484	6897	7172
Catholic	58	63	34	66	0	45	53	52	0	56	0	0	44
Public	0	0	0	0	57	0	0	0	37	0	35	49	0
Total	58	63	34	66	57	45	53	52	37	56	35	49	44

	school												
Sector	7232	7342	7345	7688	7697	7890	7919	8531	8627	8707	8854	8874	9550
Catholic	0	58	0	54	0	0	0	0	0	0	0	0	0
Public	52	0	56	0	32	51	37	41	53	48	32	36	29
Total	52	58	56	54	32	51	37	41	53	48	32	36	29

	school	
Sector	9586	Total
Catholic	59	1144
Public	0	833
Total	59	1977

> table( hs\$school) # number of observations per school

1317 1906 2208 2458 2626 2629 2639 2658 2771 3013 3610 3992 4292 4511  
 48 53 60 57 38 57 42 45 55 53 64 53 65 58  
 4530 4868 5619 5640 5650 5720 5761 5762 6074 6484 6897 7172 7232 7342  
 63 34 66 57 45 53 52 37 56 35 49 44 52 58  
 7345 7688 7697 7890 7919 8531 8627 8707 8854 8874 9550 9586  
 56 54 32 51 37 41 53 48 32 36 29 59

*tab( hs, ~school)*

*pct=0)*

> tab( ~ Sector + school, hs) # each school is in one Sector

	school	1317	1906	2208	2458	2626	2629	2639	2658	2771	3013	3610	3992
Sector		1317	1906	2208	2458	2626	2629	2639	2658	2771	3013	3610	3992
Catholic		48	53	60	57	0	57	0	45	0	0	64	53
Public		0	0	0	0	38	0	42	0	55	53	0	0
Total		48	53	60	57	38	57	42	45	55	53	64	53

	school	4292	4511	4530	4868	5619	5640	5650	5720	5761	5762	6074	6484
Sector		4292	4511	4530	4868	5619	5640	5650	5720	5761	5762	6074	6484
Catholic		65	58	63	34	66	0	45	53	52	0	56	0
Public		0	0	0	0	0	57	0	0	0	37	0	35
Total		65	58	63	34	66	57	45	53	52	37	56	35

	school	6897	7172	7232	7342	7345	7688	7697	7890	7919	8531	8627	8707
Sector		6897	7172	7232	7342	7345	7688	7697	7890	7919	8531	8627	8707
Catholic		0	44	0	58	0	54	0	0	0	0	0	0
Public		49	0	52	0	56	0	32	51	37	41	53	48
Total		49	44	52	58	56	54	32	51	37	41	53	48

	school				
Sector	8854	8874	9550	9586	Total
Catholic	0	0	0	59	1144
Public	32	36	29	0	833
Total	32	36	29	59	1977

```
> tab( ~ Sex + school, hs)
```

	school											
Sex	1317	1906	2208	2458	2626	2629	2639	2658	2771	3013	3610	3992
Female	48	27	35	57	18	0	24	27	28	19	29	21
Male	0	26	25	0	20	57	18	18	27	34	35	32
Total	48	53	60	57	38	57	42	45	55	53	64	53

	school											
Sex	4292	4511	4530	4868	5619	5640	5650	5720	5761	5762	6074	6484
Female	0	58	63	11	30	24	32	24	52	21	56	20
Male	65	0	0	23	36	33	13	29	0	16	0	15
Total	65	58	63	34	66	57	45	53	52	37	56	35

	school											
Sex	6897	7172	7232	7342	7345	7688	7697	7890	7919	8531	8627	8707
Female	29	22	30	0	29	0	11	24	16	23	24	26
Male	20	22	22	58	27	54	21	27	21	18	29	22
Total	49	44	52	58	56	54	32	51	37	41	53	48

	school				
Sex	8854	8874	9550	9586	Total
Female	17	21	19	59	1074
Male	15	15	10	0	903
Total	32	36	29	59	1977

```
> by (hs, hs$Sector, function( dd ) tab( ~ Sex + school, dd))
```

hs\$Sector: Catholic

	school											
Sex	1317	1906	2208	2458	2529	2658	3610	3992	4292	4511	4530	4868
Female	48	27	35	57	0	27	29	21	0	58	63	11
Male	0	26	25	0	57	18	35	32	65	0	0	23
Total	48	53	60	57	57	45	64	53	65	58	63	34

	school									
Sex	5619	5650	5720	5761	6074	7172	7342	7688	9586	Total
Female	30	32	24	52	56	22	0	0	59	651
Male	36	13	29	0	0	22	58	54	0	493
Total	66	45	53	52	56	44	58	54	59	1144

hs\$Sector: Public

school

Sex	2626	2639	2771	3013	5640	5762	6484	6897	7232	7345	7697	7890
Female	18	24	28	19	24	21	20	29	30	29	11	24
Male	20	18	27	34	33	16	15	20	22	27	21	27
Total	38	42	55	53	57	37	35	49	52	56	32	51

school

Sex	7919	8531	8627	8707	8854	8874	9550	Total
Female	16	23	24	26	17	21	19	423
Male	21	18	29	22	15	15	10	410
Total	37	41	53	48	32	36	29	833

Note: All Public schools are co-educational.

Try 3-way table: `> tab( hs, ~ Sex + school + Sector)`

## Summary variables and informative labels

Is a school male, female or co-ed?

```
> some( hs )
```

	X	school	mathach	ses	sector	female	Sex	Minority	Size
419	1517	2771	11.226	0.302	0	0	Male	No	415
483	1760	3013	15.741	0.192	0	0	Male	No	760
627	2969	4292	9.255	0.972	1	0	Male	Yes	1328
960	3785	5640	14.699	-0.268	0	0	Male	No	1152
1272	4953	6897	15.885	-0.388	0	0	Male	Yes	1415
1632	5696	7890	3.295	-0.118	0	1	Female	No	311
1709	6132	8531	24.418	0.592	0	0	Male	No	2190
1717	6140	8531	-1.509	0.092	0	1	Female	Yes	2190

	Sector	PRACAD	DISCLIM	HIMINTY
419	Public	0.24	1.048	0
483	Public	0.56	-0.213	0
627	Catholic	0.76	-0.674	1
960	Public	0.41	0.256	0
1272	Public	0.55	-0.361	0
1632	Public	0.21	0.845	0
1709	Public	0.58	0.132	0
1717	Public	0.58	0.132	0

Note: 'female' and 'Sex' are individual variable

*lapply* *sapply* *tapply* *vapply* 24

Generating sex composition as a variable:

group mean variable = derived variable a Level 2 variable

```
> hs$Sex.comp <- capply(hs$Sex == "Female", hs$school, mean)
> some(hs)
```

	X	school	mathach	ses	sector	female	Sex	Minority	Size
662	3004	4292	6.703	-0.138	1	0	Male	Yes	1328
929	3754	5640	9.223	-0.548	0	0	Male	No	1152
1123	4009	5762	-2.252	-1.028	0	0	Male	Yes	1826
1298	5093	7172	5.549	0.462	1	1	Female	Yes	280
1304	5099	7172	9.915	-0.628	1	0	Male	Yes	280
1383	5178	7232	16.278	-0.338	0	1	Female	Yes	1154
1536	5578	7688	9.587	0.612	1	0	Male	Yes	1410
1641	5705	7890	-2.362	-0.048	0	0	Male	Yes	311
1739	6162	8627	11.322	0.272	0	0	Male	No	2452
1922	7130	9586	7.974	0.212	1	1	Female	No	262

	Sector	PRACAD	DISCLIM	HIMINTY	Sex.comp
662	Catholic	0.76	-0.674	1	0.0000000 (Catholic boys school)
929	Public	0.41	0.256	0	0.4210526
1123	Public	0.24	0.364	1	0.5675676
1298	Catholic	0.05	1.013	1	0.5000000 (Catholic coed school)
1304	Catholic	0.05	1.013	1	0.5000000 (Catholic coed school)
1383	Public	0.20	0.975	0	0.5769231
1536	Catholic	0.65	-0.575	0	0.0000000
1641	Public	0.21	0.845	0	0.4705882
1739	Public	0.25	0.742	0	0.4528302
1922	Catholic	1.00	-2.416	0	1.0000000 (Catholic girls school)

```
> hs.sch <- up( hs , ~ school)
> dim( hs.sch )
[1] 40 6
```

```
> some( hs.sch )
```

	school	Size	Sector	PRACAD	DISCLIM	id
2771	2771	415	Public	0.24	1.048	P2771
4292	4292	1328	Catholic	0.76	-0.674	C4292
4530	4530	435	Catholic	0.60	-0.245	C4530
5720	5720	381	Catholic	0.65	-0.352	C5720
6074	6074	2051	Catholic	0.32	-1.018	C6074
6897	6897	1415	Public	0.55	-0.361	P6897
7172	7172	280	Catholic	0.05	1.013	C7172
8531	8531	2190	Public	0.58	0.132	P8531
8707	8707	1133	Public	0.48	1.542	P8707
8854	8854	745	Public	0.18	-0.228	P8854

Only Level 2 variables –  
constant within schools

```
> hs.sch.all <- up( hs , ~ school, all = T)
> dim( hs.sch.all )
[1] 40 10
```

```
> some( hs.sch.all )
```

	school	mathach	ses	Sex	Minority	Size	Sector	PRACAD	DISCLIM	id
1317	1317	13.177687	0.34533333	Female	Yes	455	Catholic	0.95	-1.694	C1317
2629	2629	14.907772	-0.13764912	Male	No	1314	Catholic	0.81	-0.613	C2629
2658	2658	13.396156	0.43844444	Female	No	780	Catholic	0.79	-0.961	C2658
3992	3992	14.645208	0.36539623	Male	No	1114	Catholic	0.73	-1.534	C3992
5640	5640	13.160105	-0.17659649	Male	No	1152	Public	0.41	0.256	P5640
5650	5650	14.273533	0.02244444	Female	Yes	720	Catholic	0.60	-0.070	C5650

Level 2 and Level 2  
summaries of Level 1  
variables

Means of numeric variables, modes of factors

? up

agg

## *Types of variables in multilevel models*

1. Variables that **vary from student to student** within schools – Level 1
2. Variables that vary between schools and **do not vary within schools** – Level 2
  - a. Variables that are characteristics of the school
  - b. Variables that are **derived** from within school variables, e.g. **group mean** ses in the sample in the school.
3. (really a version of 1) Variables that are derived by combining 1 and 2: e.g. deviations from the within group mean ses, i.e. **within school variable centered within groups (CWG)**

$$\begin{array}{ccc} X & \bar{X}_s & X - \bar{X}_s \\ \hline \text{ses} & \text{ses} & \hline \end{array}$$

## Synonyms:

### 1. Variables that vary within clusters (=groups):

Level 1<sup>1</sup> variables (if we count from the bottom as in SPSS or HLM), micro variables, within cluster variable, time-varying variables (if X is time, student-level variables)

### 2. Variables that are constant within schools:

Level 2 variables (in SPSS, HLM), macro variables, between cluster variables, *contextual variables*, time-invariant variables (if X is time), school-level variables.

---

<sup>1</sup> I believe that Pinheiro and Bates are alone counting in the opposite direction: Level 0 is the whole population, Level 1 the schools, Level 2 the students. This only matters when predicting from a multilevel model.

Cities



School



Classes



Student

World R

4

3

2

1

~~0~~  
1

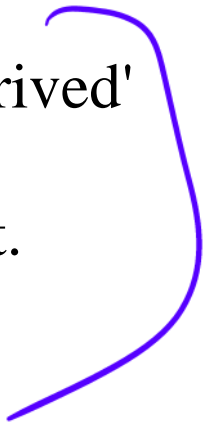
2

3

4

`predict(fit, level = 0)`

Note: The difference between a characteristic of the school and a 'derived' variable is that a derived variable could have a different value with a different sample of students. A characteristic of the school would not.



```
> hs$Sex.cat <- factor( ifelse( hs$Sex.comp == 1, "Girls",
                             ifelse( hs$Sex.comp == 0 , "Boys", "Coed")) )
> some(hs)
```

	X	school	mathach	ses	sector	female	Sex	Minority	Size
13	153	1317	12.283	0.482	1	1	Female	Yes	455
27	167	1317	6.973	0.302	1	1	Female	Yes	455
526	2284	3610	21.034	1.012	1	0	Male	No	1431
1394	5341	7342	23.271	-0.748	1	0	Male	No	1220
1417	5364	7342	12.821	-0.248	1	0	Male	No	1220
1441	5388	7342	11.664	0.862	1	0	Male	No	1220
1658	5722	7919	13.184	-0.038	0	0	Male	No	1451
1876	6504	8874	20.879	0.732	0	0	Male	Yes	2650
1884	6512	8874	24.479	0.652	0	0	Male	No	2650
1898	7106	9550	20.149	0.472	0	1	Female	No	1532

	Sector	PRACAD	DISCLIM	HIMINTY	Sex.comp	Sex.cat
13	Catholic	0.95	-1.694	1	1.0000000	Girls
27	Catholic	0.95	-1.694	1	1.0000000	Girls
526	Catholic	0.80	-0.621	0	0.4531250	Coed
1394	Catholic	0.46	0.380	1	0.0000000	Boys
1417	Catholic	0.46	0.380	1	0.0000000	Boys
1441	Catholic	0.46	0.380	1	0.0000000	Boys
1658	Public	0.50	-0.402	0	0.4324324	Coed
1876	Public	0.20	1.742	0	0.5833333	Coed
1884	Public	0.20	1.742	0	0.5833333	Coed
1898	Public	0.45	0.791	0	0.6551724	Coed

## Creating a more informative school id

```
> hs$sid <- factor( paste( substr( hs$Sector, 1,1),
                             hs$school, substr( hs$Sex.cat, 1,1), sep = ''))
# Keep each sector together, within sector order by mean ses:
> hs$sid <- reorder( hs$sid, hs$ses + 1000 * (hs$Sector == "Catholic"))
> some(hs)
```

	X	school	mathach	ses	sector	female	Sex	Minority	Size
137	789	2208	14.150	0.482	1	1	Female	No	1061
165	992	2458	7.814	-1.058	1	1	Female	Yes	545
231	1167	2626	10.350	-0.448	0	1	Female	No	2142
265	1201	2629	20.891	-0.278	1	0	Male	No	1314
358	1384	2658	9.459	0.702	1	1	Female	No	780
407	1505	2771	17.129	-0.328	0	1	Female	No	415
450	1548	2771	21.020	-1.098	0	1	Female	No	415
687	3029	4292	19.030	-0.498	1	0	Male	Yes	1328
953	3778	5640	16.212	-0.308	0	0	Male	No	1152
1617	5681	7890	0.930	-1.038	0	0	Male	No	311

	Sector	PRACAD	DISCLIM	HIMINTY	Sex.comp	Sex.cat	sid
137	Catholic	0.68	-0.864	0	0.5833333	Coed	C2208C
165	Catholic	0.89	-1.484	1	1.0000000	Girls	C2458G
231	Public	0.40	0.142	0	0.4736842	Coed	P2626C
265	Catholic	0.81	-0.613	0	0.0000000	Boys	C2629B
358	Catholic	0.79	-0.961	0	0.6000000	Coed	C2658C
407	Public	0.24	1.048	0	0.5090909	Coed	P2771C
450	Public	0.24	1.048	0	0.5090909	Coed	P2771C
687	Catholic	0.76	-0.674	1	0.0000000	Boys	C4292B
953	Public	0.41	0.256	0	0.4210526	Coed	P5640C

## *Easy manipulation of multilevel data*

Creating a multilevel data set:

1. Create a data set for each level, e.g. school and students. Or board, school and student with 3 levels.
2. Include an index variable for each level – a variable that has a unique value for each row of its data set. In each data set include the values of the index for the data set immediately above it. *key*
3. Make sure all variable names are unique across all data sets except for the index variables that need to have the same name in a data set and the data immediately below.

*How? You can use Excel and save as '.csv' file. Then read into R.*

```
> schoolfile <- read.csv("schoolfile.csv")  
> studentfile <- read.csv("studentfile.csv")
```

Merge files into a single combined file (often called a 'long' file) for analysis:

```
> combfile <- merge(schoolfile, studentfile)
```

Note: **hs** is already a long file in which Level 2 variables were entered directly in a Level 1 file. You can also do this but there are slightly higher chances of errors if Level 2 variables are entered inconsistently.

We saw above how to create a Level 2 derived variable from Level 1 data with **capply**

? merge

all = T  
all.x = T  
all.y = T

Going from the long file to the short file with 'school invariant' variables only:

```
> hs.sid <- up ( hs, ~ sid )  
> some( hs.sid )
```

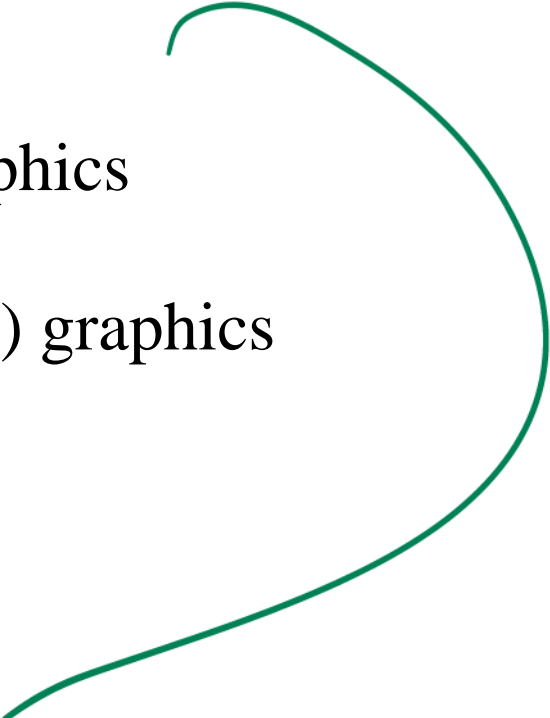
	school	sector	Size	Sector	PRACAD	DISCLIM	HIMINTY	Sex.comp
C2208C	2208	1	1061	Catholic	0.68	-0.864	0	0.5833333
C2658C	2658	1	780	Catholic	0.79	-0.961	0	0.6000000
C3610C	3610	1	1431	Catholic	0.80	-0.621	0	0.4531250
C4530G	4530	1	435	Catholic	0.60	-0.245	1	1.0000000

	Sex.cat	sid
C2208C	Coed	C2208C
C2658C	Coed	C2658C
C3610C	Coed	C3610C
C4530G	Girls	C4530G

# Looking at Hierarchical Data

*Look at relationships (mathach ~ ses) in hierarchical data*

## 3 main tools

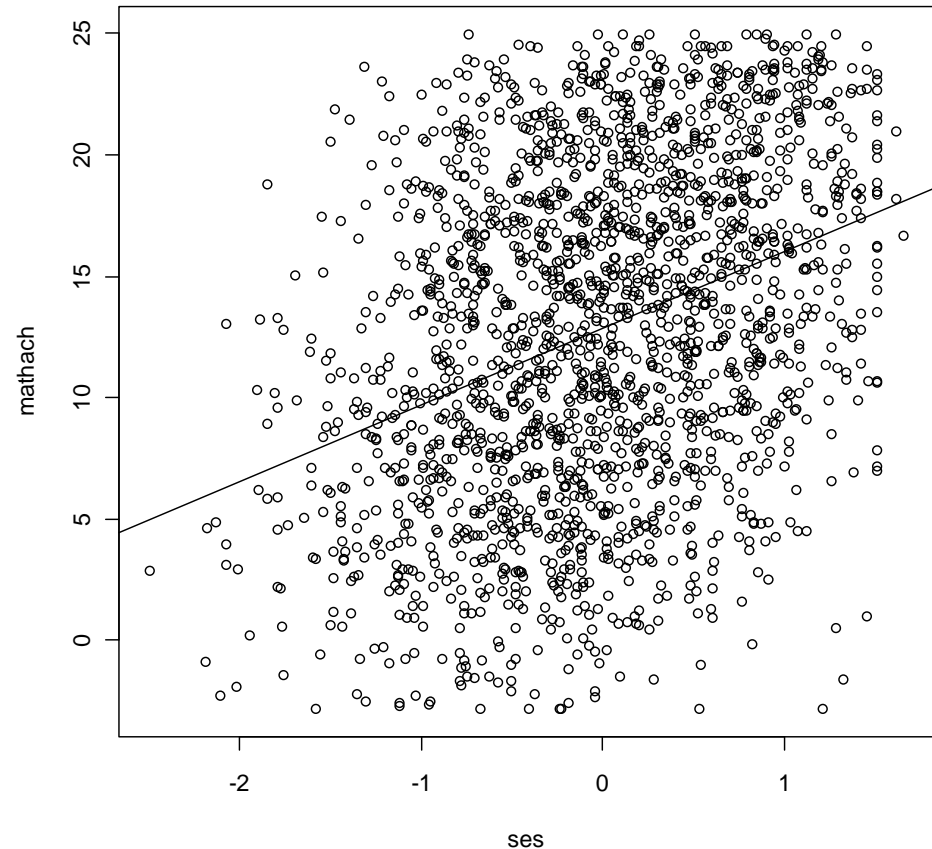
- 1) Traditional graphics
  - 2) Lattice (=trellis) graphics
  - 3) 3D graphics
- 

## Traditional graphics:

```
> fit <- lm( mathach ~ ses, hs )  
> plot( mathach ~ ses, hs )  
> abline( fit )
```

## Advantage:

- \* Easy to add new objects
- \* Intuitive
- \* Somewhat interactive

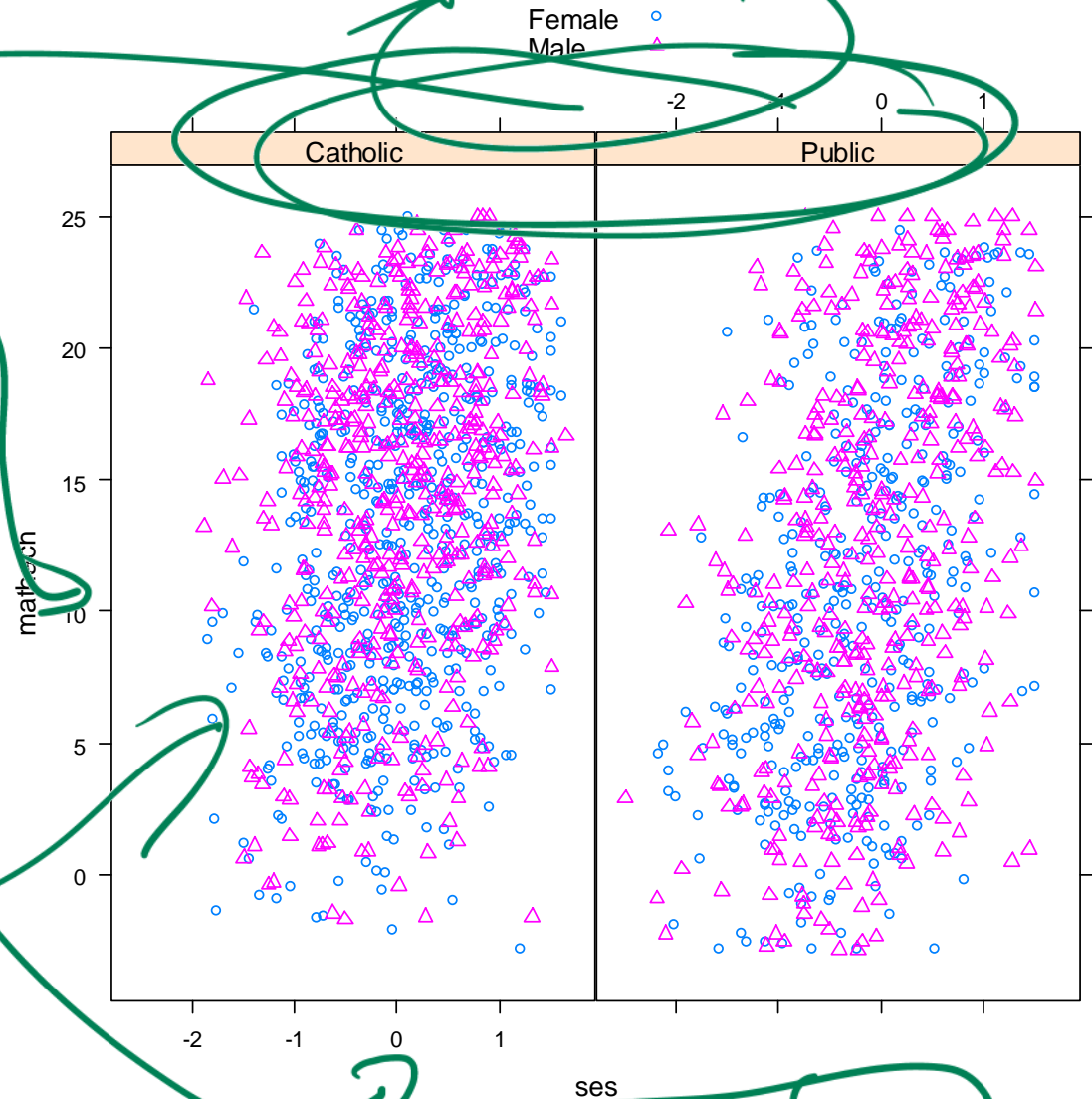


# Lattice graphics

Easy to create panels and groups within panels

```
> library(lattice)
> xyplot(mathach ~ ses
+       | Sector, hs,
+       groups = Sex,
+       auto.key = T)
```

But it's more difficult to add extra elements to the graph. This must be done in 'panel' functions that are called to generate each panel or with the 'trellis.focus' interface.

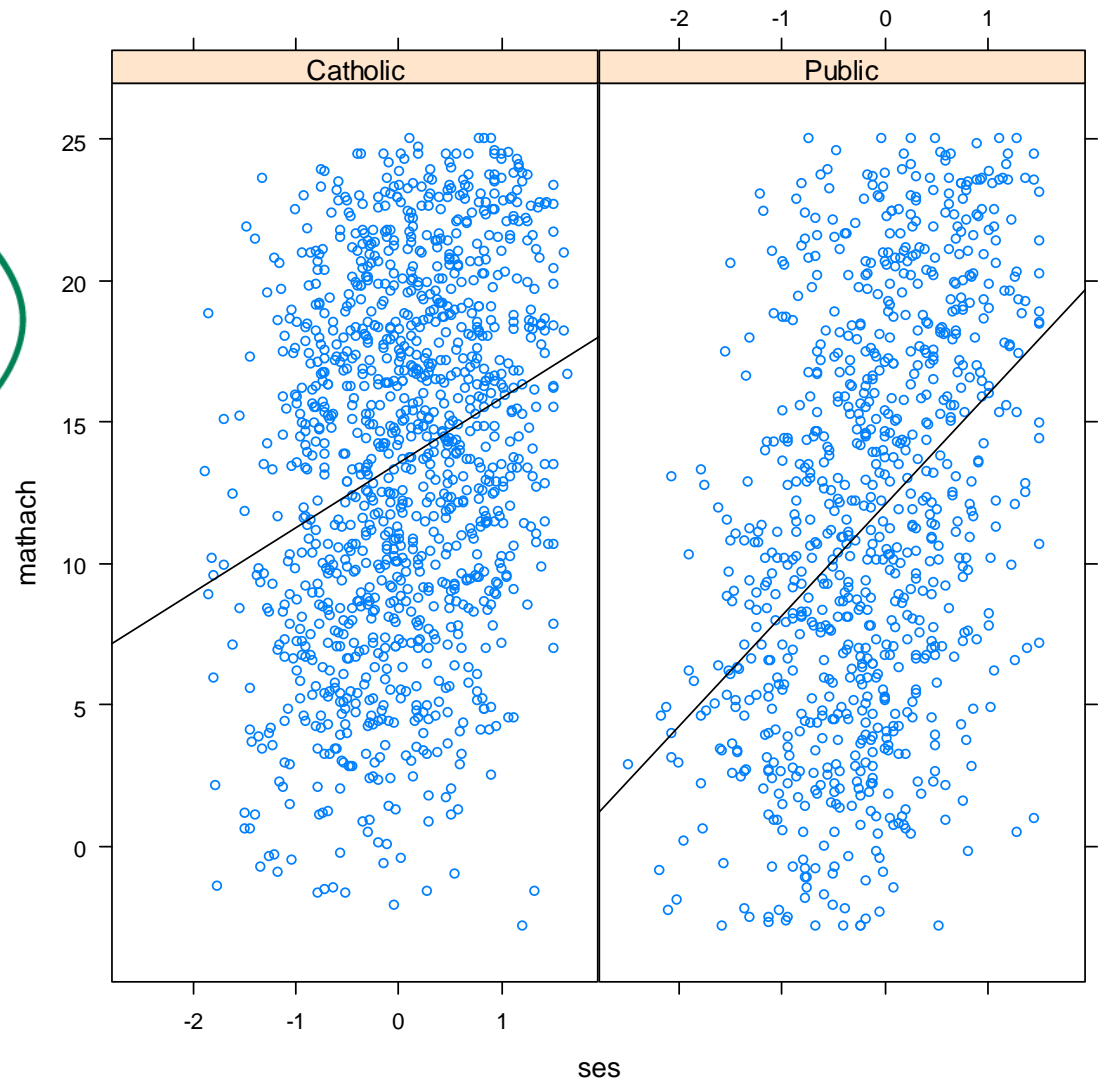


Sector \* Sex



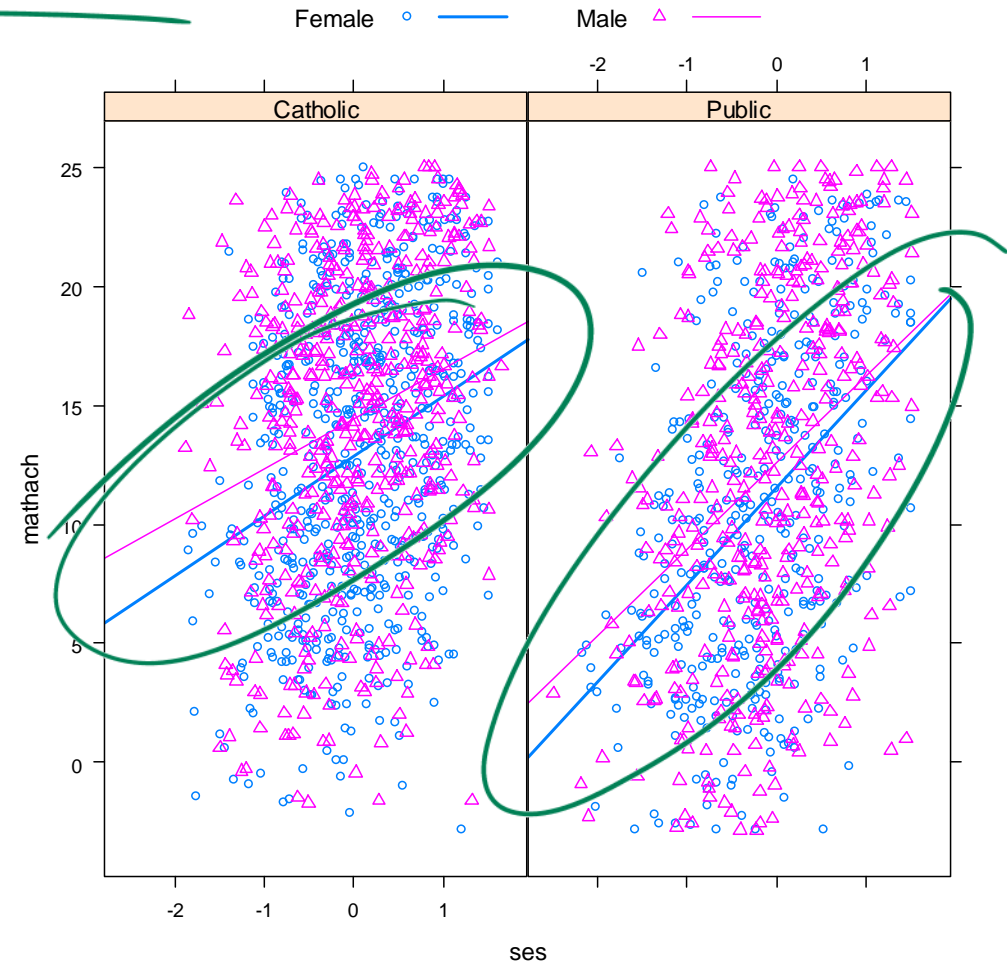
```
> xyplot(
  mathach ~ ses | Sector,
  hs,
  panel =
    function(x, y, ...) {
      panel.xyplot(x, y,
        ...)
      panel.lmline(x, y,
        ...) } )
```

The 'panel' function is defined on the fly. It uses arguments that will be passed to it automatically when it is called within xyplot to draw the panels. It uses convenience functions 'panel.xyplot' and 'panel.lmline' that are designed to work well within panels. Try `??panel`



```
xyplot( mathach ~ ses | Sector , hs,  groups = Sex,
        auto.key = list( columns = 2, lines = T, points = T),
        panel = panel.superpose.2,
        panel.groups = function(x, y, ...) {
          panel.xyplot( x, y, ...)
          panel.lmline( x, y ,...) })
```

A more complex example using groups and `panel.groups` that is called for each group within each panel.



ggplot2

## Exploring the relationship between mathach and ses

We want to explore how mathach and its relationship with ses differ between sectors.

As mentioned previously there are a number of plausible approaches:

- 1) Pooling the data: ignore schools, just pool all the data in each sector together and do an OLS regression.

```
lm( mathach ~ ses * Sector, hs)
```

- 2 a) Use a fixed effects model (version 1) to estimate relationship in each school and then compare the mean level of each sector.

```
lml <- lm( mathach ~ factor(school)/ses -1,  
          hs)
```

```
ddu <- up( hs, ~ factor(school))  
ind <- ddu$Sector == "Catholic"  
L <- rbind( "Catholic" = ind,
```

```

"Public" = 1-ind)
L <- L/apply(L,1, sum)
L <- cbind( rbind( L, 0,0), rbind( 0,0,L))
rownames( L ) <- c("Cath Int", "Pub Int",
                  "Cath Slope", "Pub Slope")

wald (lml, L)
diffmat <- rbind( "Int" = c( -1, 1, 0, 0),
                  Slope = c( 0 , 0, -1, 1))

```

```

wald (lml, diffmat %*% L)

```

	numDF	denDF	F.value	p.value
	1	2	1897	21.03533 <.00001

	Estimate	Std.Error	DF	t-value	p-val
Int	-2.027255	0.351992	1897	-5.759378	<.00001
Slope	1.109995	0.454114	1897	2.444309	0.0146

*Difference of averages that give equal weight to each school. Uses only within-school variability except for pooled estimate of  $\sigma^2$ .*

Question: Why is this so complicated? Can't we just fit a model regressing on SES, School and Sector to estimate the effect of Sector?

$$Y \sim X * (\text{School} + \text{Sector})$$

2 b) Fixed effects model (version 2): OLS regression with different intercepts in each school but common slopes in each sector.

```
fit2 <- lm( mathach ~ ses + factor(school)
            + ses:Sector, hs)
wald( fit2, "ses:Sector")
```

Coefficients	Estimate	Std.Error	DF	t-value	p-value
ses:Sector	1.204719	0.436134	1935	2.762267	0.00579

*Here we assume all slopes are the same within each sector. The average Sector slope gives more weight to schools with larger samples and more spread in ses. Between school variability in levels plays no role in SEs.*

	S2	S3	Sec Pub.	
1	0	0	0	<del>(X X X)</del>
1	0	0	0	
1	0	0	0	
1	1	0	1	
1	1	0	1	
1	1	0	1	
1	0	1	1	
1	0	1	1	
1	0	1	1	
1	0	1	1	

3) MANOVA approach. Get individual school intercepts and slopes as in 2a but then do a MANOVA to compare the two sectors.

```
lcoefs <- coef(lmList( mathach ~ ses |  
                    factor(school), hs))  
lm.mult <- lm( as.matrix(lcoefs) ~ Sector,  
              up( hs, ~ factor(school)))  
summary(lm.mult)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )	
(Intercept)	1.6672	0.3506	4.756	2.84e-05	***
SectorPublic	1.1100	0.5086	2.182	0.0353	*

*SE is measured from between school variability not within school variability. The fact that the precision of estimates varies from school to school is ignored. However inferences to generalize to the larger population. Note the larger p-value*

4) Ecological or between school model: Summarize the data from each school with the mean ses and the mean mathach from each school. Do an OLS regression on the resulting data.

```
fit.eco <- lm( mathach ~ ses,  
              up( hs, ~ factor(school), all = T))  
summary( fit.eco )
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )	
(Intercept)	12.7926	0.2898	44.150	< 2e-16	***
ses	5.7734	0.6984	8.267	5.1e-10	***

*This is estimating something totally different: the difference in between school slopes, not within school slopes*

$\gamma$

Ecological  
mean

Pooled  
 $Y \sim X$

$Y \sim X + S_c$

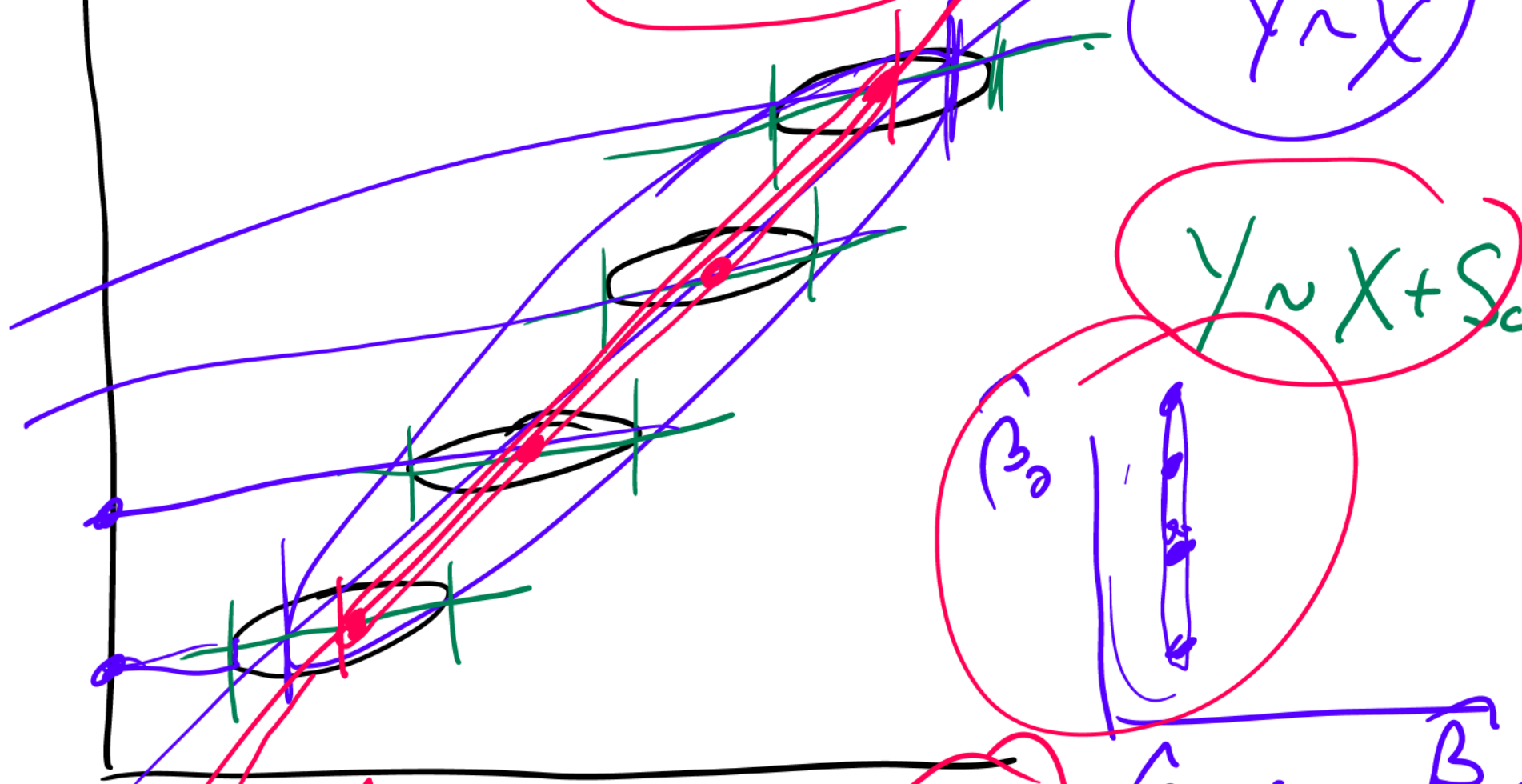
$\beta_0$

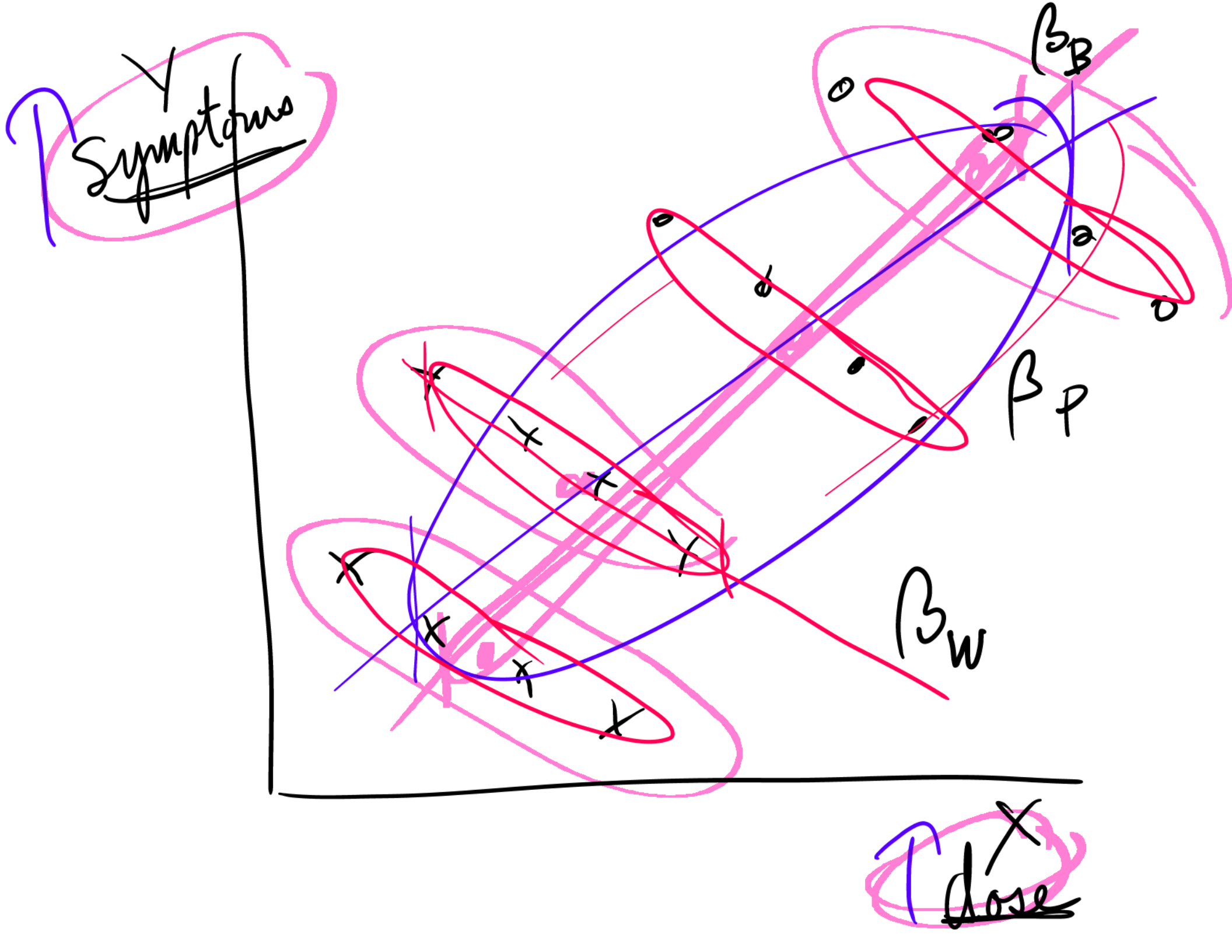
$X$

$\hat{\beta} \sim 1$

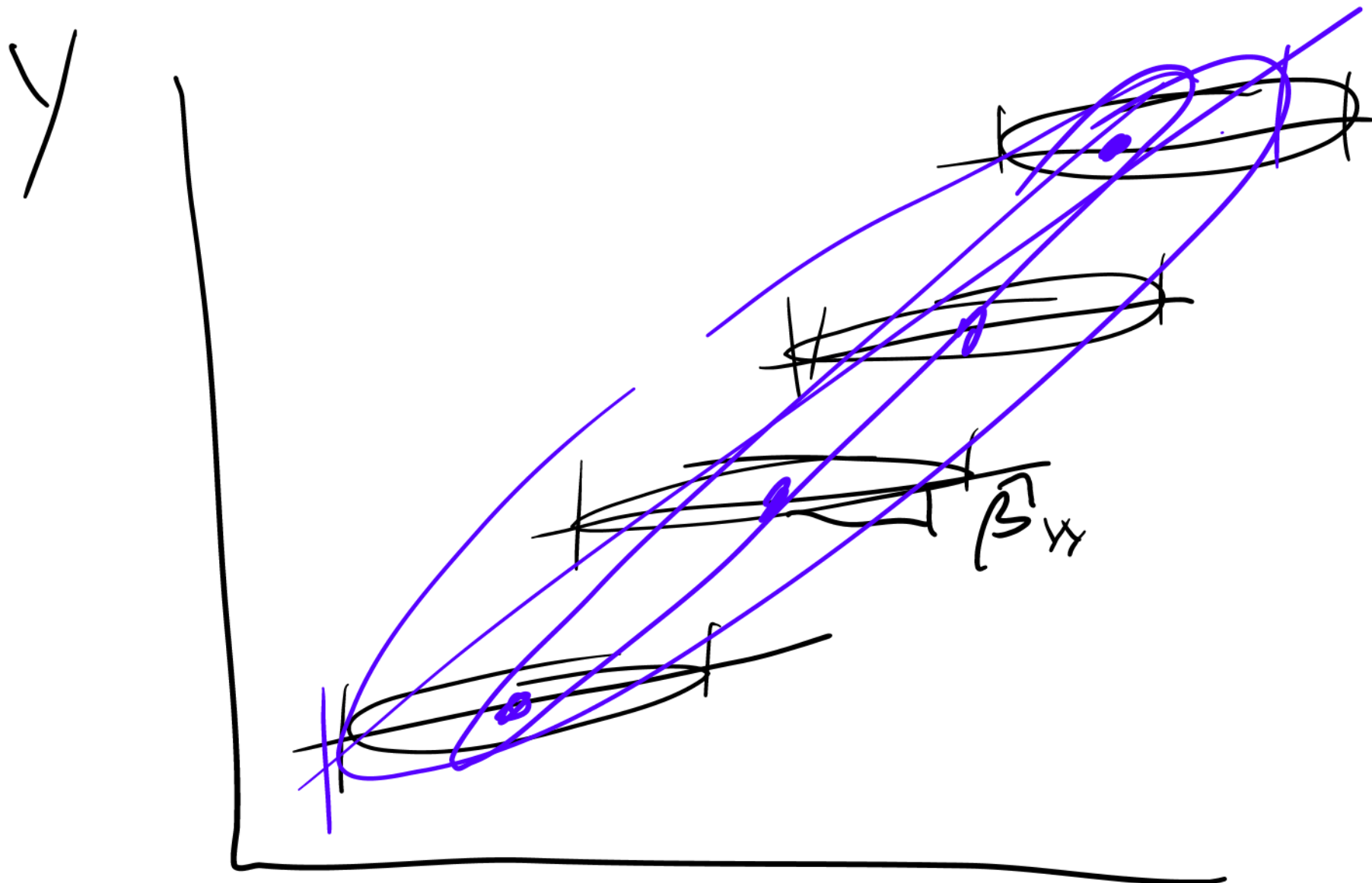
$\beta_x$

within  
between  
pooled





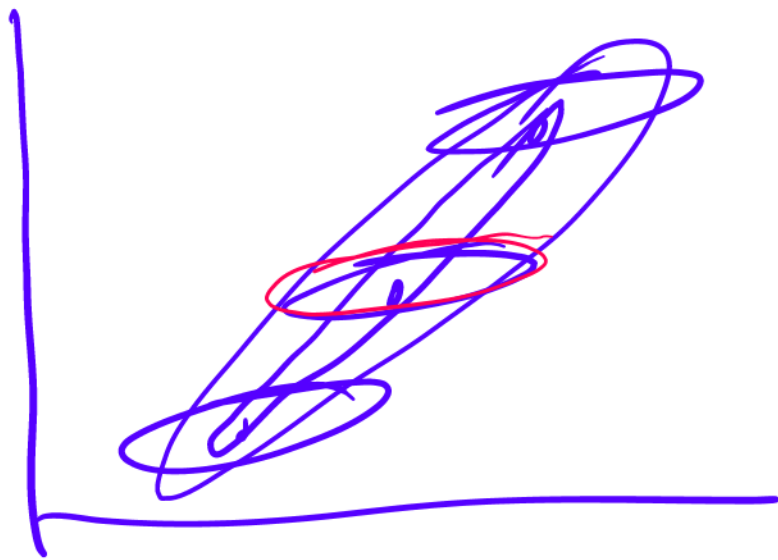




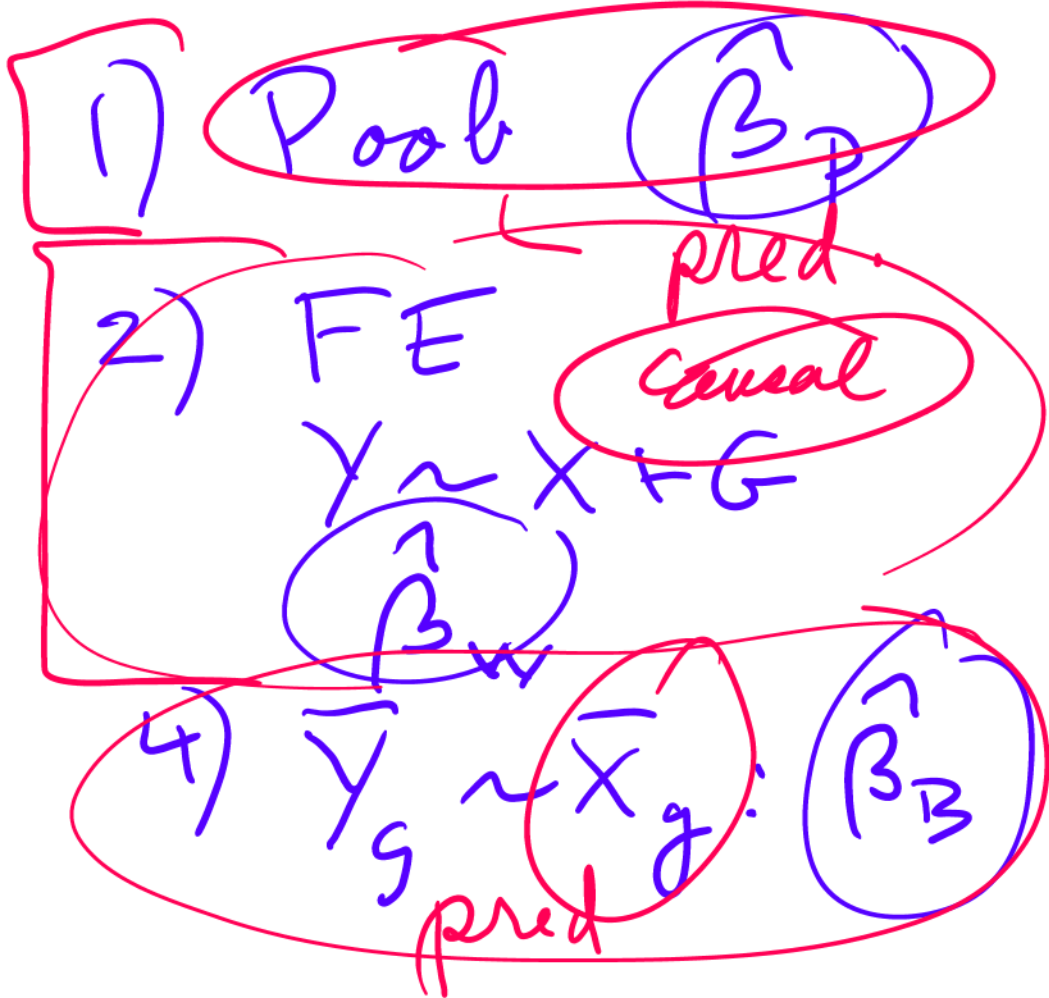
$$Y \sim X + G \sim \hat{\beta}_w$$

$$Y \sim X : \hat{\beta}_p$$

$$\left. \begin{array}{l} Y \sim \bar{X}_g \\ \bar{Y}_g \sim \bar{X}_g \end{array} \right\} \hat{\beta}_B$$



all confounding factors are constant within each level of  $G$ . Then  $\hat{\beta}_w$  estimates causal effect of  $X$ .



Problem with FE



FE ignores variability between groups.

## 5) Use a Hierarchical Linear Model

The HLM uses both between school variation and within school variation to estimate the standard error of estimates. Inference generalizes to the larger population. Some estimates in the HLM rely on the assumption that between school and within school effects are the same.

## 6) Use a Hierarchical Linear Model with appropriate contextual variables.

Using a derived contextual variable for ses (group mean ses in each school) as well as raw (or centered within school) ses allows separate unbiased estimation of both within and between school effects.



## Method 1: Pooling of data – ignore schools

```
> fit.pooled <- lm( mathach ~ ses * Sector, hs)
> summary(fit.pooled)
```

Call:

```
lm(formula = mathach ~ ses * Sector, data = hs)
```

Residuals:

Min	1Q	Median	3Q	Max
-19.1774	-4.8286	0.2949	4.9595	15.7836

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )	
(Intercept)	13.5579	0.1881	72.067	< 2e-16	***
<b>ses</b>	<b>2.2999</b>	<b>0.2582</b>	<b>8.908</b>	<b>&lt; 2e-16</b>	<b>*** [eff. of ses Cath]</b>
<b>SectorPublic</b>	<b>-1.4666</b>	<b>0.2921</b>	<b>-5.021</b>	<b>5.60e-07</b>	<b>*** [Pub-Cath ses=0]</b>
ses:SectorPublic	<b>1.6051</b>	0.3845	4.174	<b>3.12e-05</b>	<b>*** [diff. of slopes]</b>

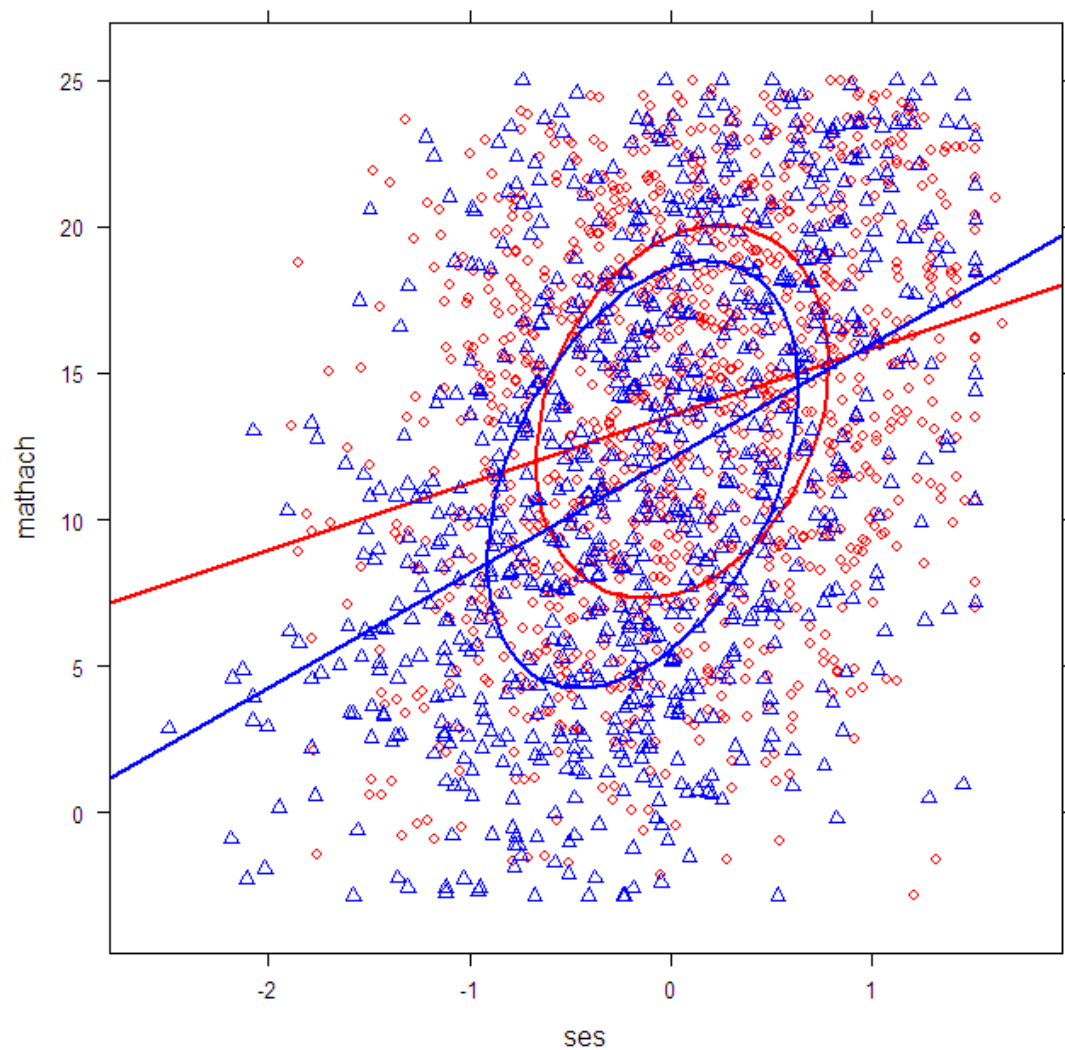
Residual standard error: 6.344 on 1973 degrees of freedom

Multiple R-squared: 0.1404, Adjusted R-squared: 0.1391

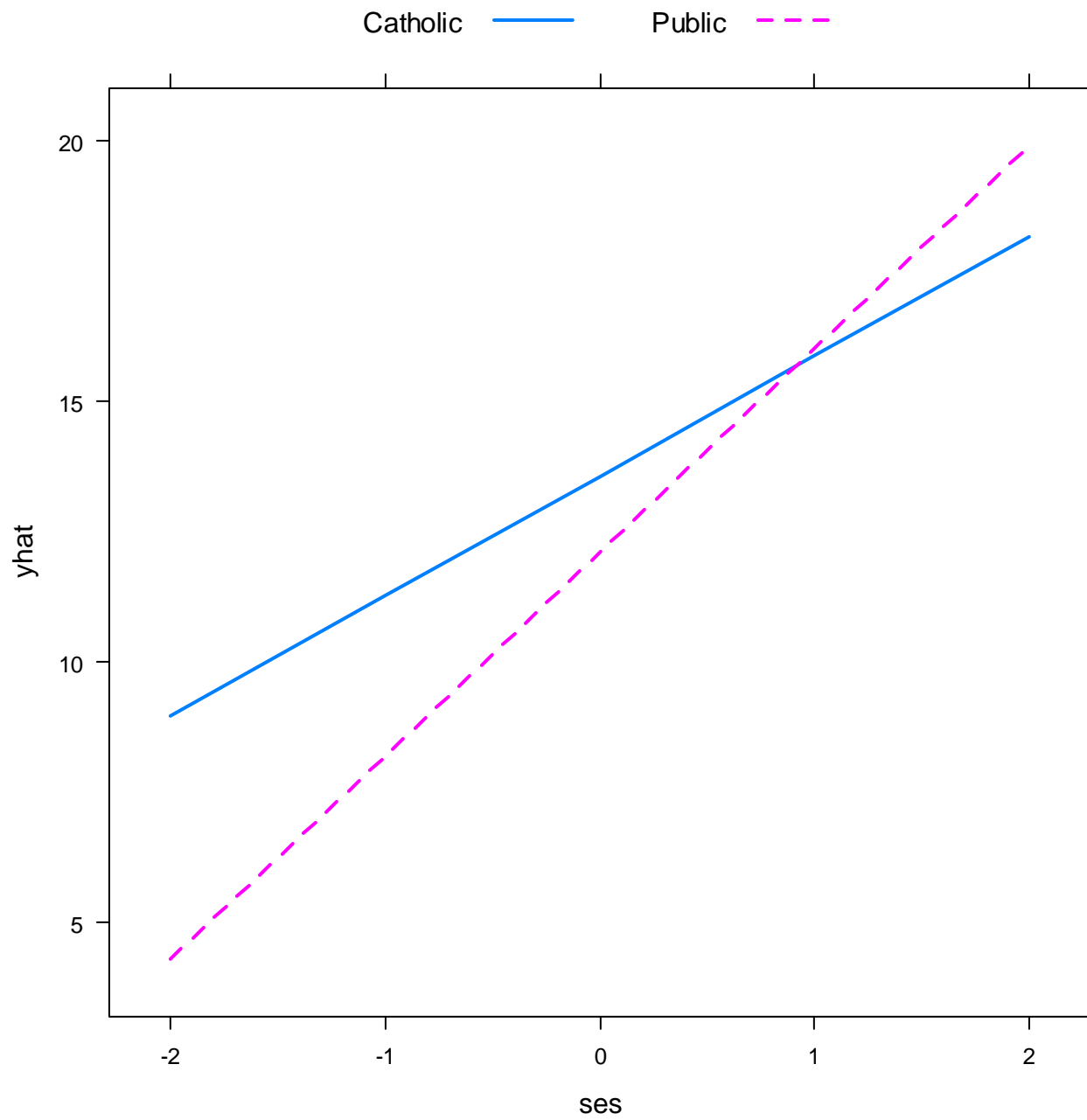
F-statistic: 107.4 on 3 and 1973 DF, p-value: < 2.2e-16

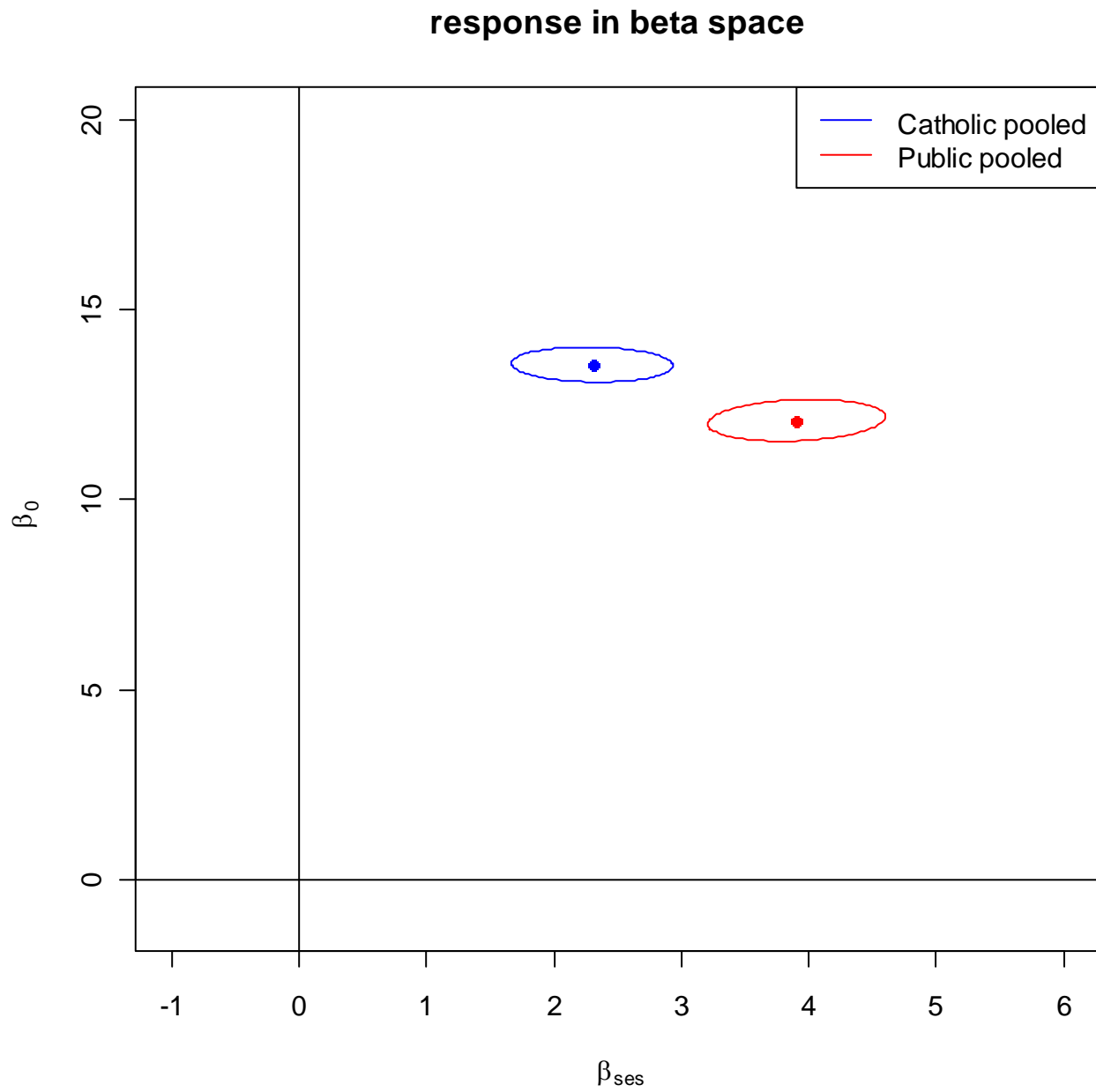
Coefficients in blue are 'marginal' to the interaction and should be interpreted – if at all – with care. The coefficient for "ses" (2.2999) is NOT "the estimated effect of ses" – it is the estimated "effect" of ses when SectorPublic = 0, i.e. in Catholic schools

## Method 1: Fitted lines



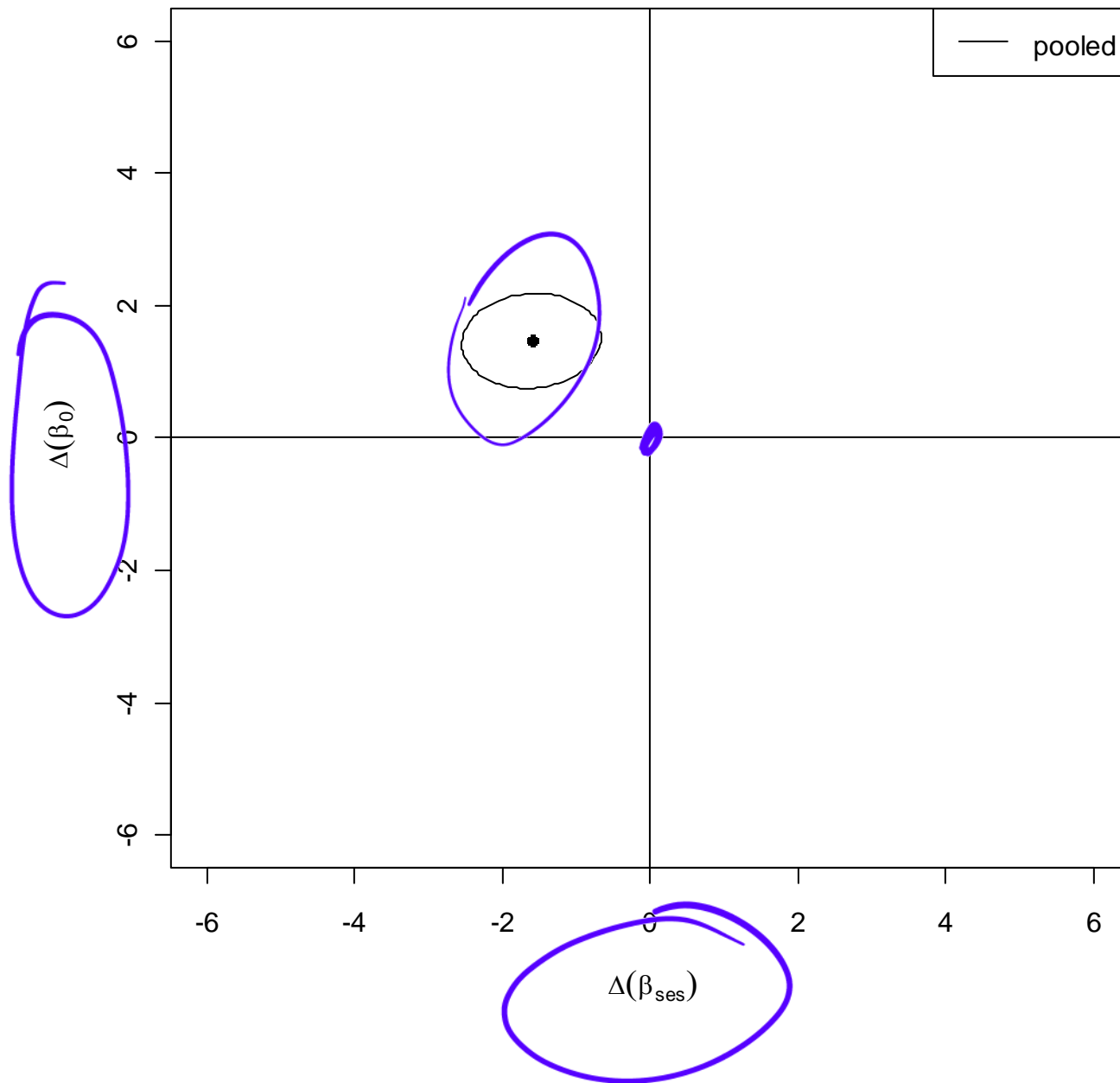
The code to produce this and following graphs is contained in the on-line appendix



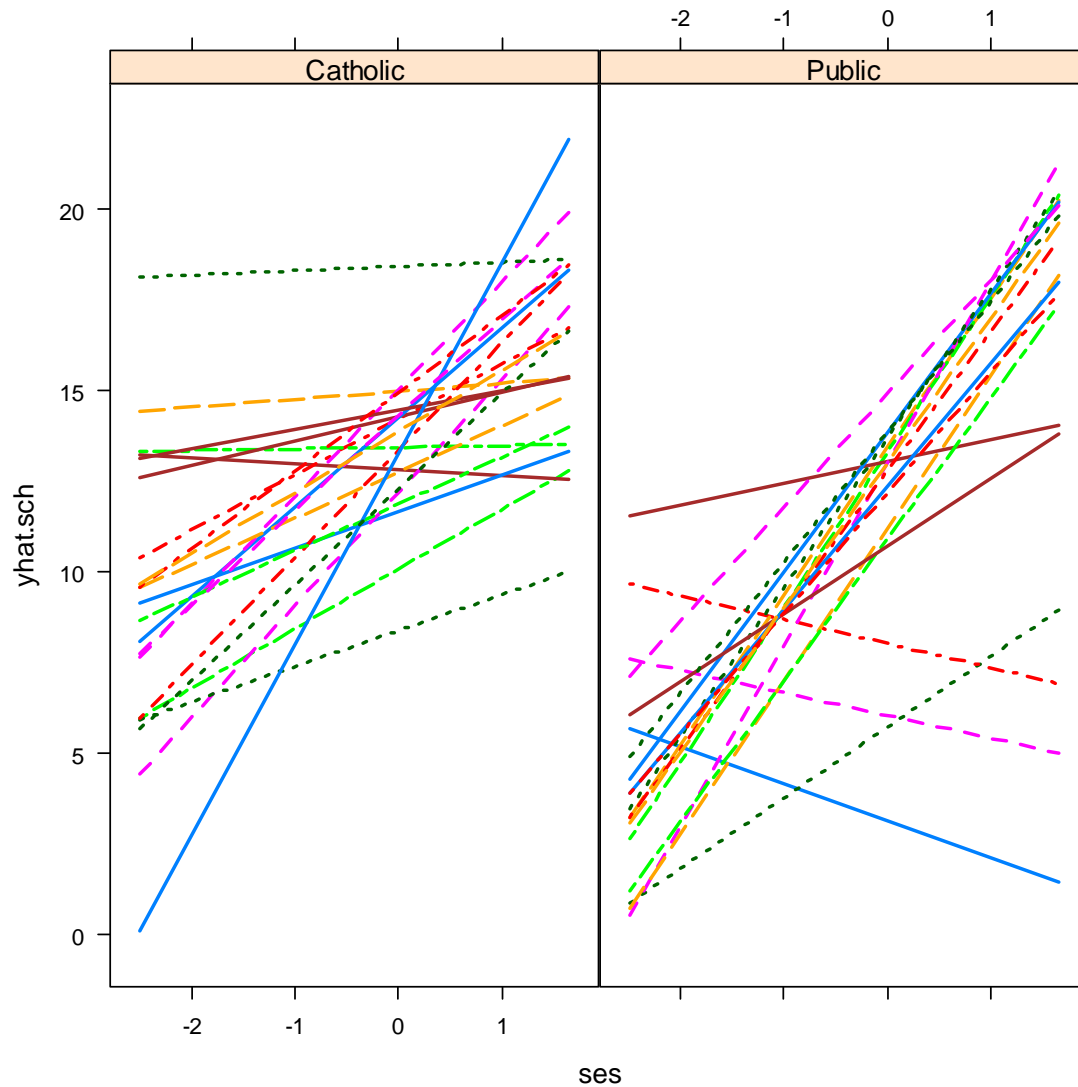


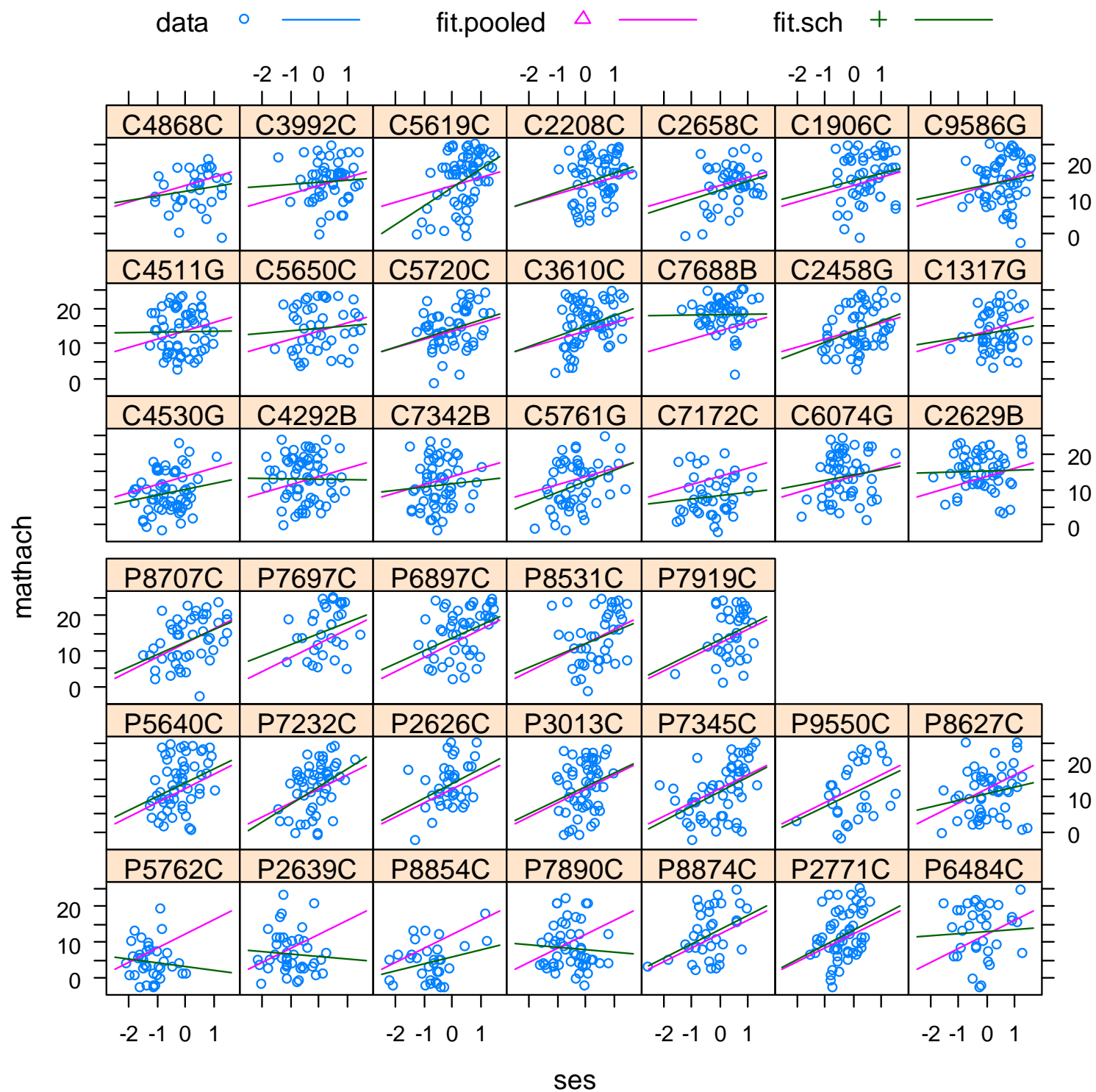
**Figure 4: 95% confidence ellipse for intercept and slope in each Sector**

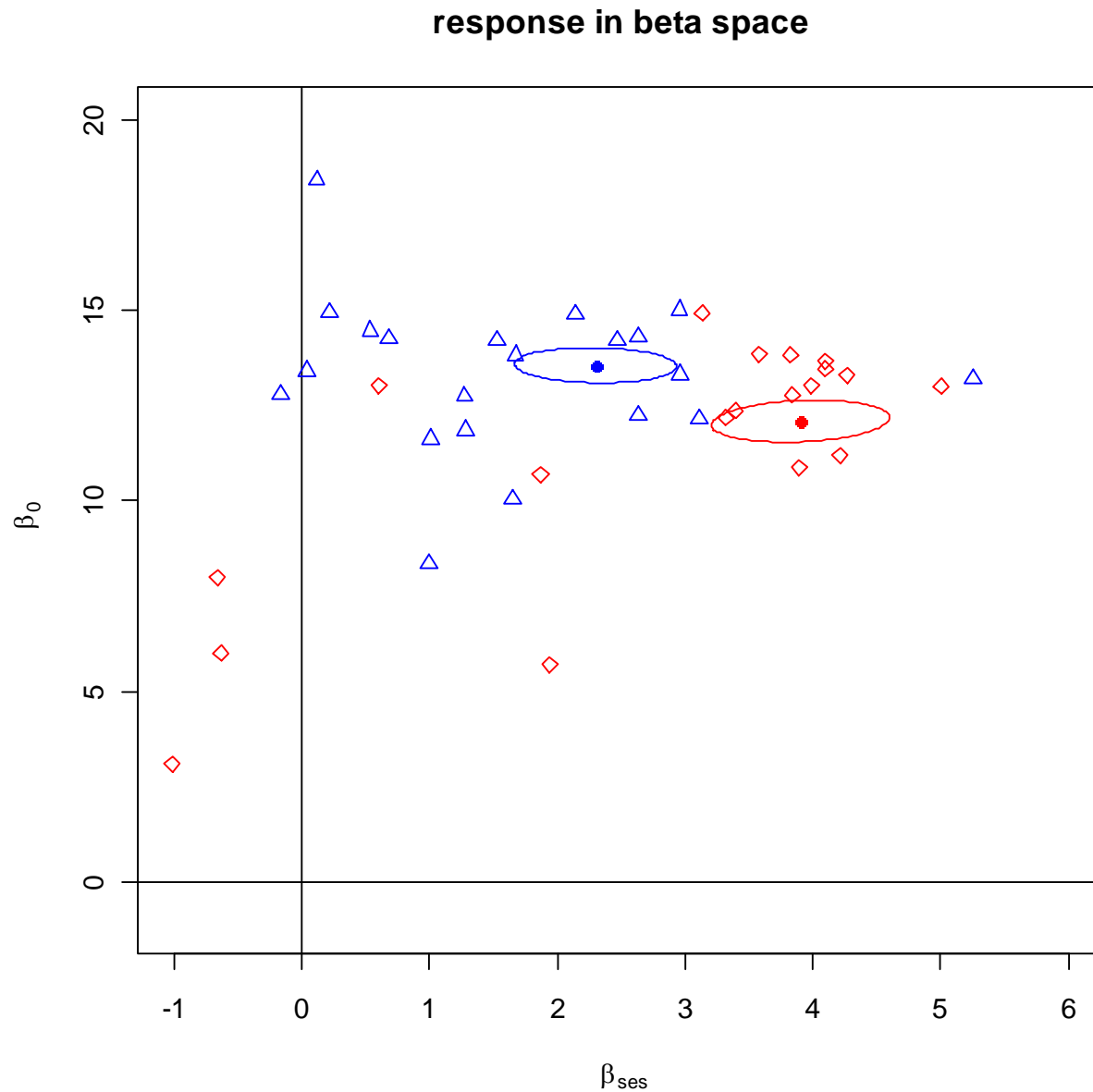
difference between sectors in beta space



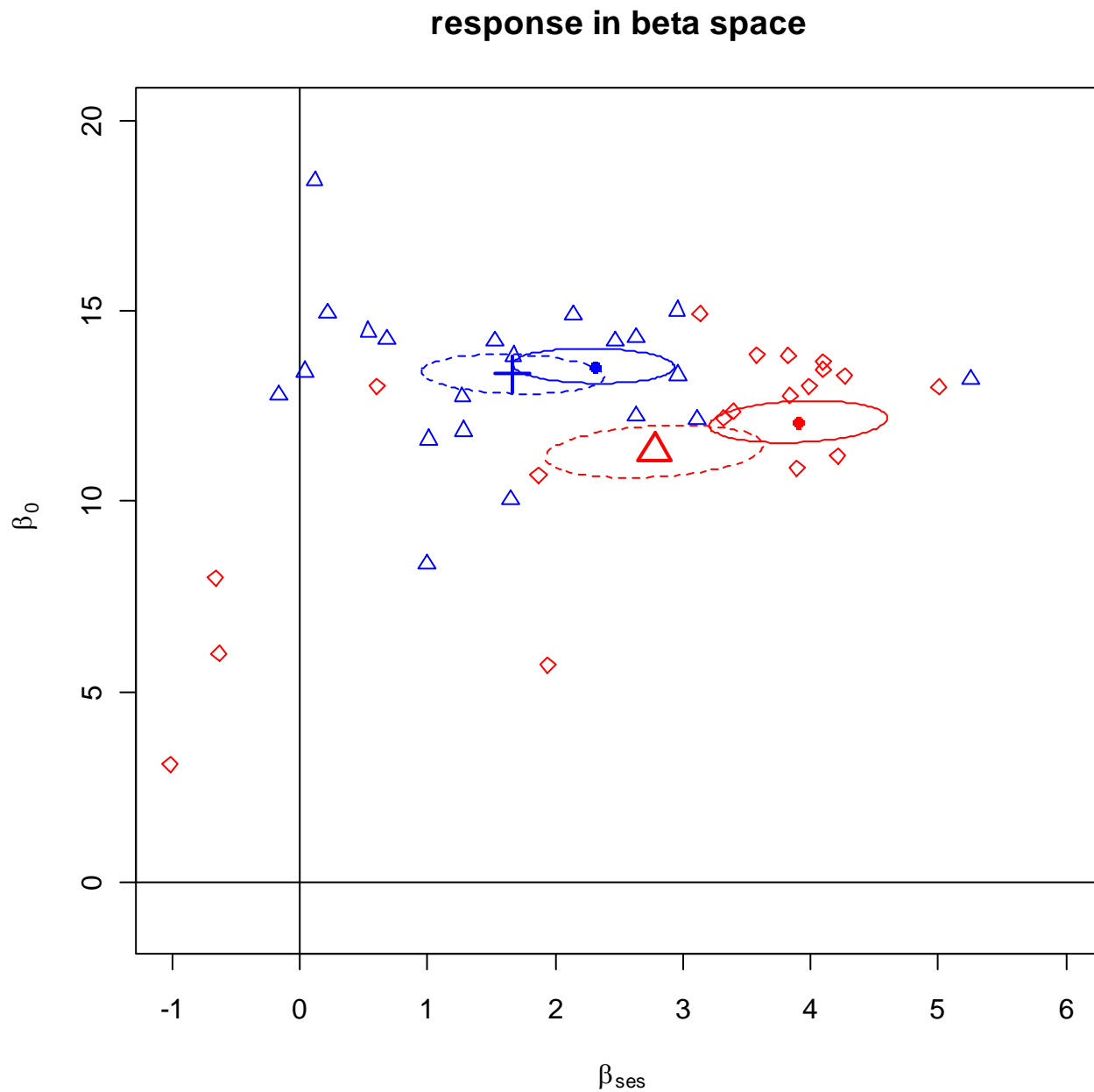
## Method 2: Fit each school then average slopes and intercepts in each sector



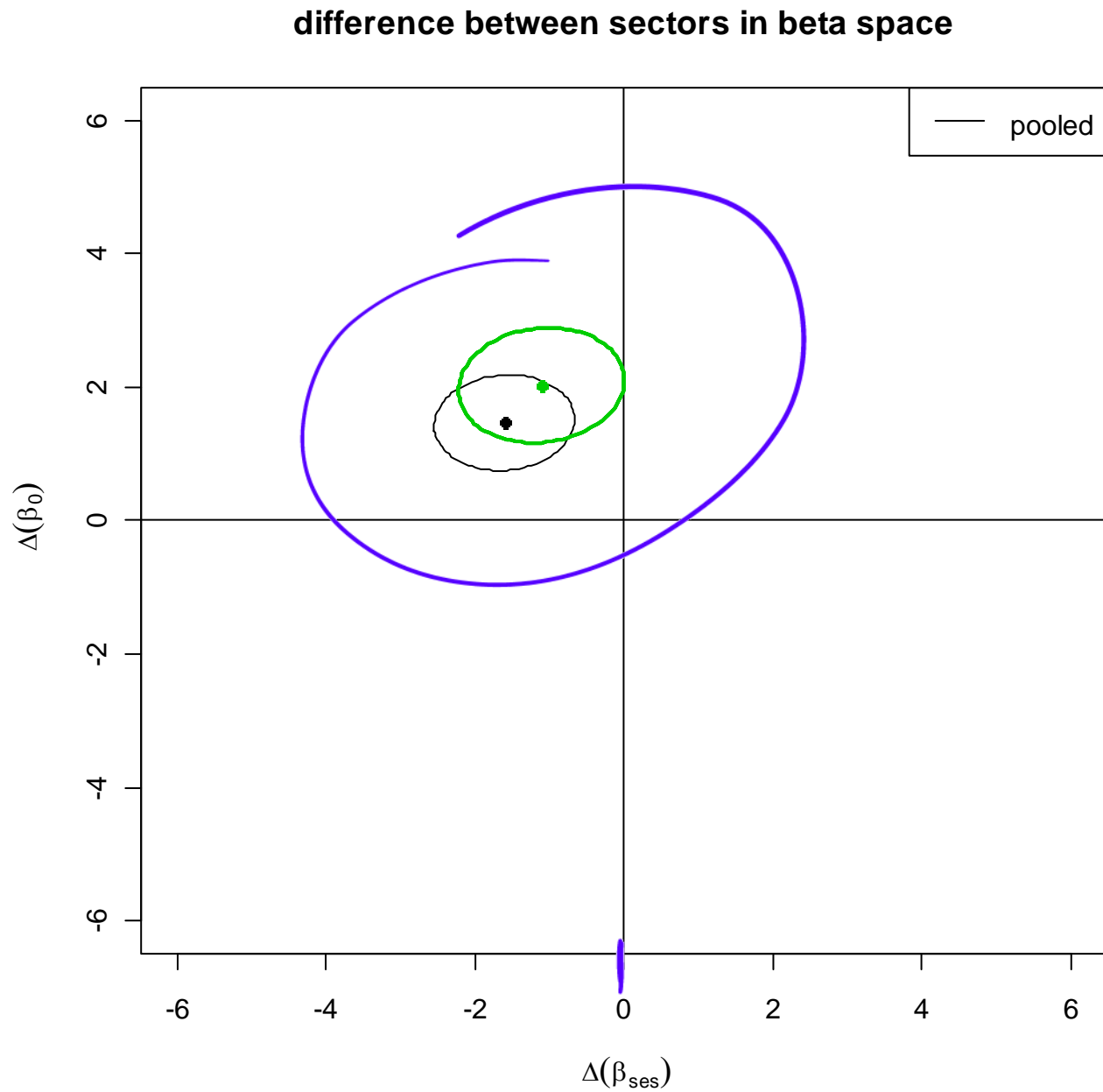




**Figure 5: Pooled data estimates with CE plus estimated line for each school**  
**Estimated lines for each sector using pooled data + estimated line for each school**



**Figure 6: Adding Sector means and CEs based on averaging the estimate for each school**



**Figure 7: Adding CE based on average of schools**

**What is the problem with this?**

**The estimated std. error depends ONLY on within school variability**

In other words if we moved the individual school arbitrarily far apart we would still have the same CE for the Sector effect.

*Principle of marginality*  $\equiv$  *Principle of invariance:*  
*Things that shouldn't matter, shouldn't matter!*

*'Principle of variance':*  
*Things that should matter, should matter!*

If this method gives us exactly the same answer regardless of the between school variability that signals that the SE can not generalize to the population of schools -- only to the putative population of new student samples within these PARTICULAR schools.

*We ignored that we shouldn't ignore? The between school variation.*

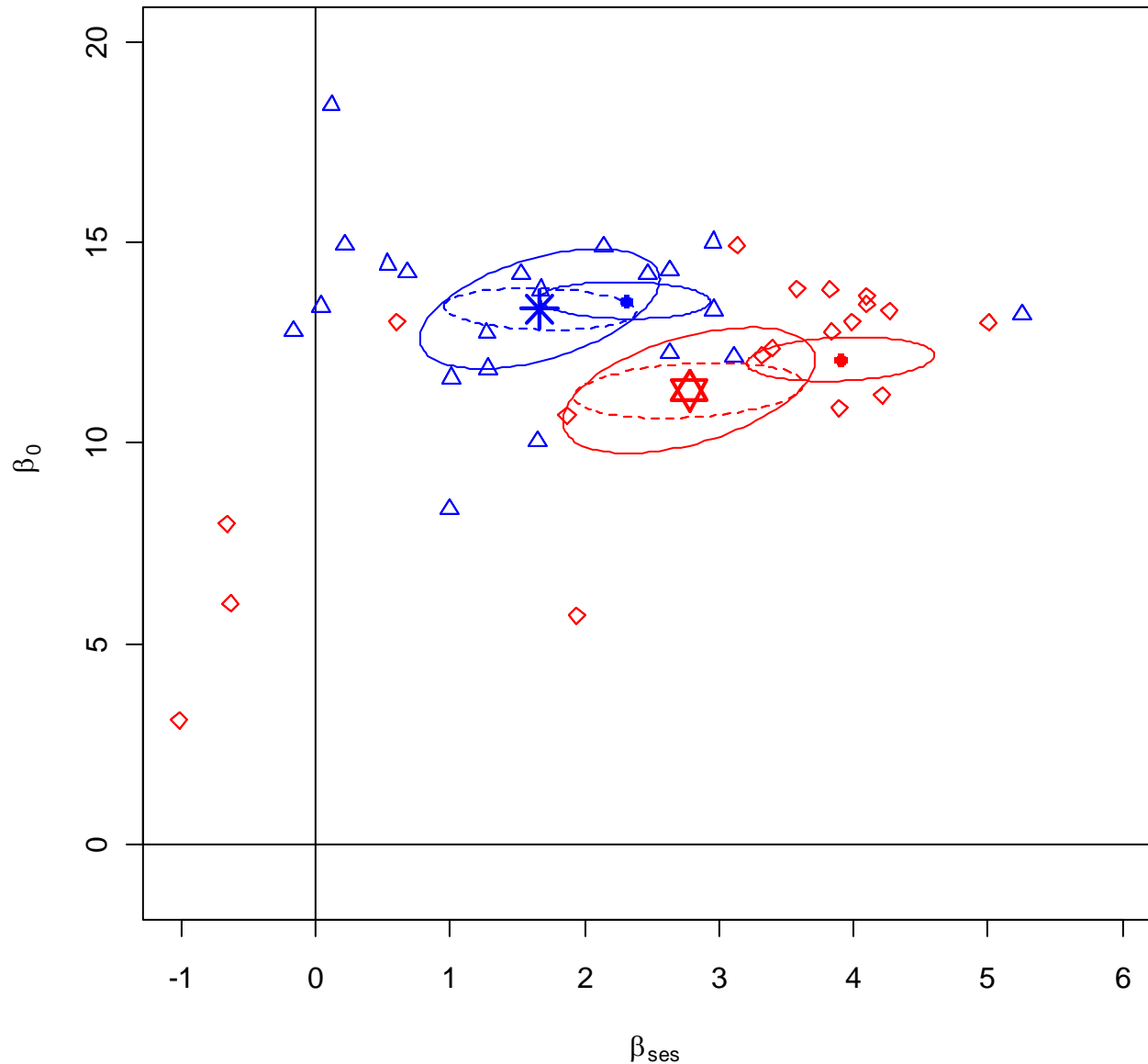
## **Method 3: Two-stage approach or 'derived variables' approach**

Idea:

**First:** Estimate slope and intercept within each school as we did in Method 2.

**Second:** Use the estimated slopes and intercepts as a multivariate sample and do a MANOVA test of equality of the two sector means.

response in beta space



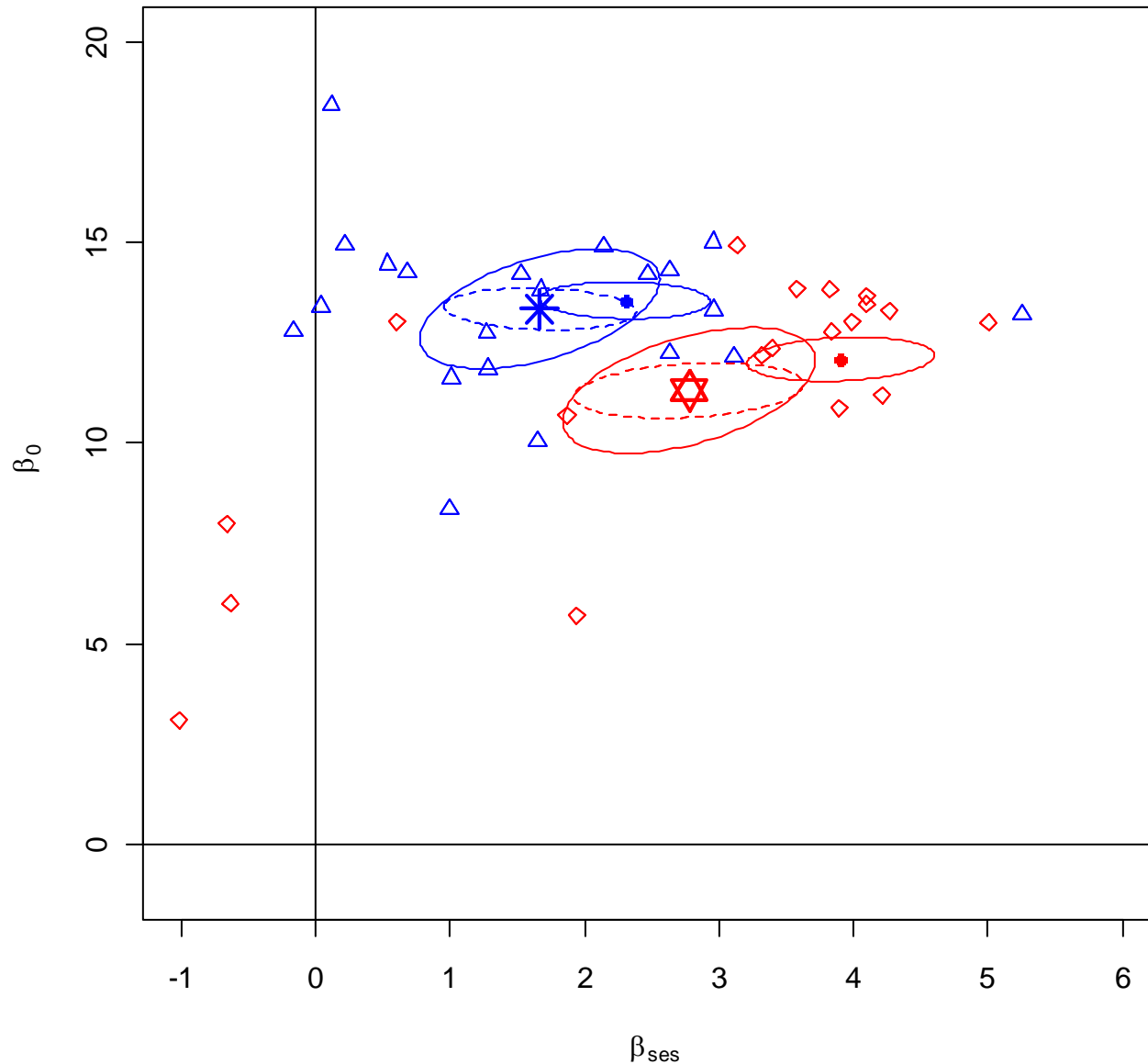
Dashed ellipse was obtained from fixed effects model.

Solid ellipse with same center, from Manova model.

Note:

- 1) both have the same centre
- 2) the latter is larger because it generalizes to new samples with **MORE** variability.

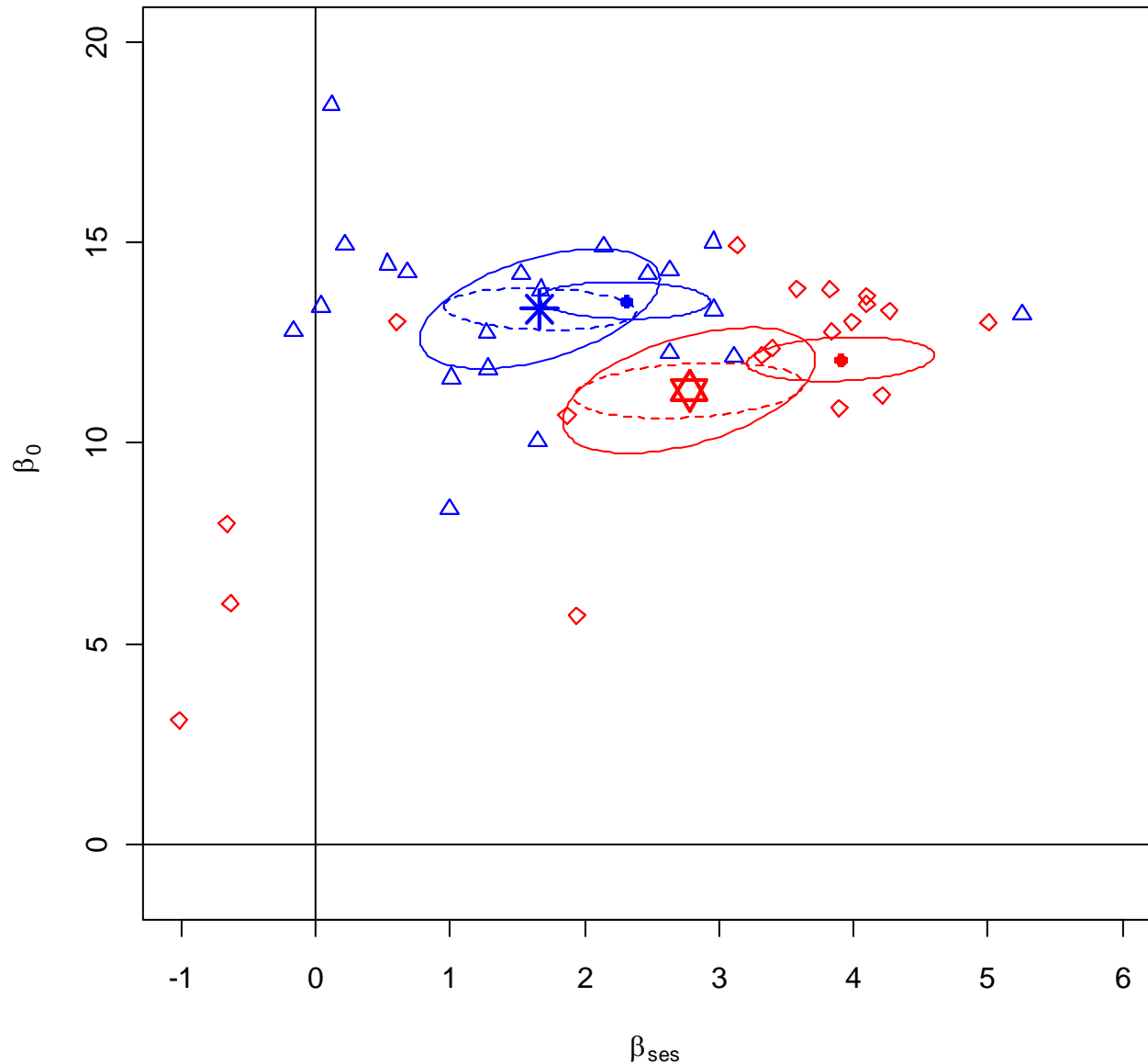
response in beta space



The fixed effects model 95% CE is valid for new samples of students from the same schools. It does not generalize to the population of schools.

The Manova model **does** generalize to new schools

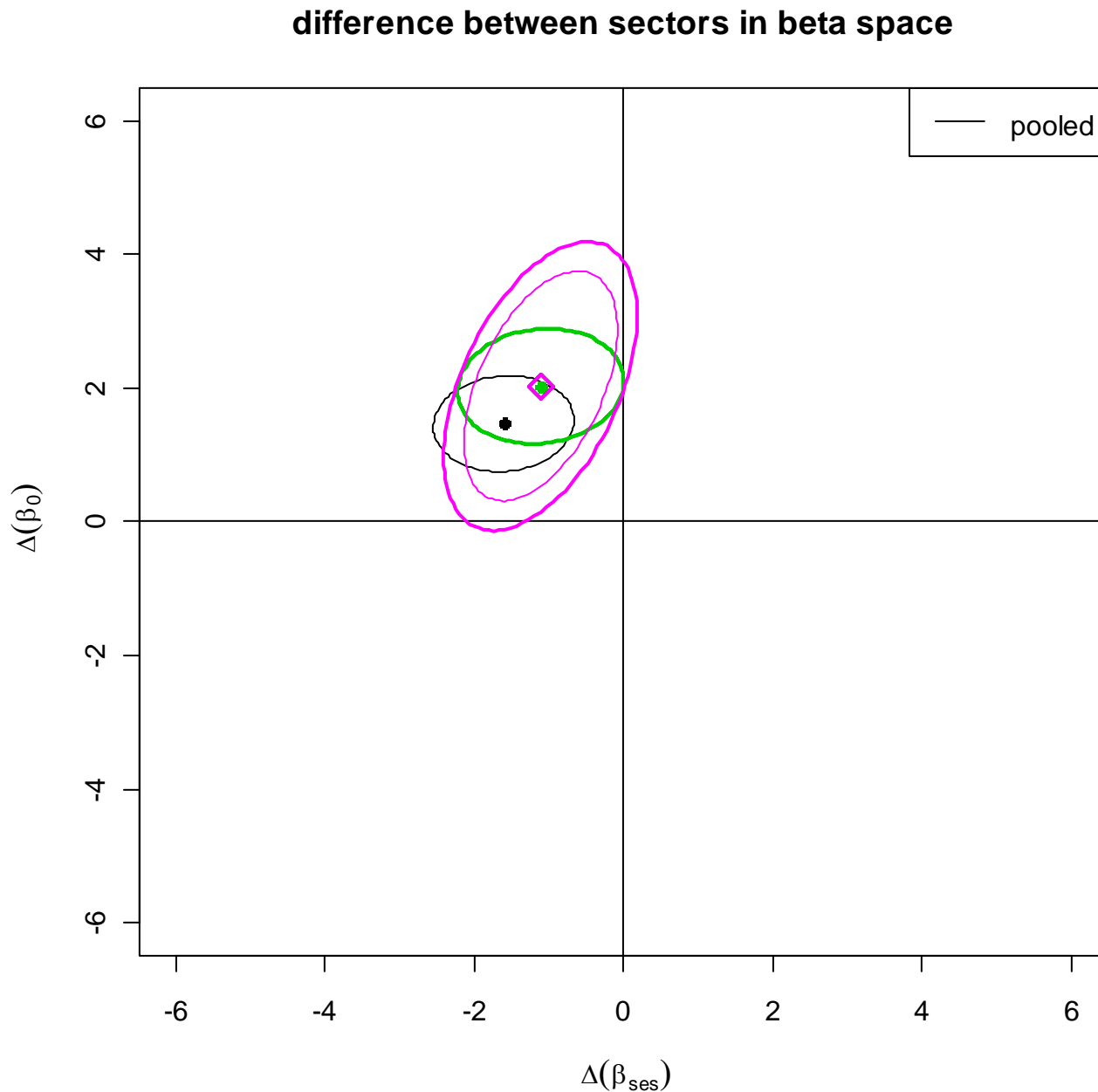
response in beta space



Disadvantages  
(often small):

- gives equal weight to all schools regardless of information in sample (n, spread of ses)

- need to discard data from schools where there are too few points to fit a model (here if  $n=1$ )

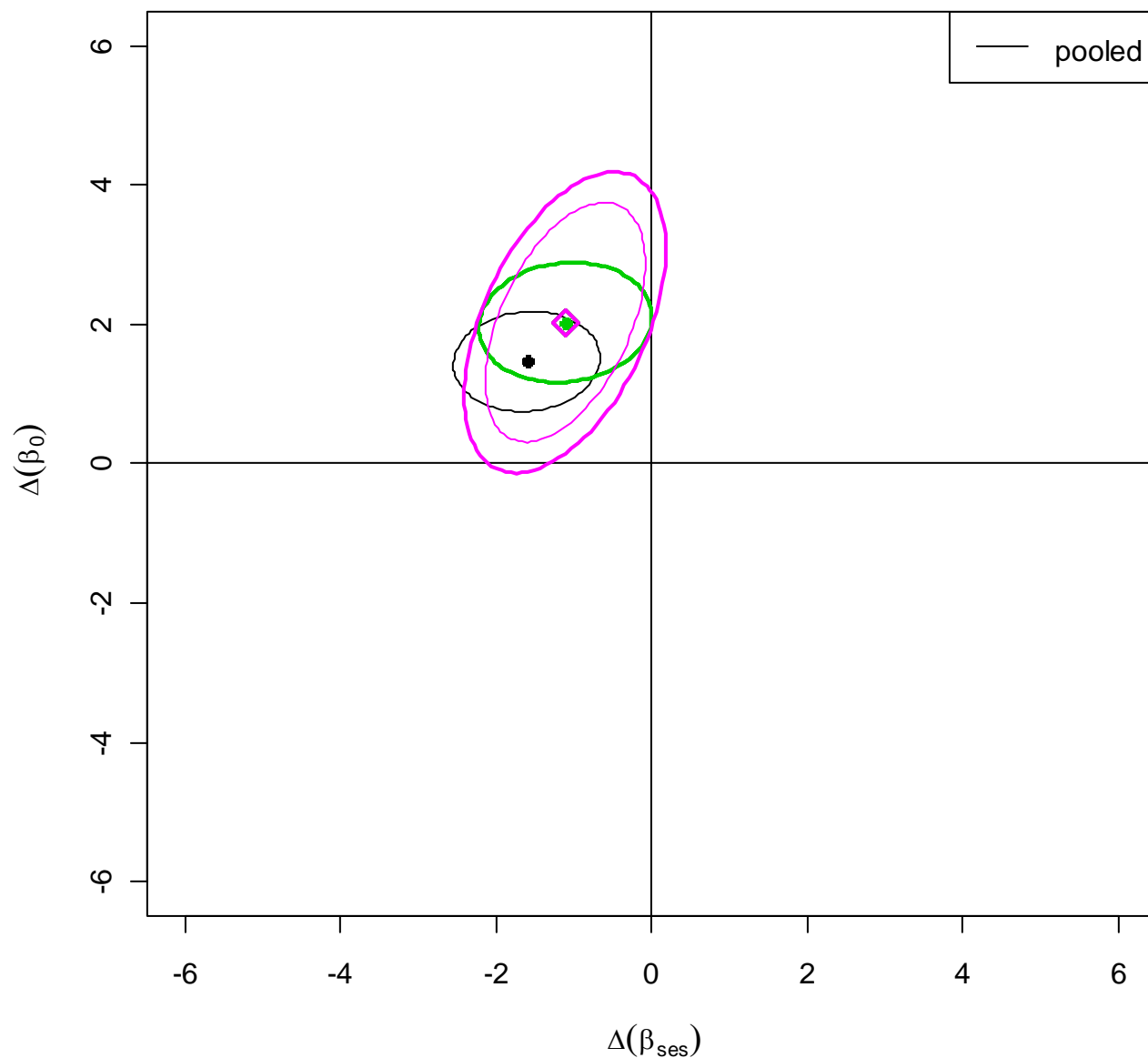


The magenta ellipses are based on the Manova model.

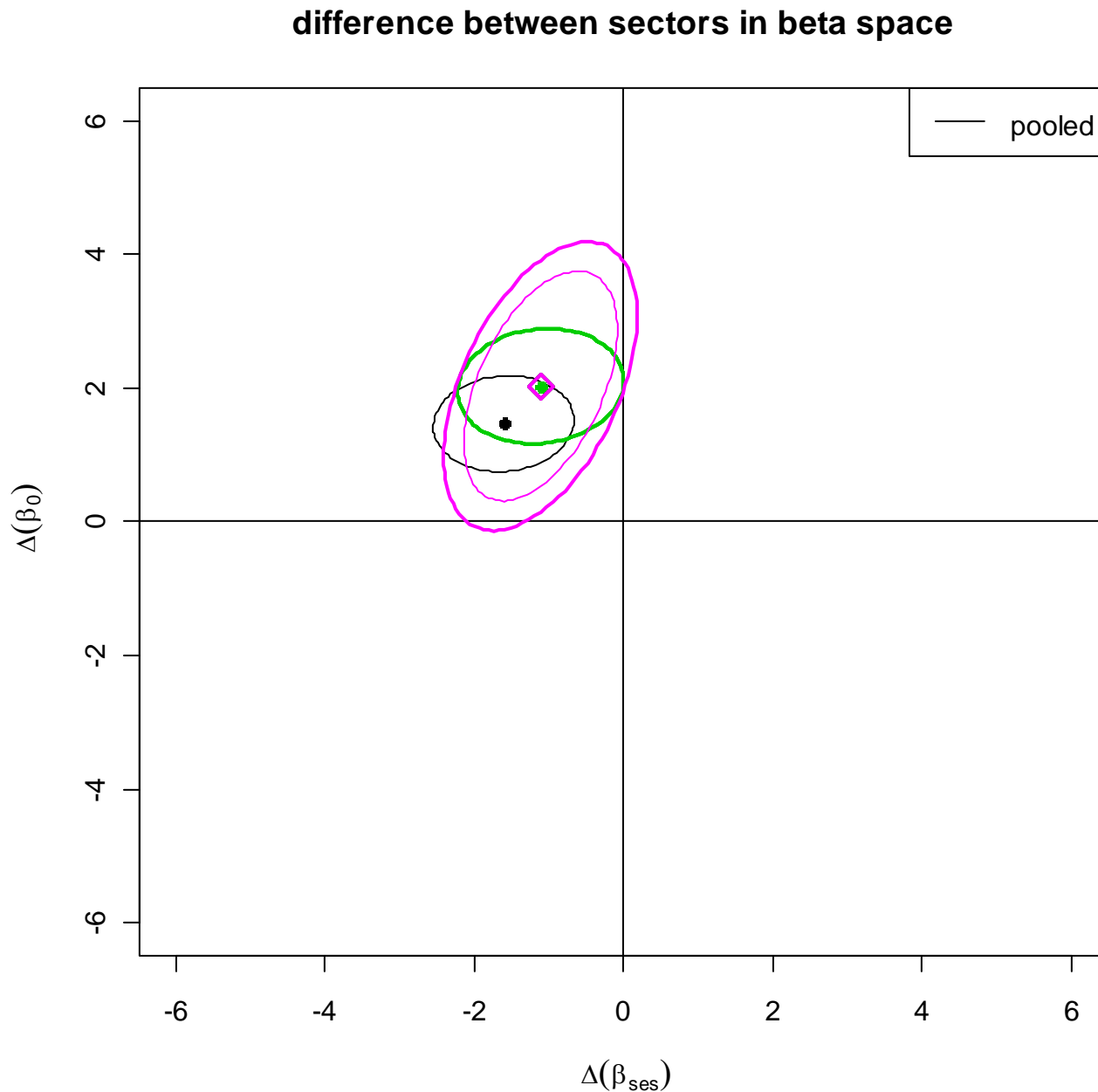
The large magenta ellipse has approximate 95% coverage.

The smaller ellipse has 95% shadows. Thus the p-value for the difference in the effect of ses in the two sectors would be just below 0.05.

difference between sectors in beta space



The magenta ellipse generalizes to the sectors, the green ellipse only to new students from the same set of schools.

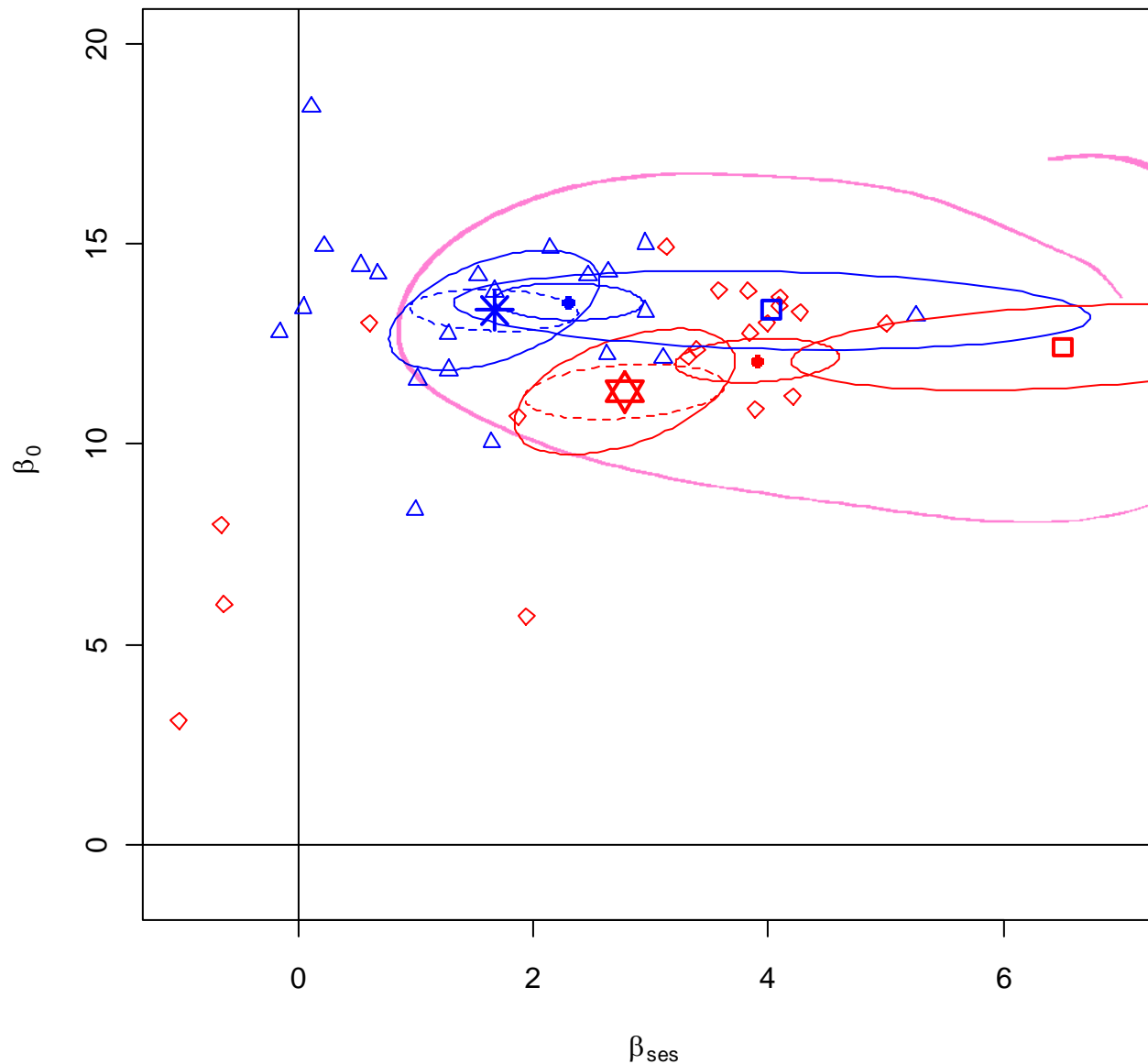


The magenta ellipses are based on the Manova model.

The large magenta ellipse has approximate 95% coverage.

The smaller ellipse has 95% shadows. Thus the p-value for the difference in the effect of ses in the two sectors would be just below 0.05.

response in beta space

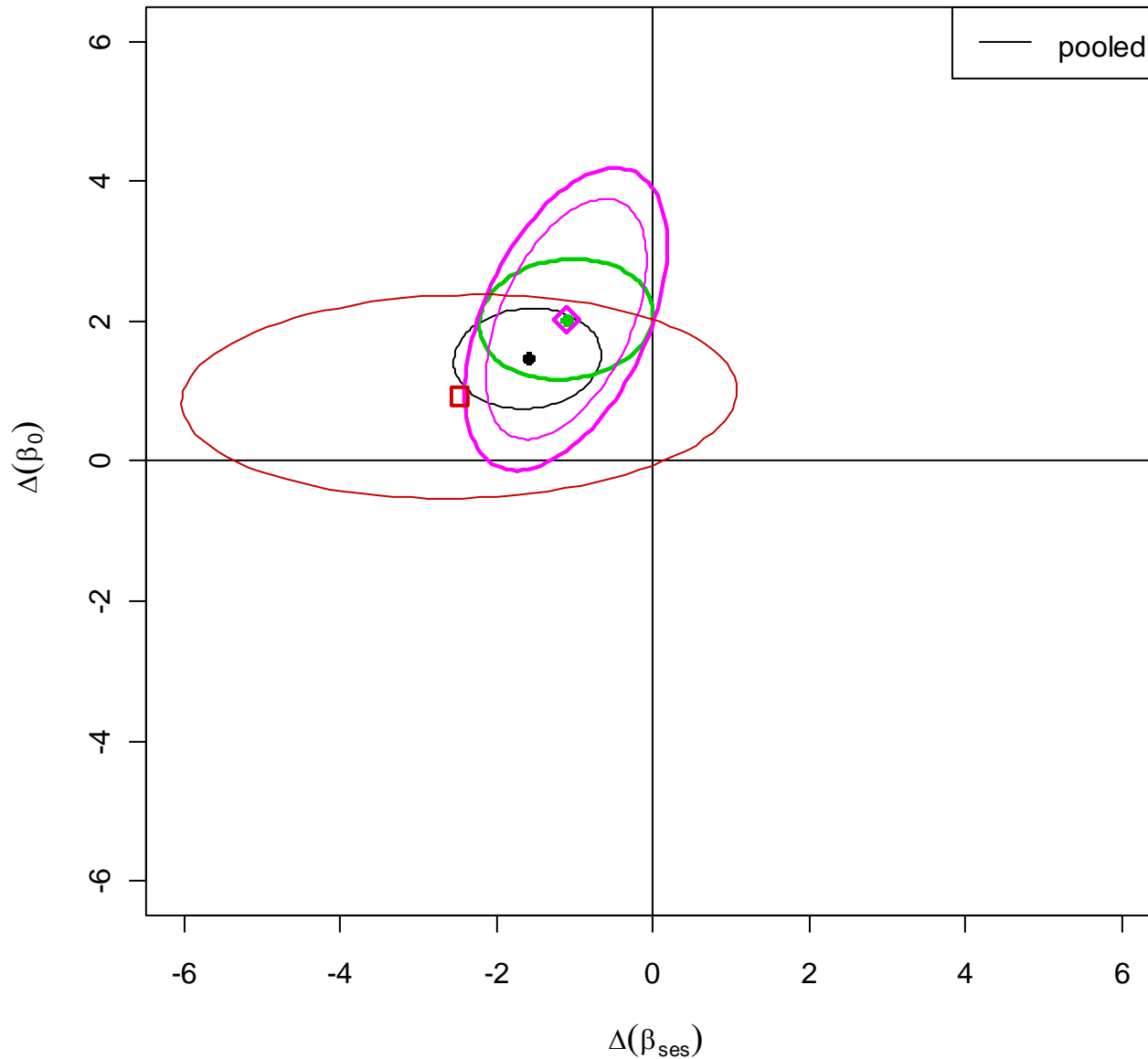


Adding the  
between school  
model

Note that the  
pooled estimates  
are somewhere  
between the 'within  
school estimates'  
and the 'between  
school estimates'

## **Method 4: The between-school model**

### difference between sectors in beta space



Note that the pooled estimate of differences also lies on an arc between the 'within estimate' and the 'between school' estimate

## Why do we get three estimates?

Because there are three effects of ses:

1. **between schools: ecological association**
2. **within schools: conditional association**
3. **across schools: marginal association**

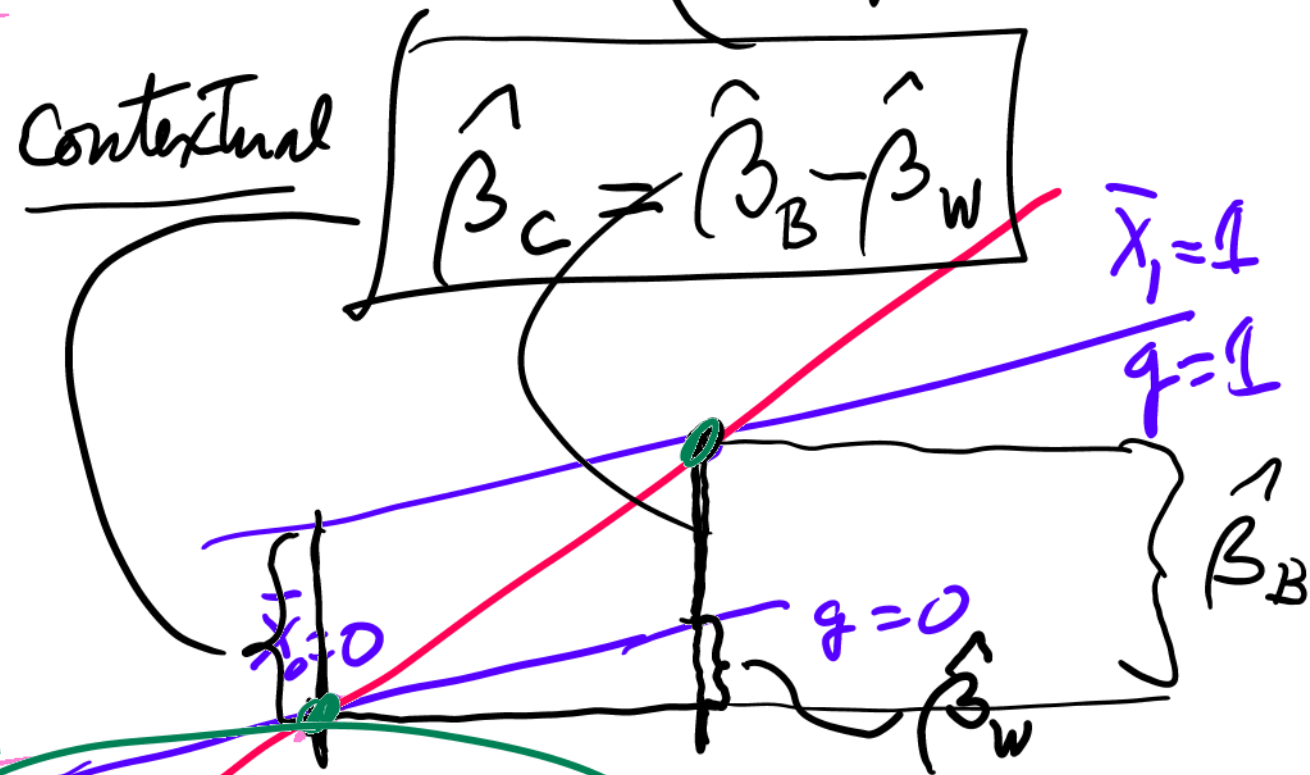
Within school  
effect

Contextual effect

$\hat{\beta}_B$   
 $\hat{\beta}_w$   
 $\hat{\beta}_p$

Between school effect

compositional



$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 \bar{X}_q$$

$\hat{\beta}_w$   $\hat{\beta}_c$

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 (X - \bar{X}_q) + \hat{\beta}_2 \bar{X}_q$$

$\hat{\beta}_w$   $\hat{\beta}_B$

$$Y \sim X + \underbrace{\text{cvar}(X, id)}_{\text{contextual}}$$

$$Y \sim \text{dvar}(X, id) + \text{cvar}(X, id)$$

Pooled effect (across schools)

Interesting fact:



## Paradoxes of Regression:

Robinson's Paradox refers to the fact that

$\beta_W$  and  $\beta_B$  can have different signs

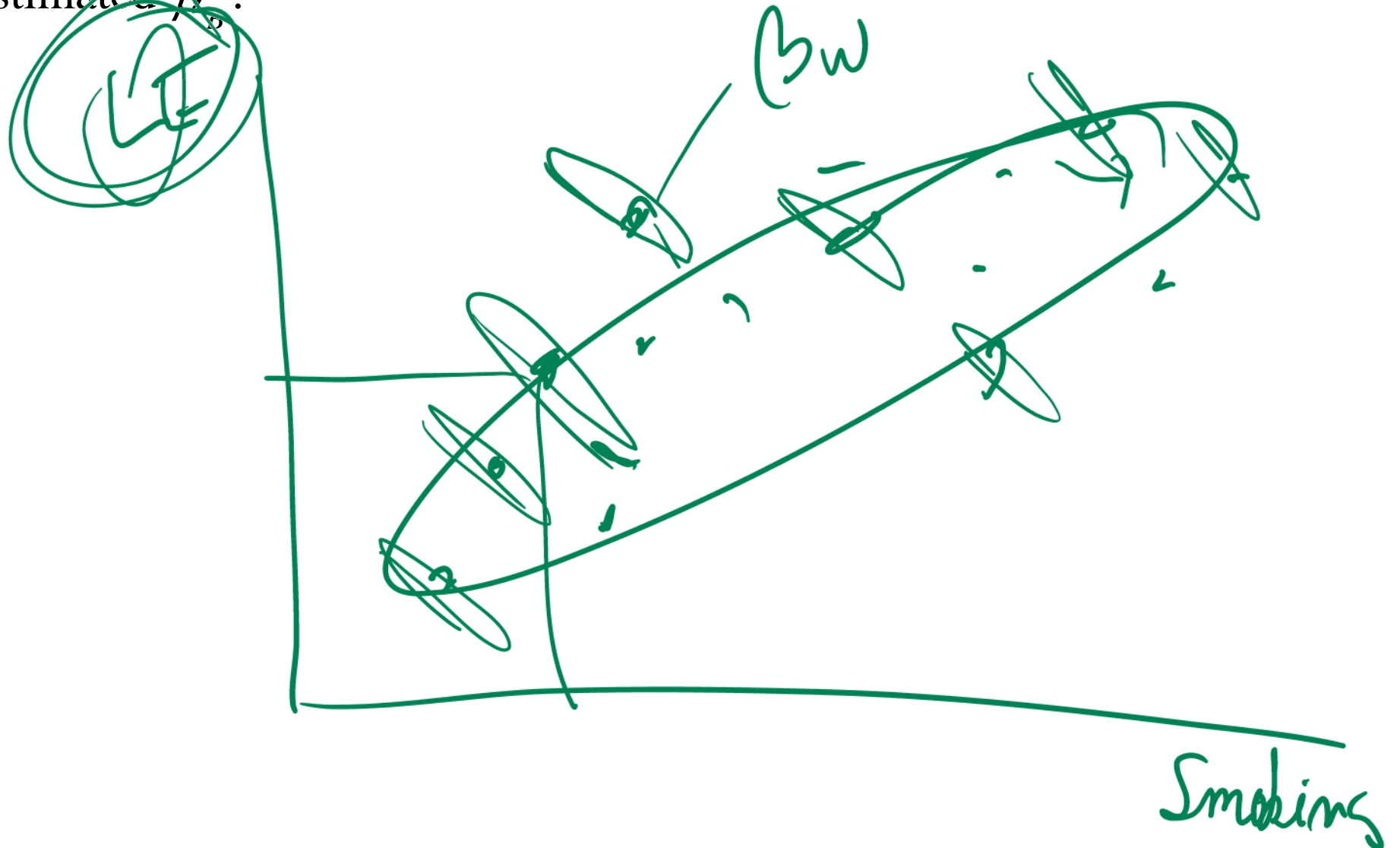
Simpson's Paradox refers to the fact that

$\beta_W$  and  $\beta_P$  can have different signs.

## Some Fallacies of Regression:

Ecological fallacy consists in estimating  $\beta_B$  and believing you have estimated  $\beta_w$ .

Atomistic fallacy consists in estimating  $\beta_w$  and believing you have estimated  $\beta_B$ .



## Summary of methods

<i>Method</i>	<i>Consistent?</i>	<i>Efficient?</i>	<i>Honest?</i>
Pooled data	Estimates a combination of $\beta_W$ and $\beta_B$ in each sector.	For what?	No. Does not take clustering into account. You might have far fewer independent pieces of information than you think.
Fixed effects	Estimates $\beta_W$ in each sector.	Yes	Only generalizes to new students from the same fixed set of schools. Does not generalize to the population of schools in each sector, i.e. to the sectors themselves. Reported SE likely to be too small to generalize to new schools

<i><b>Method</b></i>	<i><b>Consistent?</b></i>	<i><b>Efficient?</b></i>	<i><b>Honest?</b></i>
2-step method: derived variables, <b>regress then average</b>	Estimates $\beta_w$ in each sector.	No <sup>2</sup> – unless size and spread of ses is similar in each cluster. Does not give more weight to schools with more information (n or spread of ses)	Yes. Generalizes to the population of schools.

---

<sup>2</sup> Although the estimate may be similar to the fixed-effects estimate because they both estimate the same thing, it is not, in general, equal because the two estimates give different weight to each school's estimated slope.

<i>Method</i>	<i>Consistent?</i>	<i>Efficient?</i>	<i>Honest?</i>
Ecological or Between School analysis: <b>average then regress</b>	Estimates $\beta_B$ in each sector. Note that we are generally really interested in $\beta_W$	No – does not take differences in sample size and spread of data into account but it would be easy to do so.	Yes.

## *Hierarchical Models*

<i>Method</i>	<i>Consistent?</i>	<i>Efficient?</i>	<i>Honest?</i>
HLM	Yes under common tacit but unrealistic supposition that $\beta_B = \beta_W$ . Otherwise the estimate is, like the pooled estimate, a combination of $\beta_W$ and $\beta_B$ in each sector. But will be closer – generally much closer – to $\beta_W$ than the pooled estimate. It is consistent for $\beta_W$ as the cluster size increases – not as the number of clusters increases	Yes	Yes
HLM + contextual variable	Gives separate consistent separate estimates of $\beta_W$ and $\beta_B$ .	Yes	Yes

# Review of the matrix formulation of regression

You don't need to understand this in depth to use HLMs but it's useful to know where many of the results come from. If you already know regression formulated with matrices, then it's easier to see how to make the jump from OLS regression to HLM regression.

$Y = X\beta + \varepsilon$  is such a universal and convenient shorthand that we need to spell out what it means and how it is used.

Here's the equation for a single observation assuming 2 X variables:

$$Y_i = \beta_0 + x_{1i}\beta_1 + x_{2i}\beta_2 + \varepsilon_i \quad j=1, \dots, N$$

with  $\varepsilon_i$  iid  $N(0, \sigma^2)$ .

We pile these equations one on top of the other:

$$Y_1 = \beta_0 + x_{11}\beta_1 + x_{21}\beta_2 + \varepsilon_1$$

$$Y_2 = \beta_0 + x_{12}\beta_1 + x_{22}\beta_2 + \varepsilon_2$$

$$\vdots$$

$$Y_j = \beta_0 + x_{1j}\beta_1 + x_{2j}\beta_2 + \varepsilon_j$$

$$\vdots$$

$$Y_N = \beta_0 + x_{1N}\beta_1 + x_{2N}\beta_2 + \varepsilon_N$$

Note that the  $\beta$ s remain the same from line to line but  $Y$ s,  $x$ s and  $\varepsilon$ s change.

Using vectors and matrices and exploiting the rules for multiplying matrices:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{1N} & x_{2N} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{bmatrix}$$

or, in short-hand:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

In multilevel models with, say  $J$  schools indexed by  $j = 1, \dots, J$  and with the  $j$ th school having  $n_j$  students, we block students of the same school together.

We just add  $j$ s to show that this is the  $j$ th school. The big difference is that the  $\beta$ s might change from school to school and that the sample size can change from one school to the next. So we use  $n_j$  to denote the sample size for the  $j$ th school:

$j$ th

$$\begin{bmatrix} Y_{1j} \\ Y_{2j} \\ \vdots \\ Y_{n_j j} \end{bmatrix} = \begin{bmatrix} 1 & x_{11j} & x_{21j} \\ 1 & x_{12j} & x_{22j} \\ \vdots & \vdots & \vdots \\ 1 & x_{1n_j j} & x_{2n_j j} \end{bmatrix} \begin{bmatrix} \beta_{0j} \\ \beta_{1j} \\ \beta_{2j} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1j} \\ \varepsilon_{2j} \\ \vdots \\ \varepsilon_{n_j j} \end{bmatrix}$$

or, in short hand:

$$\mathbf{Y}_j = \mathbf{X}_j \boldsymbol{\beta}_j + \boldsymbol{\varepsilon}_j$$

We can stack schools on top of each other. If all schools are assumed to have the same value for  $\boldsymbol{\beta}_j = \boldsymbol{\beta}$ , then we can stack the  $\mathbf{X}$ s vertically:

$$\begin{bmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_j \\ \vdots \\ \mathbf{Y}_J \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_j \\ \vdots \\ \mathbf{X}_J \end{bmatrix} \boldsymbol{\beta} + \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_j \\ \vdots \\ \boldsymbol{\varepsilon}_J \end{bmatrix}$$

or, in shorter form:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

pooled

If the  $\beta_j$ s are different we can stack the  $X_j$ s diagonally:

$$\begin{bmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_j \\ \vdots \\ \mathbf{Y}_J \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{X}_j & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & 0 & \cdots & \mathbf{X}_J \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_j \\ \vdots \\ \boldsymbol{\beta}_J \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \vdots \\ \boldsymbol{\varepsilon}_j \\ \vdots \\ \boldsymbol{\varepsilon}_J \end{bmatrix}$$

or, in shorter form:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

again!

Something that gets used over and over again is the fact that if  $\boldsymbol{\varepsilon} \sim N(0, \sigma^2 \mathbf{I})$ , i.e. all  $\varepsilon$ 's are independent and normal with the same variance then the best estimator of  $\boldsymbol{\beta}$  is the OLS (ordinary least-squares) estimator:

$$\hat{\boldsymbol{\beta}}^{\text{OLS}} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{Y}$$

with variance

$$\sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$$

If the components of  $\boldsymbol{\varepsilon}$  are not iid but  $\boldsymbol{\varepsilon} \sim N(0, \boldsymbol{\Sigma})$  where  $\boldsymbol{\Sigma}$  is a known variance matrix (or, at least, known up to a proportional factor) then the GLS (generalized least-squares) estimator is:

$$\hat{\boldsymbol{\beta}}^{\text{GLS}} = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1} \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$$

with variance

$$(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}.$$

# The Hierarchical Model

We develop the ideas for mixed and multilevel modeling in two stages:

1. Multilevel models as presented in Bryk and Raudenbush (1992) in which the unobserved parameters at the lower level are modeled at the higher level. This is the representation used in HLM, the software developed by Bryk and Raudenbush and, to a limited extent in MLwiN.
2. Mixed models in which the levels are combined into a combined equation with two parts: one for 'fixed effects' and the other for 'random effects.' This is the form used in R, SAS and in many other packages.

Although the former is more complex, it is more natural and intuitive. It also gives us important insights into the structure of these models.

We will use the high school Math Achievement data for an extensive example. We think of our data as structured in two levels: **students within schools** and **between schools**.

We also have two types of predictor variables:

- 1.within-school Level 1 variables:** Individual student variables: SES, Sex, individual minority status. These variables are also known by many other names, e.g. inner variables, micro variables, level-1 variables<sup>3</sup>, time-varying variables in the longitudinal context.
- 2.between-school Level 2 variables:** Sector: Catholic or Public, school meanses, size, mean ses of sample, sample size. These variables are also known as outer variables, macro variables, level-2 variables, or time-invariant variables in a longitudinal context. A between-school variable can be created from a within-school variable by taking the

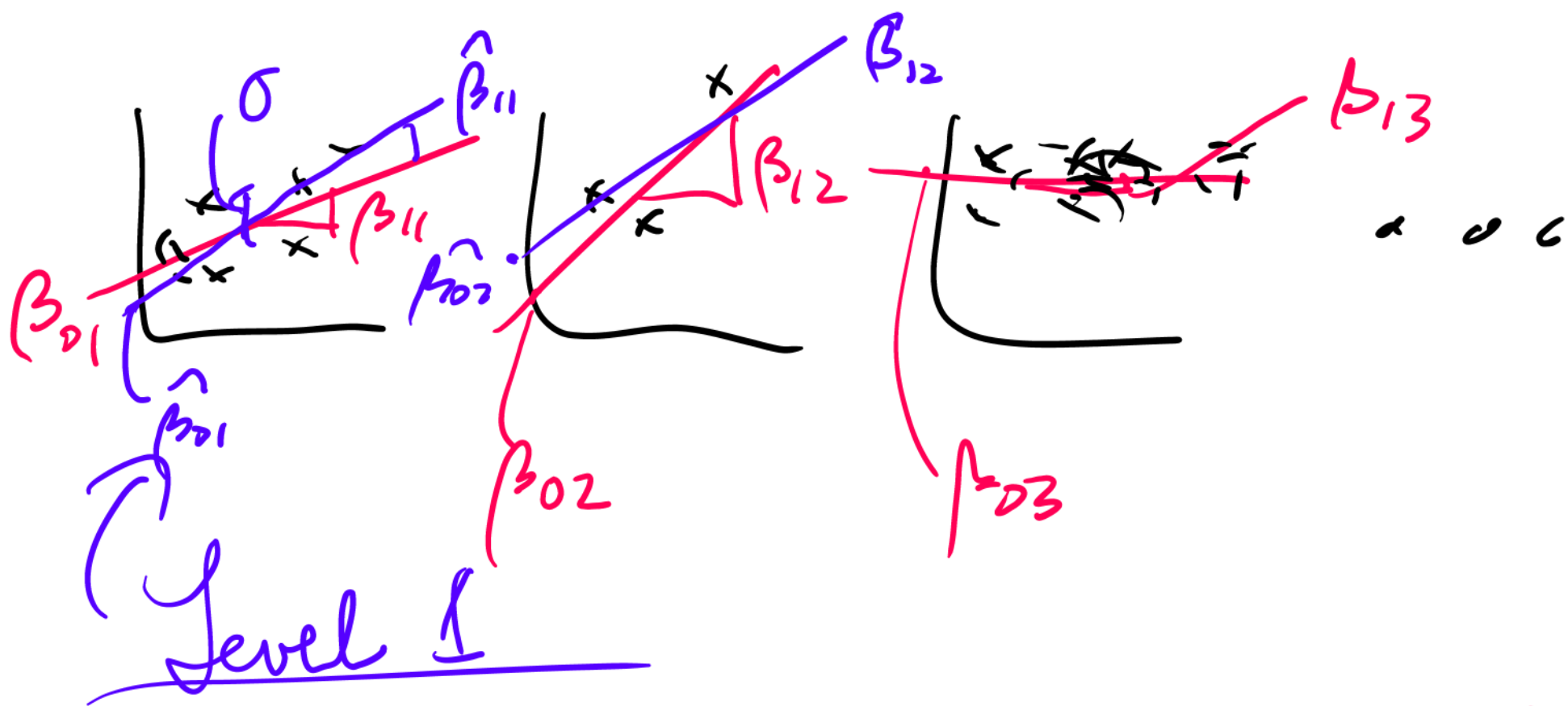
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<sup>3</sup> In some hierarchical modeling traditions, e.g. R, the numbering of levels is reversed going from the top down instead of going from the bottom up. One needs to check which approach an author or package is using.

average of the within-school variable within each school. Such a derived between-school variable is known as a 'contextual' variable. These variables are useful only if the average differs from school to school. Balanced data in which the set of values of within-school variables is the same in each school does not give rise to contextual variables.

### ***Basic structure of the model:***

1. Each school has a true regression line that is not directly observed
2. The observations from each school are generated by taking random observations generated with the school's true regression line
3. The true regression lines for each school come from a population or populations of regression lines



Level 2

$$\begin{pmatrix} \beta_{01} \\ \beta_{11} \end{pmatrix} \quad \begin{pmatrix} \beta_{02} \\ \beta_{12} \end{pmatrix} \quad \dots \quad \begin{pmatrix} \beta_{0j} \\ \beta_{1j} \end{pmatrix} \dots$$

$$\sim N \left( \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix}, G_{2 \times 2} \right)$$

## ***Within School model:***

For school  $i$ : (For now we suppose all schools come from the same population, e.g. only one Sector)

1) True but unknown  $\boldsymbol{\beta}_j = \begin{bmatrix} \beta_{0j} \\ \beta_{\text{SES } i} \end{bmatrix} = \begin{bmatrix} \beta_{0j} \\ \beta_{1j} \end{bmatrix}$  for each school

2) The data are generated as

$$Y_{ij} = \beta_{0j} + \beta_{0j} X_{ij} + \varepsilon_{ij}$$

$$\varepsilon_{ij} \sim N(0, \sigma^2) \text{ independent of } \boldsymbol{\beta}_j \text{'s}$$

### ***Between School model:***

We start by supposing that the  $\boldsymbol{\beta}_j = \begin{bmatrix} \beta_{0j} \\ \beta_{\text{SES}j} \end{bmatrix} = \begin{bmatrix} \beta_{0j} \\ \beta_{1j} \end{bmatrix}$  are sampled from a single population of schools. In vector notation:

$$\boldsymbol{\beta}_j = \boldsymbol{\gamma} + \mathbf{u}_j \quad \mathbf{u}_j \sim N(\mathbf{0}, \mathbf{G})$$

where

$$\mathbf{G} = \begin{bmatrix} g_{00} & g_{10} \\ g_{10} & g_{11} \end{bmatrix}$$

is a variance matrix.

Writing out the elements of the vectors:

$$\boldsymbol{\beta}_j = \begin{bmatrix} \beta_{0j} \\ \beta_{1j} \end{bmatrix} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix} + \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix}, \quad \begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix}\right)$$

Note:

$$\text{Var}(\beta_{0i}) = g_{00}$$

$$\text{Var}(\beta_{1i}) = g_{11}$$

$$\text{Cov}(\beta_{0i}, \beta_{1i}) = g_{10} = g_{01}$$

## *A simulated example*

To generate an example we need to do something with SES although its distribution is not part of the model. In the model the values of SES are taken as given constants.

We will take:

$$\gamma = \begin{bmatrix} 12 \\ 2 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 16 & 8 \\ 8 & 25 \end{bmatrix}, \sigma^2 = 20$$

Once we have generated  $\beta_j$  we generate  $N_j \sim \text{Poisson}(30)$  and  $SES \sim N(0,1)$

Here's our first simulated school in detail:

For  $j=1$ :

SES :

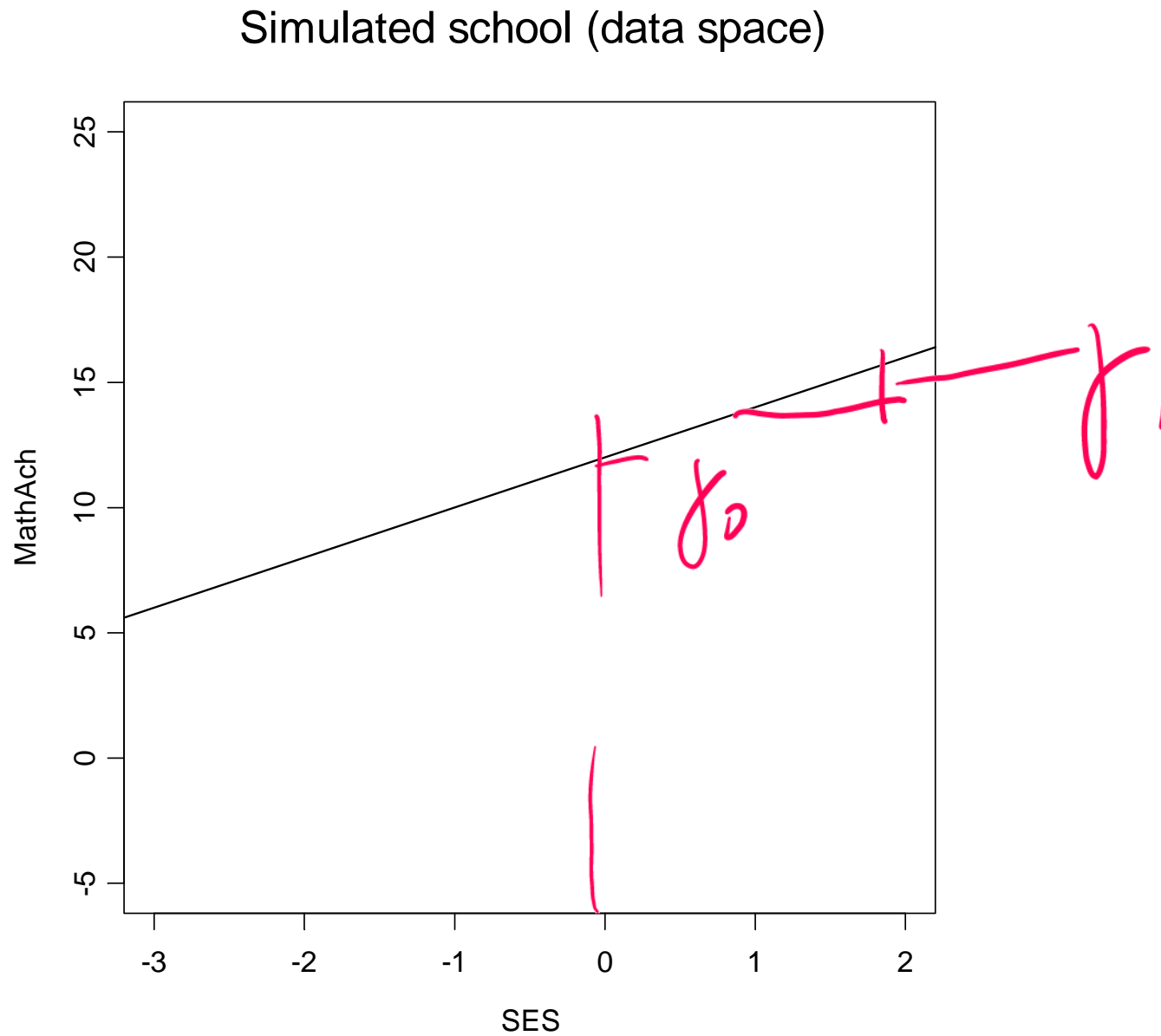
-1.05	-0.78	1.05	-1.01	0.77	1.85	0.87	-1.18	0.18	2.08	-1.14	-1.71
-0.64	-0.41	0.86	1.29	0.04	0.23	0.90	0.50	-2.10	-1.89	0.38	

$\varepsilon_j$  :

4.46	-0.73	0.30	7.63	-7.03	1.20	-6.23	-4.66	6.17	0.75	-1.43	0.46
3.64	-2.39	2.24	2.60	3.96	0.71	-3.74	3.30	4.42	-4.59	-3.61	

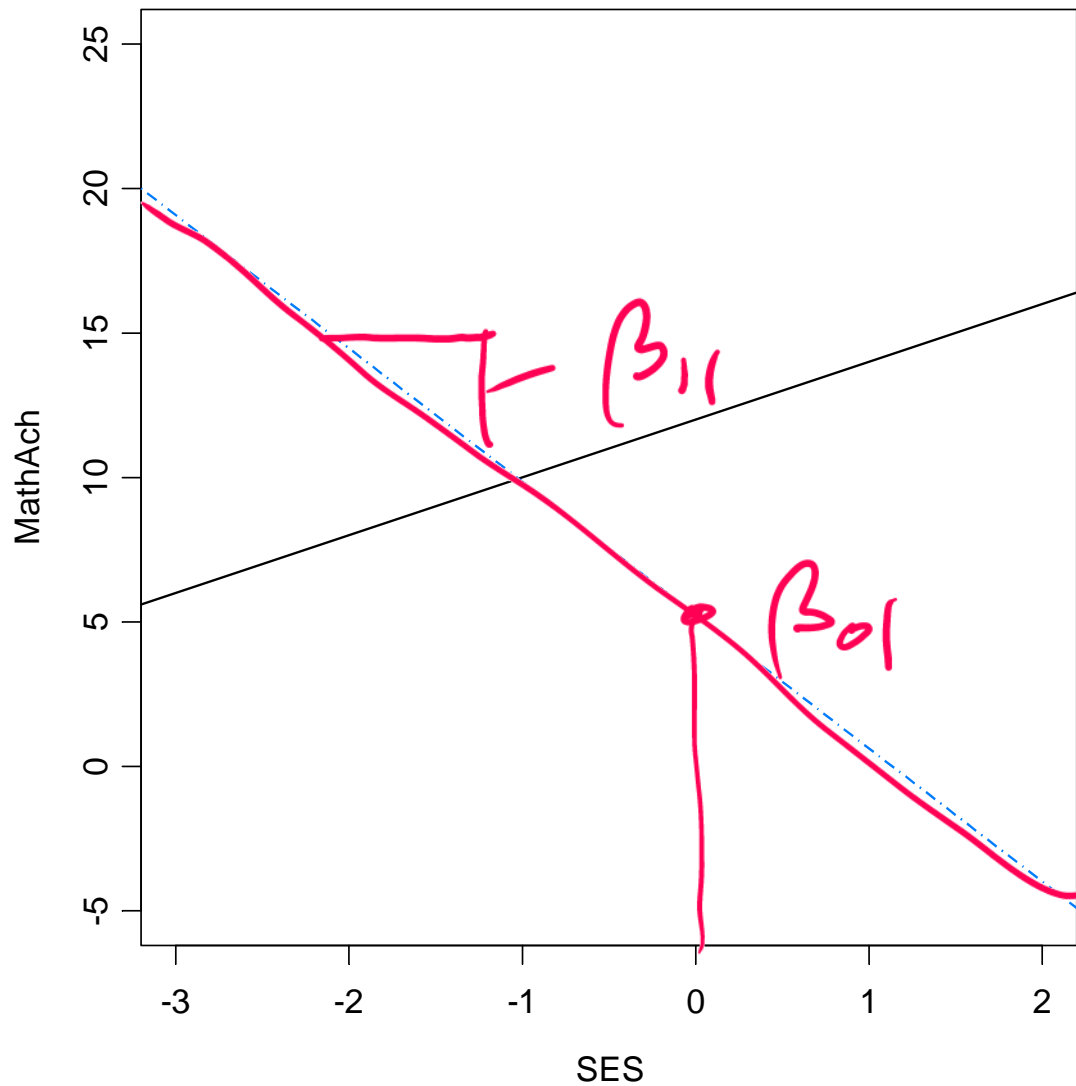
$$Y_{ij} = \beta_{0j} + \beta_{1j}SES_{ij} + \varepsilon_{ij}:$$

14.53	8.09	0.70	17.56	-5.34	-2.10	-4.99	6.03	10.58	-3.59	
9.09	13.57	11.83	4.75	3.51	1.88	9.00	4.91	-2.66	6.24	
19.37	9.38	-0.13								



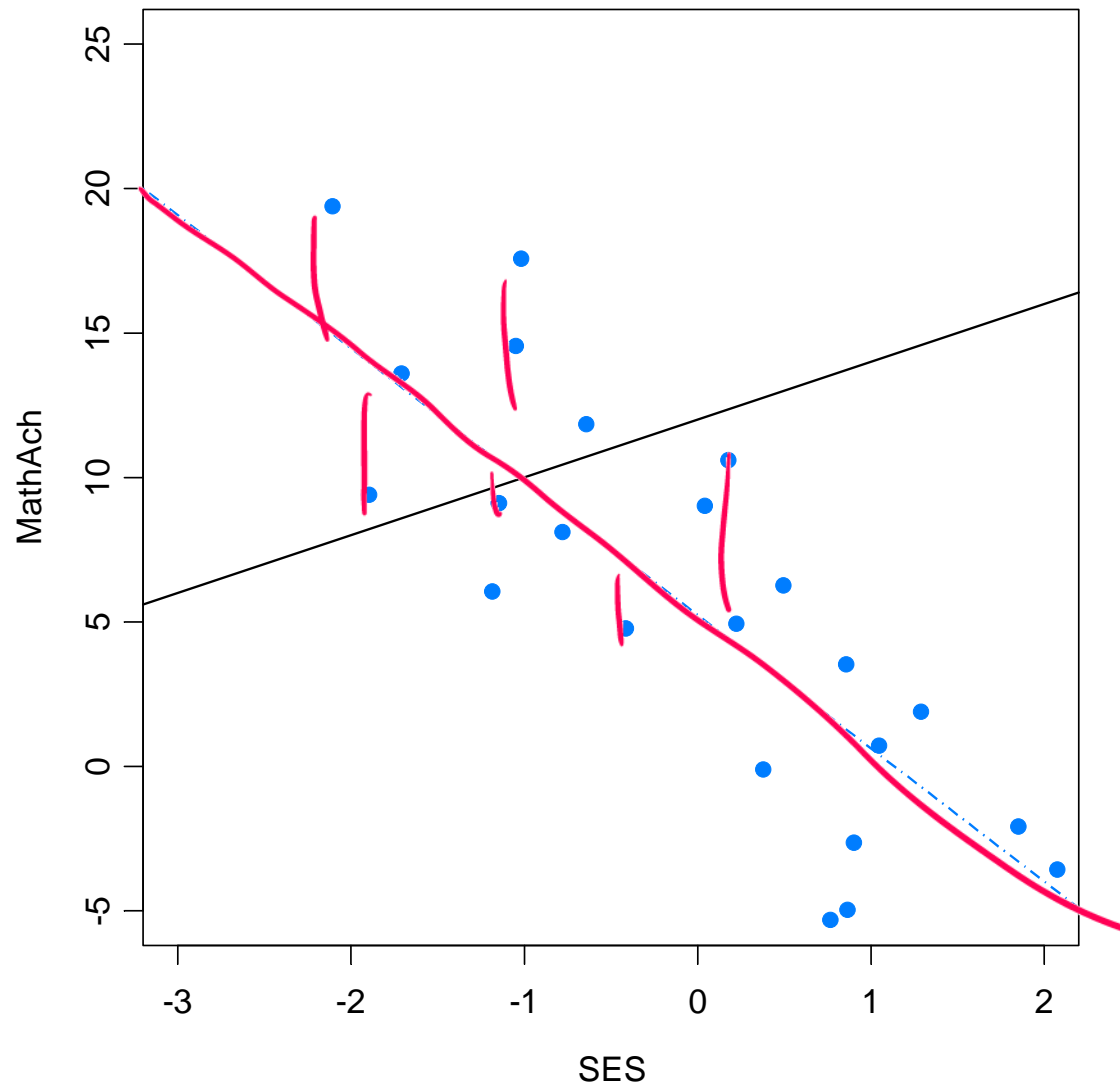
**Figure 8: Simulation: mean population regression line  $\gamma$**

### Simulated school (data space)



**Figure 9: Simulated school: True regression line in School 1:  $\beta_j = \gamma + u_j$**

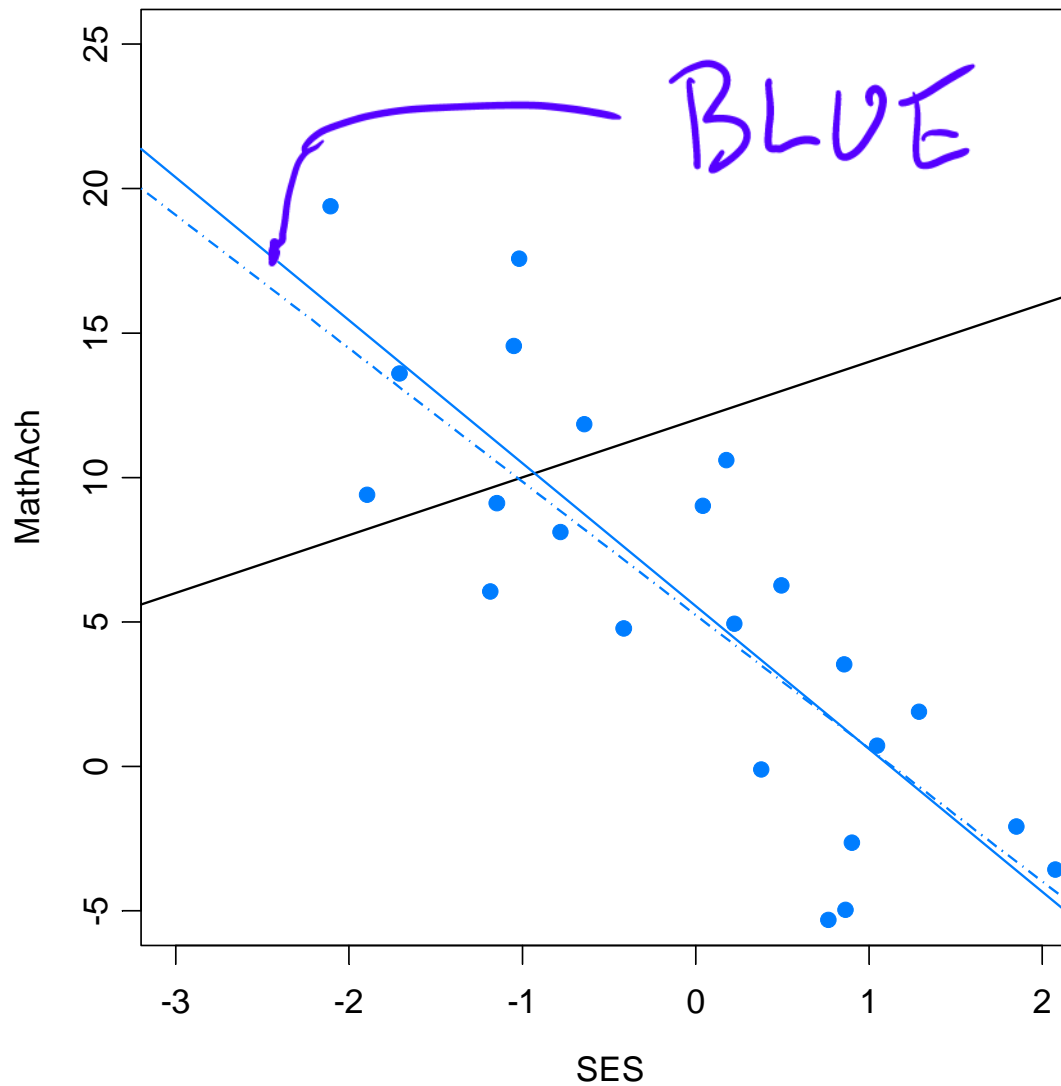
# Simulated school (data space)



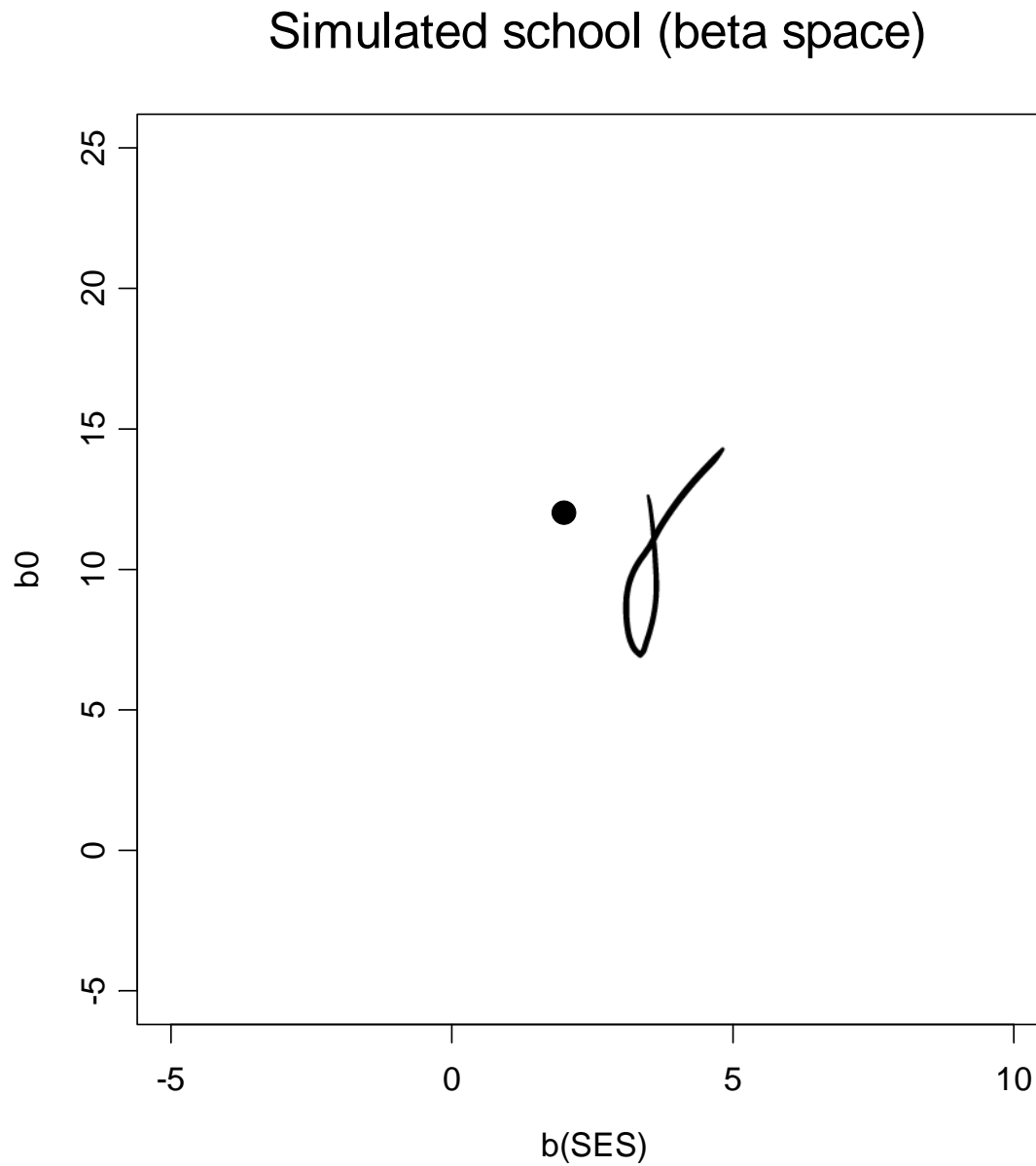
**Figure 10: School 1 regression line with data generated by**

$$Y_{ij} = \beta_{0i} + \beta_{1i}SES_{ij} + \varepsilon_{ij}$$

# Simulated school (data space)



**Figure 11: Simulated school: True regression line  $\beta_i$ , data, and least-squares line  $\hat{\beta}_i$**



**Figure 12: Simulated school in beta space with true mean line represented by a point.**

### Simulated school (beta space)

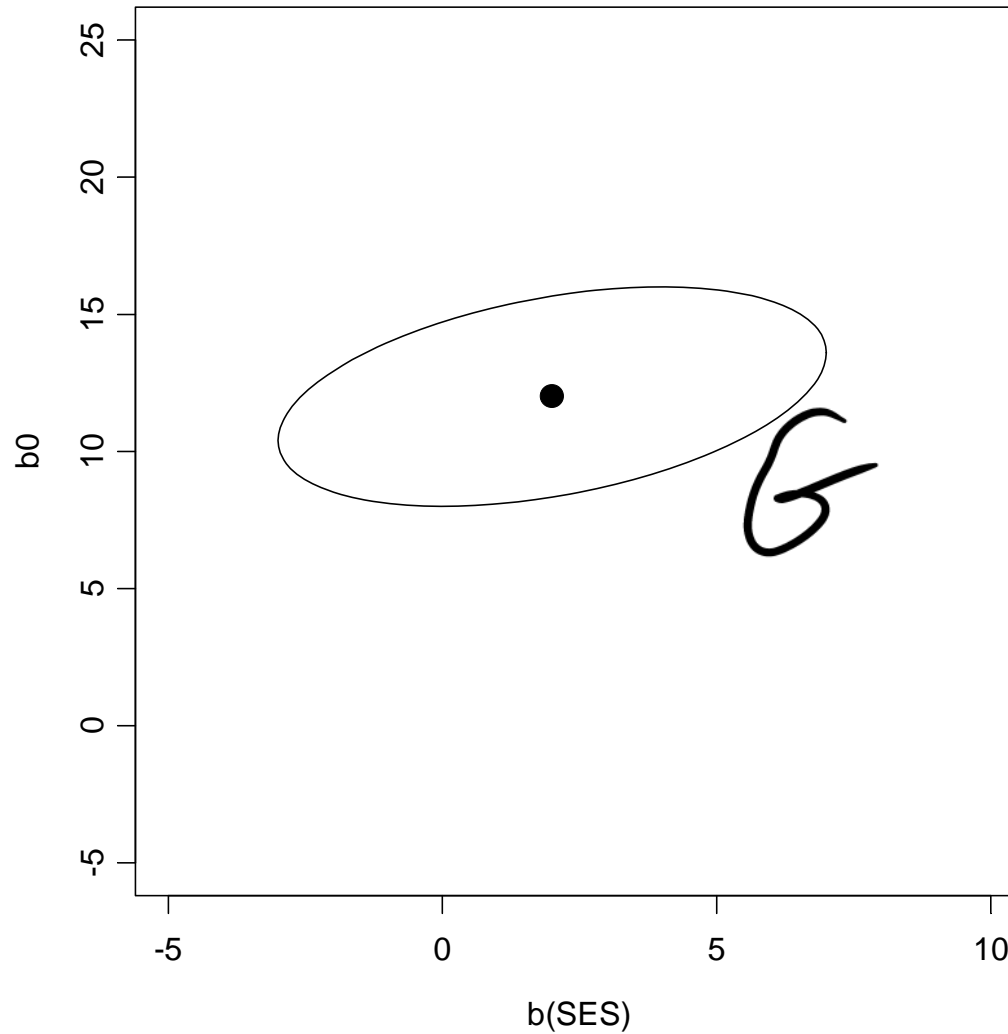


Figure 13: Simulated school: population mean line in beta space with dispersion ellipse with matrix  $G$  for random slopes and intercepts. Note that shadows of the ellipse yield the mean plus or minus 1 standard deviation

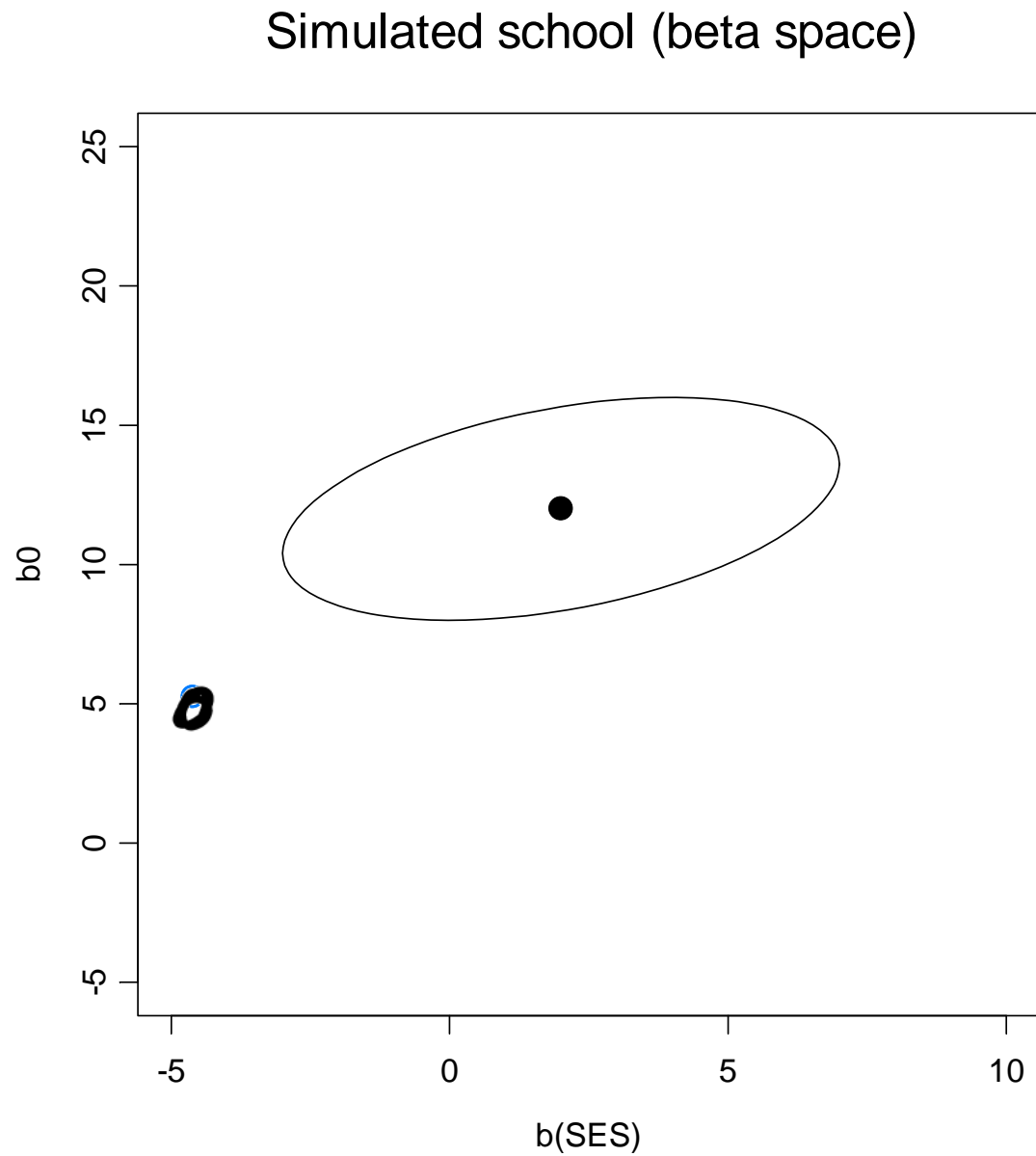
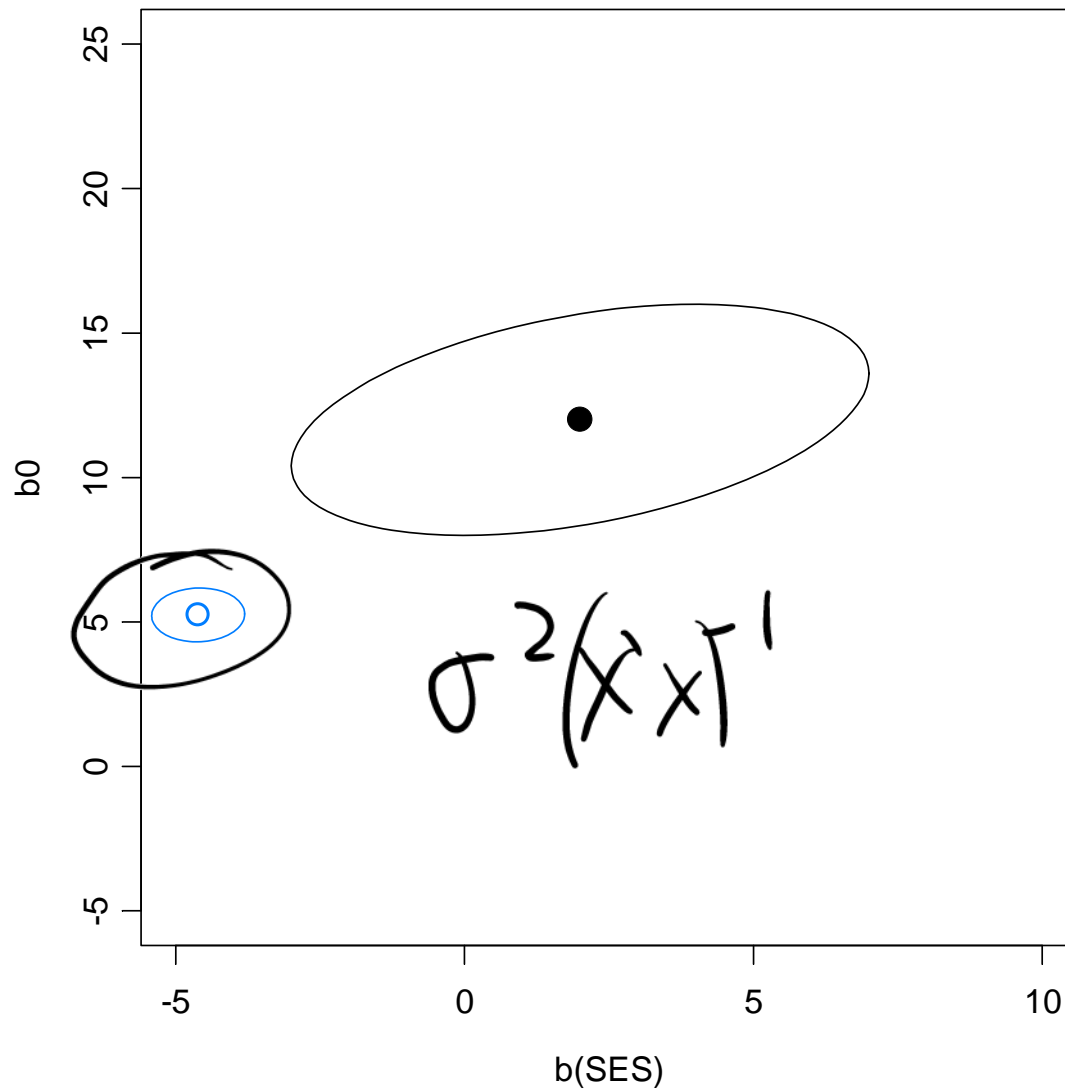
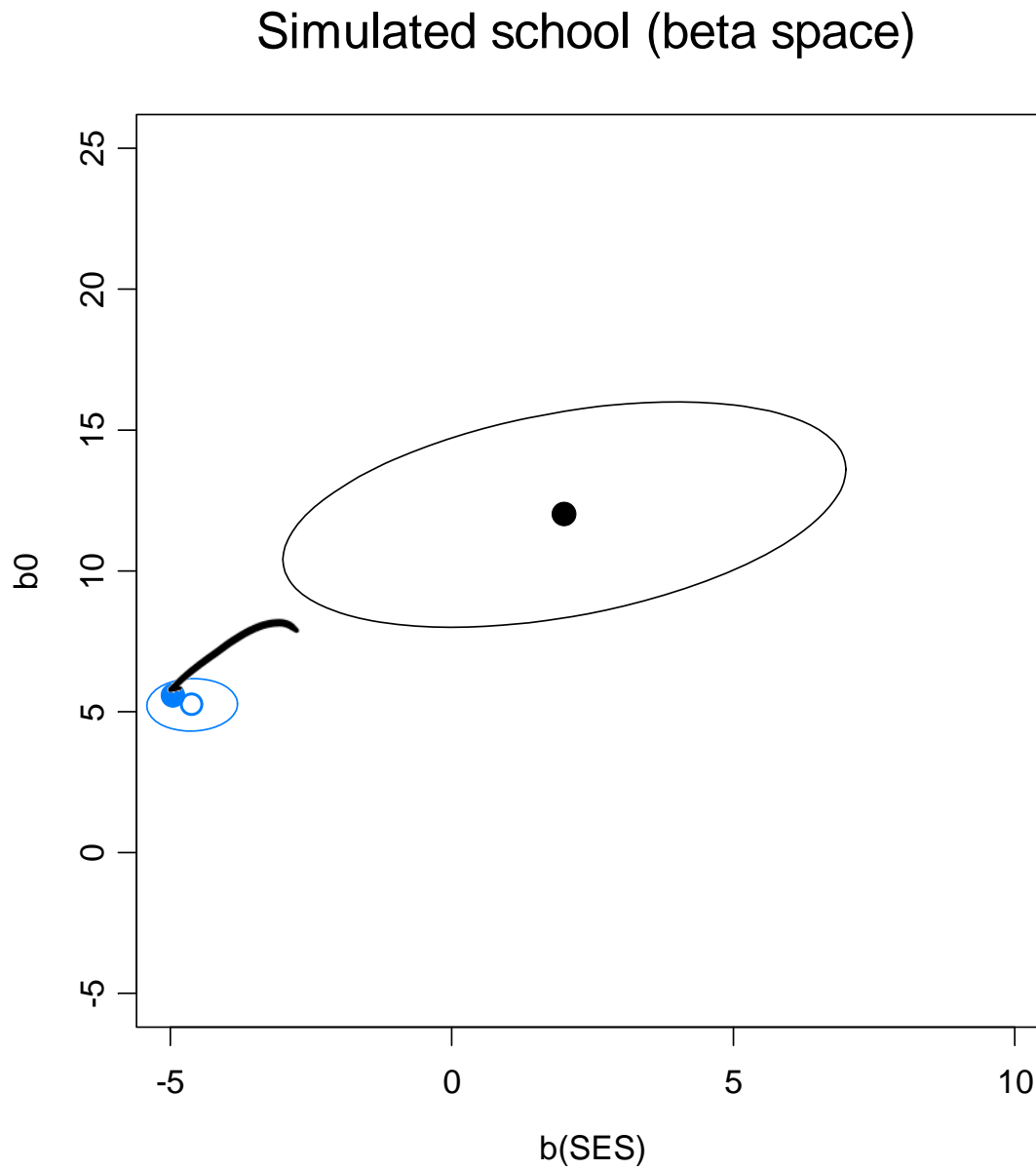


Figure 14: A random ‘true’ intercept and slope from the population. This one happens to be somewhat atypical but not wholly implausible.

# Simulated school (beta space)

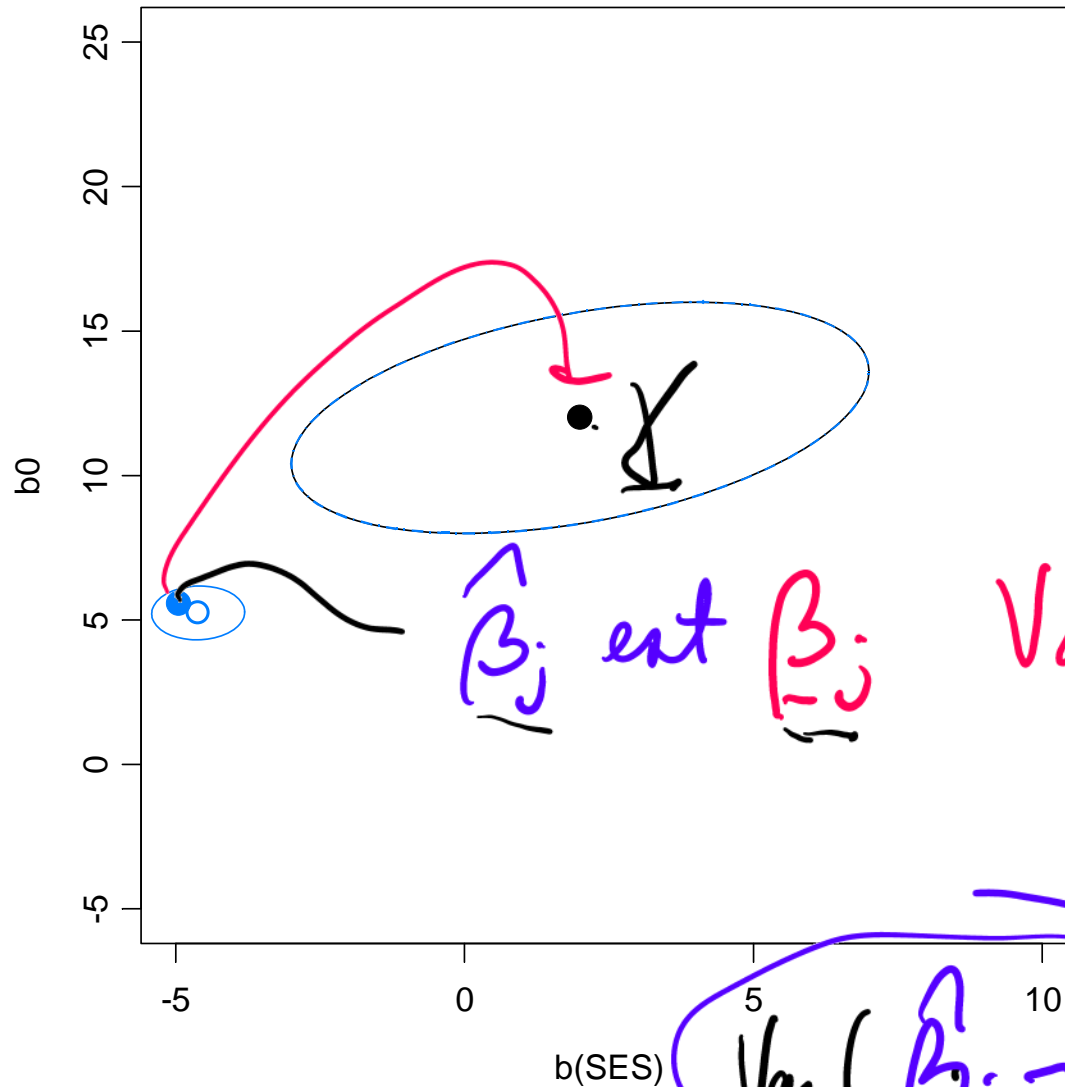


**Figure 15: ‘True’ intercept and slope with dispersion ellipse with matrix  $\sigma^2 (\mathbf{X}_j' \mathbf{X}_j)^{-1}$  for  $\hat{\beta}_j$ .**



**Figure 16: Observed value of  $\hat{\beta}_j$ .**

# Simulated school (beta space)

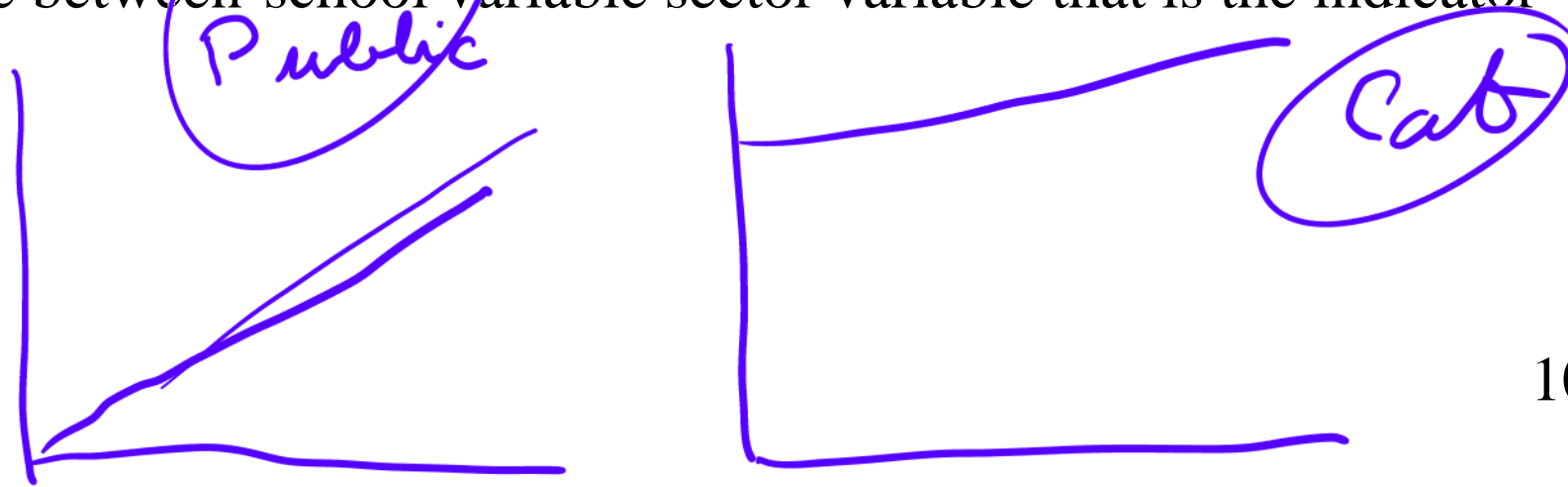


**Figure 17: The blue dispersion ellipse with matrix  $V_j = G + \sigma^2 (X_j' X_j)^{-1}$  is almost coincident with the dispersion ellipse with matrix  $T$ .**

Note that with smaller  $N$ , larger  $\sigma^2$  or smaller dispersion for SES, these dispersion ellipse for the true  $\beta_j$  (with matrix  $\tau$ ) and the dispersion ellipse for  $\hat{\beta}_j$  as an estimate of  $\gamma$  (with matrix  $V_j = G + \sigma^2(X_j'X_j)^{-1}$ ) could differ much more than they do here. Also note that the statistical design of the study can make  $\sigma^2(X_j'X_j)^{-1}$  smaller but, typically, not  $G$ .

### ***Between-School Model: What $\gamma$ means***

Instead of supposing that we have a single population of schools we now add the between-school model that will allow us to suppose that there are two populations of schools: Catholic and Public and that the population mean slope and intercept may be different in the two sectors. Let  $W$  represent the between-school variable sector variable that is the indicator



variable for Catholic schools:  $w_j$  is equal to 1 if school  $j$  is Catholic and 0 if it is public.<sup>4</sup>

We have two regression models, one for intercepts and one for the slopes:

$$\begin{aligned}\beta_{0j} &= \gamma_{00} + \gamma_{01}W_j + u_{0j} \\ \beta_{1j} &= \gamma_{10} + \gamma_{11}W_j + u_{1j}\end{aligned}$$

$\begin{matrix} 0 & \text{Pub} \\ 1 & \text{Cath} \end{matrix}$

We can work out the following interpretation of the  $\gamma_{ij}$  coefficients by setting  $w_j$  to 0 for Public schools and then to 1 for Catholic schools. The interpretation is analogous to that of the ordinary regression to compare two schools except that we are now comparing the two sectors.

---

<sup>4</sup> Between-school variables are not limited to indicator variables. Any variables suitable as a predictor in a linear model could be used as long as it is a function of schools, i.e. has the same value for every subject within each school.

In Public schools:

$$W_j = 0$$

$$\beta_{0j} = \gamma_{00} + \gamma_{01} \times 0 + u_{0j} = \gamma_{00} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11} \times 0 + u_{1j} = \gamma_{10} + u_{1j}$$

In Catholic schools:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} \times 1 + u_{0j} = \gamma_{00} + \gamma_{01} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11} \times 1 + u_{1j} = \gamma_{10} + \gamma_{11} + u_{1j}$$

Thus:

1.  $\gamma_{00}$  is the mean achievement intercept for Public schools, i.e. the mean achievement when SES is 0.
2.  $\gamma_{00} + \gamma_{01}$  is the mean achievement intercept for Catholic schools so that  $\gamma_{01}$  is the difference in mean intercepts between Catholic and Public schools.
3.  $\gamma_{10}$  is the mean slope in Public schools.

4.  $\gamma_{10} + \gamma_{11}$  is the mean slope in Catholic schools so that  $\gamma_{11}$  is the mean difference in (or difference in mean) slopes between Catholic and Public schools.
5.  $u_{0j}$  is the unique “effect” of school  $j$  on the achievement intercept, conditional given  $W$ .
6.  $u_{1j}$  is the unique “effect” of school  $j$  on the slope, conditional given  $W$ .

Now,  $u_{0j}$  and  $u_{1j}$  are Level 2 random variables (random effects) which we assume to have 0 mean and variance-covariance matrix:

$$\mathbf{G} = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix}$$

This is a multivariate model with the complication that the dependent variables,  $\beta_{0j}$ ,  $\beta_{1j}$  are not directly observable.

As mentioned above, one way to proceed would be to use a two-stage process:

1. Estimate  $\beta_{0j}$ ,  $\beta_{1j}$  with least-squares within each school, and
2. use the estimated values in a Level-2 analysis with the model above.

**Some problems with this approach are:**

1. Each  $\hat{\beta}_{0i}$ ,  $\hat{\beta}_{1i}$  might have a different variance due to differing  $n_j$ s and different predictor matrices  $x_i$  in each school. A Level 2 analysis that uses OLS will not take these factors in consideration.
2. Even if  $x_i$  (thus  $n_j$ ) is the same for each school, we might be interested in getting information on T itself, not on

$$\text{var}(\hat{\beta}_i) = \mathbf{G} + \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

3.  $\hat{\beta}_{0i}$ ,  $\hat{\beta}_{1i}$  might be reasonable estimates of the 'parameters'  $\beta_{0i}$  and  $\beta_{1i}$  but, as 'estimators' of the random variables  $\beta_{0i}$  and  $\beta_{1i}$  they ignore the information contained in the distribution of  $\beta_{0i}$  and  $\beta_{1i}$ .
4. Some level 1 models might not be estimable, so information from these schools would be entirely lost.

# Mixed or Combined or Composite model

*From the multilevel model to the mixed model*

Since

$$\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11}W_j + u_{1j}$$

**Between School  
Model**

We combine the models by substituting the *between school model* above into the *within school model*:

$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + r_{ij}$$

**Within School Model**

Substituting, we get

$$\begin{aligned} Y_{ij} &= \beta_{0j} + \beta_{1j}X_{ij} + r_{ij} \\ &= (\gamma_{00} + \gamma_{01}W_j + u_{0j}) \\ &\quad + (\gamma_{00} + \gamma_{11}W_j + u_{1j})X_{ij} + r_{ij} \end{aligned}$$

We then rearrange the term to separate fixed parameters from random coefficients:

Same as previous page

Grouping fixed and random parts together

$$Y_{ij} = \left( \beta_{0j} \right) + \left( \beta_{1j} \right) X_{ij} + r_{ij}$$

$$= \left( \gamma_{00} + \gamma_{01} W_j + u_{0j} \right)$$

$$+ (\gamma_{10} + \gamma_{11} W_j + u_{1j}) X_{ij} + r_{ij}$$

$$= \gamma_{00} + \gamma_{01} W_j + \gamma_{10} X_{ij} + \gamma_{11} W_j X_{ij}$$

$$+ u_{0j} + u_{1j} X_{ij} + r_{ij}$$

$$\sim W + X + X * W$$

$$\sim X * W$$

$$\sim 1 + X | school$$

The last two lines looks like the sum of two linear models:

- 1) an ordinary linear model with coefficients that are *fixed* parameters:

$$\gamma_{00} + \gamma_{01} W_j + \gamma_{10} X_{ij} + \gamma_{11} W_j X_{ij}$$

with fixed parameters  $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}$ , and

2) a linear model with *random* coefficients and an error term:

$$u_{0j} + u_{1j}X_{ij} + r_{ij}$$

with random ‘parameters’  $u_{0j}$  and  $u_{1j}$ .

Note the following:

1. the **fixed model** contains both outer variables and inner variables as well as an interaction between inner and outer variables. This kind of interaction is called a ‘cross-level’ interaction. It allows the effect of X to be different in each Sector.
2. the **random effects** model only contains an intercept and an inner variable. There are *very arcane* situations in which it might make sense to include an outer variable in the random effects portion of the model which we will consider briefly later.

Understanding the connection between the multilevel model and the combined model is useful because some packages require the model to be specified in its multilevel form (e.g. MLWin) while others require the model to be specified in its combined form as two models: the fixed effects model and the random effects model (e.g. SAS PROC MIXED, R and S-Plus lme() and nlme()).

### *GLS form of the model*

Another way of looking at this model is to see it as a linear model with a complex form of error. Let  $\delta_{ij}$  represent the combined error term – also known as the composite error term:

$$\delta_{ij} = u_{0j} + u_{1j}X_{ij} + r_{ij}$$

We can then write the model as:

$$Y_{ij} = \gamma_{00} + \gamma_{01}W_j + \gamma_{10}X_{ij} + \gamma_{11}W_jX_{ij} + \delta_{ij}$$

$\text{Var}(\delta)$

~~$V$~~

$$\hat{\beta} = (X'V^{-1}X)^{-1} (X'V^{-1}y)$$

---

This looks like an ordinary linear model except that the  $\delta_{ij}$ s are **not** identically  $N(0, \sigma^2)$  and are **not** independent since the same  $u_{0j}$  and  $u_{1j}$  contribute to the random error for all  $\delta_{ij}$ s in the  $j$ th school. If we let  $\delta_j$  be the vector of errors in the  $j$ th school we can express the distribution of the combined errors as follows:

$$\delta_j \sim N(0, \mathbf{G} + \sigma^2 (\mathbf{X}_j' \mathbf{X}_j)^{-1}), \quad \delta_j \text{ and } \delta_k \text{ are independent for } j \neq k.$$

If  $\mathbf{T}$  and  $\sigma^2$  were known then the variance-covariance matrix of the random errors could be computed and the model fitted with Generalized Least-Squares (GLS).

With  $\mathbf{T}$  and  $\sigma^2$  unknown, we can iteratively estimate them and use the estimated values to fit the linear parameters,  $\gamma_{st}$  by GLS. There are variants depending on the way in which  $\mathbf{T}$  and  $\sigma^2$  are estimated. Using full likelihood yields what is often called “IGLS,” “ML,” or “FIML.” Using

the conditional likelihood of residuals given  $\hat{Y}$  yields “RIGLS” or “REML” (R for restricted or reduced).

### ***Matrix form***

Take all observations in school  $j$  and assemble them into vectors and matrices: (this is called the Laird-Ware formulation of the model from Laird and Ware (1982))

$$\mathbf{Y}_j = \mathbf{X}_j \boldsymbol{\gamma} + \mathbf{Z}_j \mathbf{u}_j + \mathbf{r}_j$$

where

$$\mathbf{Y}_j = \begin{bmatrix} Y_{1j} \\ \vdots \\ Y_{n_j j} \end{bmatrix}, \quad \mathbf{X}_j = \begin{bmatrix} 1 & W_j & X_{1j} & W_j X_{1j} \\ 1 & W_j & X_{2j} & W_j X_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & W_j & X_{n_j j} & W_j X_{n_j j} \end{bmatrix}, \quad \mathbf{Z}_j = \begin{bmatrix} 1 & X_{1j} \\ 1 & X_{2j} \\ \vdots & \vdots \\ 1 & X_{n_j j} \end{bmatrix}$$

$$\mathbf{u}_j = \begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \gamma_{10} \\ \gamma_{11} \end{pmatrix}, \quad \mathbf{r}_j = \begin{pmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{n_j j} \end{pmatrix}, \quad j=1, \dots, J$$

The distribution of the random elements is:  $\mathbf{u}_j \sim N(0, \mathbf{G})$ ,  $\mathbf{r}_j \sim N(0, \sigma^2 \mathbf{I})$  with  $u_j$  independent of  $r_j$ .

Now we put the school matrices together into big matrices:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\gamma} + \mathbf{Z}\mathbf{u} + \mathbf{r}$$

where

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_J \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_J \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_J \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_J \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{Z}_J \end{bmatrix}$$

with

$$\mathbf{u} \sim N \left( \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{G} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{G} \end{bmatrix} \right)$$

and

$$\mathbf{r} \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$$

which might be deceptive because the “ $\mathbf{I}$ ” is now much larger than before. The new block diagonal matrix for the variance of  $\mathbf{u}$  is often with the same symbol as the variance of  $\mathbf{u}_j$ . To avoid confusion we can use  $\ddot{\mathbf{G}}$ .



## ***Notational Babel***

Mixed models were simultaneously and semi independently developed by researchers in many different disciplines, each developing its own notation. The notation we are using here is that of Bryk and Raudenbush (1992) which has been very influential in social research. Many publications use this notation. It differs from the notation used in SAS documentation whose development was more influenced by seminal statistical work in animal husbandry. It is, of course, perfectly normal to fit models in SAS but to report findings using the notation in common use in the subject matter area. A short bilingual dictionary follows. Fortunately, **Y**, **X** and **Z** are used with the same meaning.

Line 4

	Bryk and Raudenbush	SAS help files	Pinheiro and Bates	My current preference
Fixed effects parameters	$\gamma$	$\beta$	$\beta$	$\gamma$
Cluster random effect	$\beta$	$b$		$\beta$
Cluster random effect (centered)	$u$	$\gamma$	$b$	$u$
Variance of random effects	$T$	$G$	$\Psi$	$G$
Within cluster error variance	$\Sigma$	$R$	$\sigma^2 \Lambda$	$R$

For example in Bryk and Raudenbush the Mixed Model is:

$$Y_i = X_i \gamma + Z_i u_i + \varepsilon_i$$

$$u_i \sim N(0, T) \quad \varepsilon_i \sim N(0, \Sigma_i)$$

GEE

$\sigma^2 I$

In Pinheiro and Bates:

$$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{b}_i + \boldsymbol{\varepsilon}_i$$

$$\mathbf{b}_i \sim N(\mathbf{0}, \boldsymbol{\Psi}); \quad \boldsymbol{\varepsilon}_i \sim N(\mathbf{0}, \sigma^2 \boldsymbol{\Lambda}_i)$$

### *The GLS fit*

With the matrix formulation of the model, it is easy to Express the GLS estimator of  $\boldsymbol{\gamma}$ . First denote:

$$\mathbf{V} = \text{Var}(\boldsymbol{\delta}) = \mathbf{Z}\ddot{\mathbf{G}}\mathbf{Z}' + \sigma^2\mathbf{I}$$

Then the GLS estimator is:

$$\hat{\boldsymbol{\gamma}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y}$$

We will see that the presence of  $\mathbf{V}^{-1}$  can result in an estimate that is very different from its OLS analogue<sup>5</sup>

---

<sup>5</sup> One ironic twist concerns small estimated values of  $\sigma^2$ . Normally this would be a cause for rejoicing; however it can result in a nearly singular  $\mathbf{V}$ .

The model we just derived has every important component we want:

1. a within-cluster variable  $X$  with a fixed effect
2. a between cluster variable  $W$  with a fixed effect
3. a cross-level interaction  $X*W$  with a fixed effect
4. a random intercept varying from cluster to cluster
5. a random slope varying from cluster to cluster.

Fitting this model in R:

```
> library(nlme)
> fit.mixed <- lme( Y ~ X * W, dd,
  random = ~ 1 + X | school)
```

only  
level  
variables

in long form

```
> fit.mixed <- lme( Y ~ 1 + X + W + X:W, dd,
  random = ~ 1 + X | school)
```

---

Although this need not imply that  $X'V^{-1}X$  is nearly singular. Algorithms do not yet take advantage of this. .

## From the simple to the complex

Traditional name	fixed part	random part
One way ANOVA with random effects	$Y \sim 1$	$\sim 1 \mid \text{school}$
Means as outcomes	$Y \sim 1 + W$	$\sim 1 \mid \text{school}$
One way ANCOVA	$Y \sim 1 + X$	$\sim 1 \mid \text{school}$
Random coefficients	$Y \sim 1 + X$	$\sim 1 + X \mid \text{school}$
Intercepts and slopes as outcomes	$Y \sim 1 + X + W + X:W$	$\sim 1 + X \mid \text{school}$
Non random slopes	$Y \sim 1 + X + W + X:W$	$\sim 1 \mid \text{school}$
Parallel mean slopes	$Y \sim 1 + X + W$	$\sim 1 + X \mid \text{school}$

$\delta_{H111}$  between  $\delta_W$  &  $\delta_P$

Contextual cluster mean variable with CWG variable and random CWG slopes	$Y \sim 1 + \text{cvar}(X, \text{school}) + \text{dvar}(X, \text{school})$	$\sim 1 + \text{dvar}(X, \text{school})   \text{school}$
Contextual cluster mean variable with raw variable and random CWG slopes	$Y \sim 1 + \text{cvar}(X, \text{school}) + X$	$\sim 1 + \text{dvar}(X, \text{school})   \text{school}$
Intercepts and slopes as outcomes with contextual cluster mean variable with CWG variable and random CWG effect	$Y \sim 1 + (\text{cvar}(X, \text{school}) + \text{dvar}(X, \text{school})) * W$	$\sim 1 + \text{dvar}(X, \text{school})   \text{school}$

## The simplest models

We have now built up the notation and some theory for a fairly general form of the linear mixed model with both Level 1 and Level 2 variables and a random effects model with a random intercept and a random slope. We will now consider the interpretation of simpler models in which we keep only some components of the more general model. Even when we are interested in the larger model, it is important to understand the simple ‘sub-models’ because they are used for hypothesis testing in the larger model. We will also consider some extensions of the concepts we have seen so far in the context of some of these simpler models.

### *One-way ANOVA with random effects*

This is the simplest random effects models and provides a good starting point to illustrate the special characteristics of these models.

Level 1 model:

$$Y_{ij} = \beta_{0j} + r_{ij}$$

Level 2 model:

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

Combined model:

$$Y_{ij} = \gamma_{00} + u_{0j} + r_{ij}$$

$$\text{Var}(Y_{ij}) = \text{Var}(u_{0j} + r_{ij}) = g_{00} + \sigma^2$$

Note the intraclass correlation coefficient:

$$\rho = g_{00} / (g_{00} + \sigma^2)$$

Also note that within each school:

$$E(\bar{Y}_{.j}) = \gamma_{0j}$$

$$\text{Var}(\bar{Y}_{.j} | \beta_{0j}) = \frac{\sigma^2}{n_j}$$

but across the population:

$$E(\bar{Y}_{.j}) = \gamma_{0j}$$

$$\text{Var}(\bar{Y}_{.j} | \beta_{0j}) = g_{00} + \frac{\sigma^2}{n_j}$$

This is an example of two very useful facts:

1. the unconditional (sometimes called ‘marginal’ but not by economists) mean is equal to the **mean conditional mean**,
2. the unconditional variance is equal to the **mean of the conditional variance** plus the **variance of the conditional mean**, i.e.:

$$\begin{aligned}
\text{Var}(\bar{Y}_{.j}) &= \text{E}(\text{Var}(\bar{Y}_{.j} | \beta_{0j}) + \text{Var}(\text{E}(\bar{Y}_{.j} | \beta_{0j}))) \\
&= \sigma^2 + \text{Var}(\beta_{0j}) \\
&= \sigma^2 + g_{00}
\end{aligned}$$

$$\begin{aligned}
\text{Var}(Y_{.j}) &= \text{E}(\text{Var}(Y_{.j} | \beta_{0j}) + \text{Var}(\text{E}(Y_{.j} | \beta_{0j}))) \\
&= \sigma^2 + \text{Var}(\beta_{0j}) \\
&= \sigma^2 + g_{00}
\end{aligned}$$

### ***Estimating the one-way ANOVA model***

There are three kinds of parameters that need to be estimated:

1. **fixed effect parameters** : in this case there is only one:  $\gamma_{00}$ ,
2. **variance-covariance components**:  $g_{00}$  and  $\sigma^2$ ,
3. **random effects**:  $\beta_{0j}$  or, equivalently, combined with  $\tau_{00} : u_{0j}$ .

We use a different approach for each type of parameter.

The **fixed effects parameters** are like linear regression parameters except that they are estimated from observations that are not independent. Instead of using OLS (ordinary least-squares) we use **GLS (generalized least-squares)** using the estimates of the variance-covariance components as the variance matrix in the GLS procedure.

The **variance-covariance parameters** are estimated using **ML (maximum likelihood)** or **REML (restricted maximum likelihood)**.

Note that each step above assumes that the other one has been completed. What really happens is that estimation goes back and forth between the two steps until convergence.

The **random effects** are not just parameters. They are realizations of random variables. This means that we have two sources of information

about them: we can ‘estimate’ them from the observed data and we can ‘guess’ them from their distribution. Putting these two sources of information together is the essence of Bayesian estimation, or empirical Bayesian estimation because the distribution of the random effects, determined by  $\mathbf{G}=[g_{oo}]$ , is estimated from the data and model. The random effects are **predicted** (in contrast with ‘estimated’) using **EBLUPs (Empirical Best Linear Unbiased Predictors)** with the empirical posterior expectation:

$$E(\beta_{01}, \dots, \beta_{0J} | Y_1, \dots, Y_n)$$

i.e. the expected value of what is unknown given what is known.

We will look at the estimation of the three types of parameters in detail in this example.

First we consider the analysis of the data using OLS in which we treat  $\beta_{01}, \dots, \beta_{0J}$  as non-random parameters. **The coding of the school effect**

**determines what is estimated by the intercept term.** It is a weighted linear combination of the  $\beta_{0j}$ s:

$$\psi_w = \sum_{j=1}^J w_j \beta_{0j}$$

If the coding uses “true” contrasts (each column of the **coding matrix** sums to 0) the weights are all equal to  $1/J$  and  $\psi_w$  is the ordinary mean of  $\beta_{0j}$ s:

$$\psi_w = \frac{1}{J} \sum_1^J \beta_{0j}$$

In this case

$$\hat{\psi}_w = \frac{1}{J} \sum_1^J \bar{Y}_j = \bar{Y}_{Schools}$$

With “sample size” coding, e.g.

	$V_1$	$V_2$	$V_3$	$\dots$	$V_{J-1}$
$School_1$	$n_j$	0	0	$\dots$	0
$School_2$	0	$n_j$	0	$\dots$	0
$School_3$	0	0	$n_j$	$\dots$	0
$School_4$	0	0	0	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$School_{J-1}$	0	0	0	$\vdots$	$n_j$
$School_J$	$-n_1$	$-n_2$	$-n_3$	$\dots$	$-n_{J-1}$

each column of the design matrix sums to 0 and the intercept will estimate:

$$\psi_w = \frac{\sum_{j=1}^J n_j \beta_{0j}}{\sum_{j=1}^J n_j}$$

which weights each school according to its sample size. This can be thought of as the mean of the population of **students** instead of the population of **schools**. The estimator would be the overall average of  $Y$ :

$$\psi_w = \frac{\sum_{j=1}^J n_j \bar{Y}_j}{\sum_{j=1}^J n_j} = \bar{Y}_{..} = \bar{Y}_{Students}$$

We are not limited to these two obvious choices. A more appropriate set of weights could be school size, with coding:

	$V_1$	$V_2$	$V_3$	$\cdots$	$V_{J-1}$
$School_1$	$s_J$	0	0	$\cdots$	0
$School_2$	0	$s_J$	0	$\cdots$	0
$School_3$	0	0	$s_J$	$\cdots$	0
$School_4$	0	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$School_{J-1}$	0	0	0	$\vdots$	$s_J$
$School_J$	$-s_1$	$-s_2$	$-s_3$	$\cdots$	$-s_{J-1}$

the intercept would estimate:

$$\psi_s = \frac{\sum_{j=1}^J s_j \beta_{0j}}{\sum_{j=1}^J s_j}$$

In each case the form of the estimate is a weighted mean of the individual school averages:

$$\hat{\psi}_w = \sum_{j=1}^J w_j \bar{Y}_j$$

with variance:

$$\text{Var}(\hat{\psi}_w | \beta_{01}, \dots, \beta_{0J}) = \sum_{j=1}^J w_j^2 \frac{\sigma^2}{n_j}$$

where the weights,  $w_j$ , sum to 1. Note that the variance is minimized when the weights are proportional to  $n_j$ , i.e.  $w_j = n_j / n$  where  $n$  is the total sample size:  $n = \sum_j n_j$ . In this case the variance is  $\sigma^2 / n$ . Thus, the **student mean** is the parameter estimated with the least variance.

## ***Mixed model approach***

With a mixed model we want to estimate  $\gamma_{00}$  instead of a particular linear combination of  $\beta_{0j}$ s. Any weighted mean  $\hat{\psi}_w = \sum_j w_j \bar{Y}_j$  of  $\bar{Y}_j$ s will be unbiased for  $\gamma_{00}$  because

$$\begin{aligned} E(\hat{\psi}_w) &= E(\sum_j w_j \bar{Y}_j) \\ &= \sum_j w_j E(\beta_{0j}) \\ &= \sum_j w_j \gamma_{00} \\ &= \gamma_{00} \end{aligned}$$

if the  $w_j$  s are weights with  $\sum_j w_j = 1$ .

Now, to calculate the variance of  $\hat{\psi}_w$  as an estimator of  $\gamma_{00}$ , we first need the variance of  $\bar{Y}_j$  as an estimator of  $\gamma_{00}$  with  $\beta_{0j}$  random:

$$\text{Var}(\bar{Y}_j) = g_{00} + \sigma^2 / n_j$$

Thus:

$$\text{Var}(\hat{\psi}_w) = \sum_j w_j^2 (g_{00} + \sigma^2 / n_j)$$

The optimal estimator is obtained by taking weights **inversely proportional** to  $(g_{00} + \sigma^2 / n_j)$ .

Consider the implications:

1. If  $g_{00}$  is much larger than  $\sigma^2$ , the weights will be nearly constant and  $\hat{\psi}_w$  will be close to  $\bar{Y}_{Schools}$ .
2. Conversely, if  $g_{00}$  is much smaller than  $\sigma^2$ , the weights will be nearly proportional to  $n_j$  and the estimator will be close to  $\bar{Y}_{Students}$ .

If it is not reasonable to treat the  $\beta_{0j}$ s as a random sample from the same  $N(0, g_{00})$  distribution then these two estimators could estimate two quantities with very different meanings. Consider, for example, what would happen if there is a strong relationship between  $\beta_{0j}$  and  $n_j$ s. What gets estimated is governed by the ratio  $g_{00} / \sigma^2$  – a purely statistical consideration quite disconnected from any interpretation of the estimator. It is important to appreciate that your estimator is determined by considerations that might not be relevant.

In R the command is:

```
lme ( y ~ 1 , hs, random = ~ 1 | school )
```

In SAS, the (minimal) commands would be<sup>6</sup>:

```
PROC MIXED DATA = MIXED.HS;  
  CLASS SCHOOL;  
  MODEL Y = ;  
  RANDOM INTERCEPT / SUBJECT=SCHOOL;  
  RUN;
```

---

<sup>6</sup> To use the HS data set, download the self-extracting file following the link at the course website. Save it in a convenient directory. Click on its icon to create the SAS data set HS.SD2. From SAS, create a library named MIXED that points to this directory. You can then use the data set using the syntax in this example.

## ***EBLUPs***

This interesting topic can, alas, be skipped. It played a central role in the early development of mixed models for animal husbandry where an important practical problem was estimating the reproductive qualities of a bull from the characteristics of its progeny. In most applications of mixed models in the social sciences, the focus is on the estimation of the fixed parameters and much less so on the ‘prediction’ of the random effects.

Estimating the  $u_{0j}$ s involves using two sources of information: the data and their distribution as random variables. First consider the OLS estimator for  $\beta_{0j}$ :

$$\hat{\beta}_{0j} = \bar{Y}_{.j}$$

Now, to get the *Empirical Best Linear Unbiased Predictor* of  $u_{0j}$ s, we pretend that the estimated values of  $\gamma_{00}$  and  $\sigma^2$  are the “true” values and we calculate the conditional expectation of  $u_{0j}$ s given  $y_{ij}$ s. This is done most

easily using the matrix formulation of the model and a formula for the conditional expectation in the multivariate case. We use partitioned matrices to express the joint distribution of  $\mathbf{Y}$  and  $\mathbf{u}$ :

$$\begin{bmatrix} \mathbf{Y} \\ \mathbf{u} \end{bmatrix} \sim N \left( \begin{bmatrix} \mathbf{X}\boldsymbol{\gamma} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{Z}\ddot{\mathbf{G}}\mathbf{Z}' + \sigma^2\mathbf{I} & \mathbf{Z}\ddot{\mathbf{G}} \\ \ddot{\mathbf{G}}\mathbf{Z}' & \ddot{\mathbf{G}} \end{bmatrix} \right)$$

A “well-known” formula gives:

$$\hat{\mathbf{E}}(\mathbf{u} | \mathbf{Y}) = \ddot{\mathbf{G}}\mathbf{Z}\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\gamma})$$

where  $\mathbf{V} = \mathbf{Z}\ddot{\mathbf{G}}\mathbf{Z}' + \sigma^2\mathbf{I}$ . This formula with a bit more mechanical work will give us the EBLUP below, but we will derive it intuitively:

1. We could estimate  $u_{0j}$  with the “obvious” OLS estimate:

$$\hat{u}_{0j} = \hat{\beta}_{0j} - \hat{\gamma}_{00} = \bar{Y}_{.j} - \hat{\gamma}_{00}$$

as an estimate of  $u_{0j}$  this has variance  $\sigma^2/n_j$ .

2. We could also guess that  $u_{0j}$  is equal to 0 (the mean of its distribution) and our guess would have variance  $g_{00}$ .

How can we “best” combine these independent sources of information? By using weights proportional to inverse variance! This gives us the **EBLUP** of  $u_{0j}$ :

$$\tilde{u}_{0j} = \frac{\frac{1}{\sigma^2/n_j} \hat{u}_{0j} + \frac{1}{g_{00}} 0}{\frac{1}{\sigma^2/n_j} + \frac{1}{g_{00}}} = \frac{\hat{u}_{0j}}{1 + \frac{\sigma^2/n_j}{g_{00}}}$$

This has the effect of **shrinking**  $\hat{u}_{0j}$  towards 0 by a factor of

$$\frac{\frac{1}{\sigma^2/n_j}}{\frac{1}{\sigma^2/n_j} + \frac{1}{\tau_{00}}} = \frac{1}{1 + \frac{\sigma^2/n_j}{\tau_{00}}}$$

Consider how the amount of shrinking depends on the relative values of  $\sigma^2$ ,  $g_{00}$  and  $n_j$ . There will be more shrinkage if

1.  $g_{00}$  is small: i.e. the distribution of  $u_{0j}$  is known to be close to 0.
2.  $\sigma^2$  is large: i.e.  $\bar{Y}_{0j}$  has large variation as an estimate of  $\beta_{0j}$ .
3.  $n_j$  is small: ditto.

The EBLUP estimator of  $\beta_{0j}$  (we'll call it  $\tilde{\beta}_{0j}$  works exactly the same way with the OLS estimator (analyzing each school separately) which gets shrunk towards the overall estimator  $\hat{\gamma}_{00}$ . This is in exactly the same spirit as shrinkage estimators derived from Bayesian, Empirical Bayes or frequentist approaches. Bradley Efron and Carl Morris wrote an interesting article on the topic in *Scientific American*, Efron and Morris(1977).

## Slightly more complex models

### *Means as outcomes regression*

Level 1 model:

$$Y_{ij} = \beta_0 + \beta_{0j} + r_{ij}$$

Level 2 model:

$$\beta_{0j} = \gamma_{00} + \gamma_{01}W_j + u_{0j}$$

Combined model:

$$Y_{ij} = \gamma_{00} + \gamma_{01}W_j + u_{0j} + r_{ij}$$

Note that

$$\text{Var}(Y_{ij}) = \text{Var}(u_{0j} + r_{ij})$$

as above but, in this model,  $\text{Var}(Y_{ij})$  is a conditional variance, conditional given  $W$ .

In R the command is:

```
lm ( y ~ w , hs, random = ~ 1 | school )
```

In SAS, the commands for the means as outcomes model would be:

```
PROC MIXED DATA = MIXED.HS;  
CLASS SCHOOL;  
MODEL Y = W ;  
RANDOM INTERCEPT / SUBJECT = SCHOOL;  
RUN
```

### ***One-way ANCOVA with random effects***

Level 1 model:

$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + r_{ij}$$

Level 2 model:

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$\beta_{1j} = \gamma_{10}$$

Combined model:

$$Y_{ij} = \gamma_{00} + \gamma_{10}X_{ij} + u_{0j} + r_{ij}$$

In R the command is:

```
lm ( y ~ x, hs, random = ~ 1 | school )
```

In SAS, the commands for one-way ANCOVA with random effects are:

```
PROC MIXED DATA = MIXED.HS;  
CLASS SCHOOL;  
MODEL Y = X ;  
RANDOM INTERCEPT / SUBJECT = SCHOOL;  
RUN;
```

## ***Random coefficients model***

Level 1 model:

$$Y_{ij} = \beta_{0j} + \beta_{1j}X_{ij} + r_{ij}$$

Level 2 model:

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$\beta_{1j} = \gamma_{10} + u_{1j}$$

with:

$$\text{Var}\left(\begin{bmatrix} u_{0j} \\ u_{1j} \end{bmatrix}\right) = \mathbf{T} = \begin{bmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{bmatrix}$$

Combined model:

$$Y_{ij} = \tau_{00} + \tau_{10}X_{ij} + u_{0j} + u_{1j}X_{ij} + r_{ij}$$

In R the command would be:

```
lm ( y ~ x , hs, random = ~ 1 + x | school )
```

In SAS, the commands for the random coefficients model are:

```
PROC MIXED DATA = MIXED.HS;  
CLASS SCHOOL;  
MODEL Y = X ;  
RANDOM INTERCEPT X / SUBJECT = SCHOOL TYPE =  
UN;  
RUN;
```

### *Intercepts and Slopes as outcomes*

This corresponds to the full model presented in 0 above.

In R the command would be:

```
lme ( y ~ x * w , hs, random = ~ 1 + x | school )
```

The SAS commands for this model are:

```
PROC MIXED DATA = MIXED.HS;  
  CLASS SCHOOL;  
  MODEL Y = X W X*W;  
  RANDOM INTERCEPT X / SUBJECT = SCHOOL TYPE = UN;  
  RUN;
```

Note the  $X*W$  term. It is called a *cross-level interaction*. It has the function of allowing the mean slope with respect to  $X$  to vary with  $W$ . Note that R automatically generates the marginal terms,  $x$  and  $w$ .

### ***Nonrandom slopes***

Consider the full model but with  $\tau_{11} = 0$  (hence  $\tau_{01} = 0$  also, otherwise  $T$  would not be a variance matrix). This is a model in which the variation in

$\hat{\beta}_{1j}$  from school to school is wholly consistent with the expected variation within schools and there is no need to postulate that  $\tau_{11} > 0$ .

In R the command would be:

```
lm ( y ~ x * w , hs, random = ~ 1 | school )
```

The SAS commands are left as an exercise.

## *Contextual effects*

A major – and underexploited – advantage of multilevel models is that it is easy to separately estimate the between-cluster and the within-cluster effects of a variable. The advantages of this approach are:

1. Including both effects in the model allows each to be estimated without contamination from the other. Many classical applications of mixed models are based on the assumption that the between effect and the within effect are equal. If the assumption is not satisfied the estimate is biased.
2. Effects at both levels can be estimated simultaneously with SEs that allow inference to appropriate populations. In contrast, the fixed effects model only allows generalization to new samples from the same clusters. The between-cluster model did not provide an estimate of the within-cluster effect.

3. Both between-cluster and within-cluster variables as well as cross-level interactions can be included in the same model.

Fixed part of the model with contextual cluster mean variable:

# Fitting the models

## *One way anova with random effect*

```
> fit.oneway.re <- lme( mathach ~ 1, hs, random = ~ 1 | sid)
> summary(fit.oneway.re)
```

Linear mixed-effects model fit by REML

Data: hs

AIC	BIC	logLik
12985.94	13002.71	-6489.969

Random effects:

Formula: ~1 | sid

(Intercept) Residual

StdDev: 2.836278 6.296759

Fixed effects: mathach ~ 1

	Value	Std.Error	DF	t-value	p-value
(Intercept)	12.60468	0.4711941	1937	26.75049	0

Standardized Within-Group Residuals:

Min	Q1	Med	Q3	Max
-2.78262694	-0.74562760	0.03825124	0.78826675	2.51105403

Number of Observations: 1977

Number of Groups: 40

```

>
> intervals( fit.oneway.re )
Approximate 95% confidence intervals

Fixed effects:
              lower      est.      upper
(Intercept) 11.68057 12.60468 13.52878
attr(,"label")
[1] "Fixed effects:"

Random Effects:
Level: sid
              lower      est.      upper
sd((Intercept)) 2.214072 2.836278 3.633338

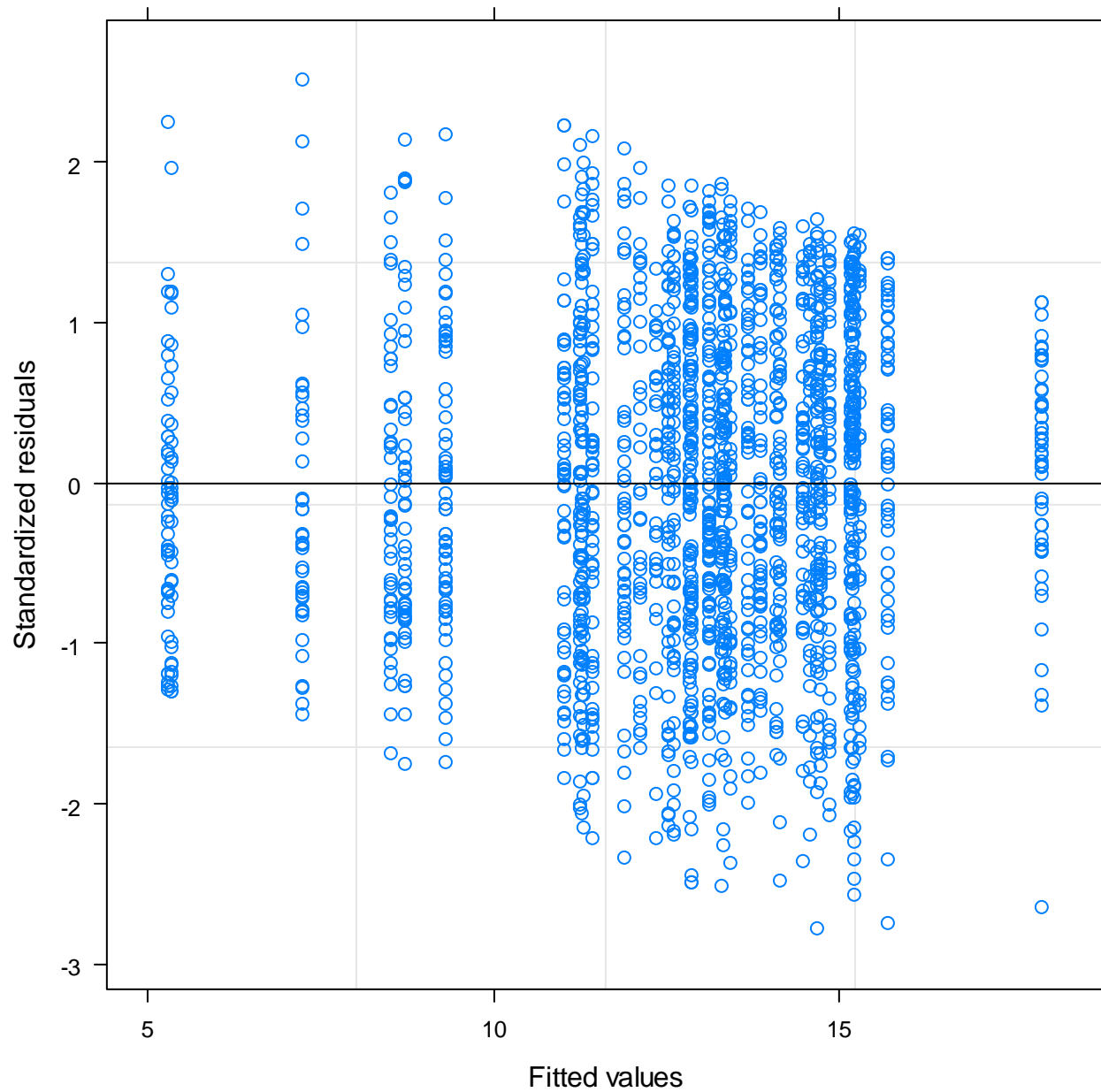
Within-group standard error:
      lower      est.      upper
6.101522 6.296759 6.498242

> glh( fit.oneway.re )
numDF denDF F.value p.value
    1   1937 715.589 <.00001

Coefficients  Estimate Std.Error   DF  t-value p-value Lower 0.95 Upper 0.95
(Intercept) 12.60468    0.47119 1937 26.75049 <.00001   11.68057   13.52878

```

Note: this could use a better approximation for degrees of freedom, e.g. the Satterthwaite algorithm that SAS uses.



```
plot(fit.oneway.re )
```

Note pattern in fitted residuals in contrast with OLS

```

> fixef( fit.oneway.re )    # estimation of fixed part of model
(Intercept)
  12.60468

> ranef( fit.oneway.re )    # BLUP of error in random portion
(Intercept)
P5762C -7.30651445
P2639C -5.36017663
P8854C -7.24846197
P6484C  0.26973942
. . .
C2208C  2.58744359
C2658C  0.71334861
C1906C  3.09104215
C9586G  2.08485465

> coef( fit.oneway.re )    # BLUP combining fixed and random parts
(Intercept)
P5762C    5.298161
P2639C    7.244499
P8854C    5.356214
. . .
C5619C   15.220870
C2208C   15.192119

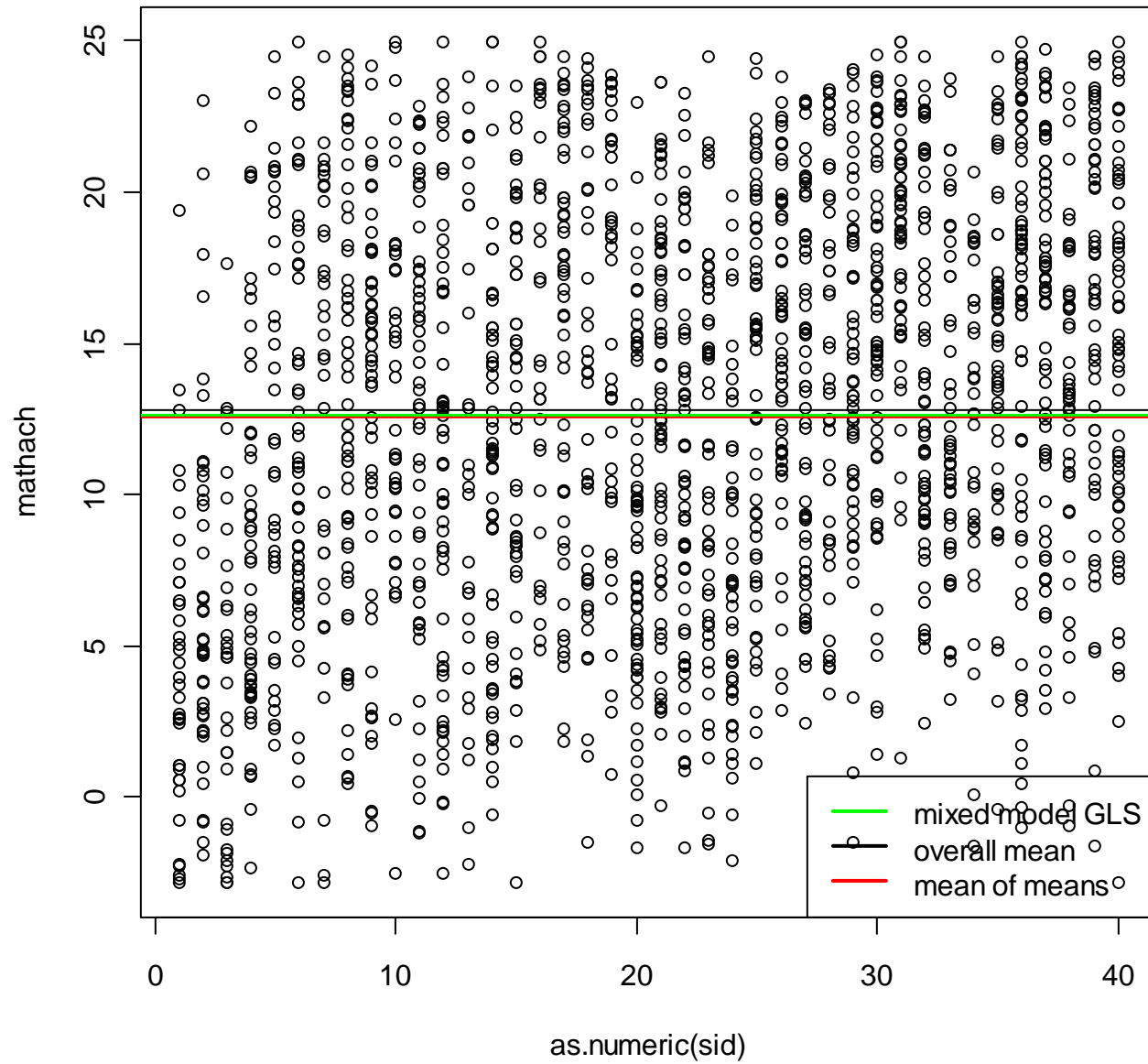
```

C2658C	13.318024
C1906C	15.695718
C9586G	14.689530

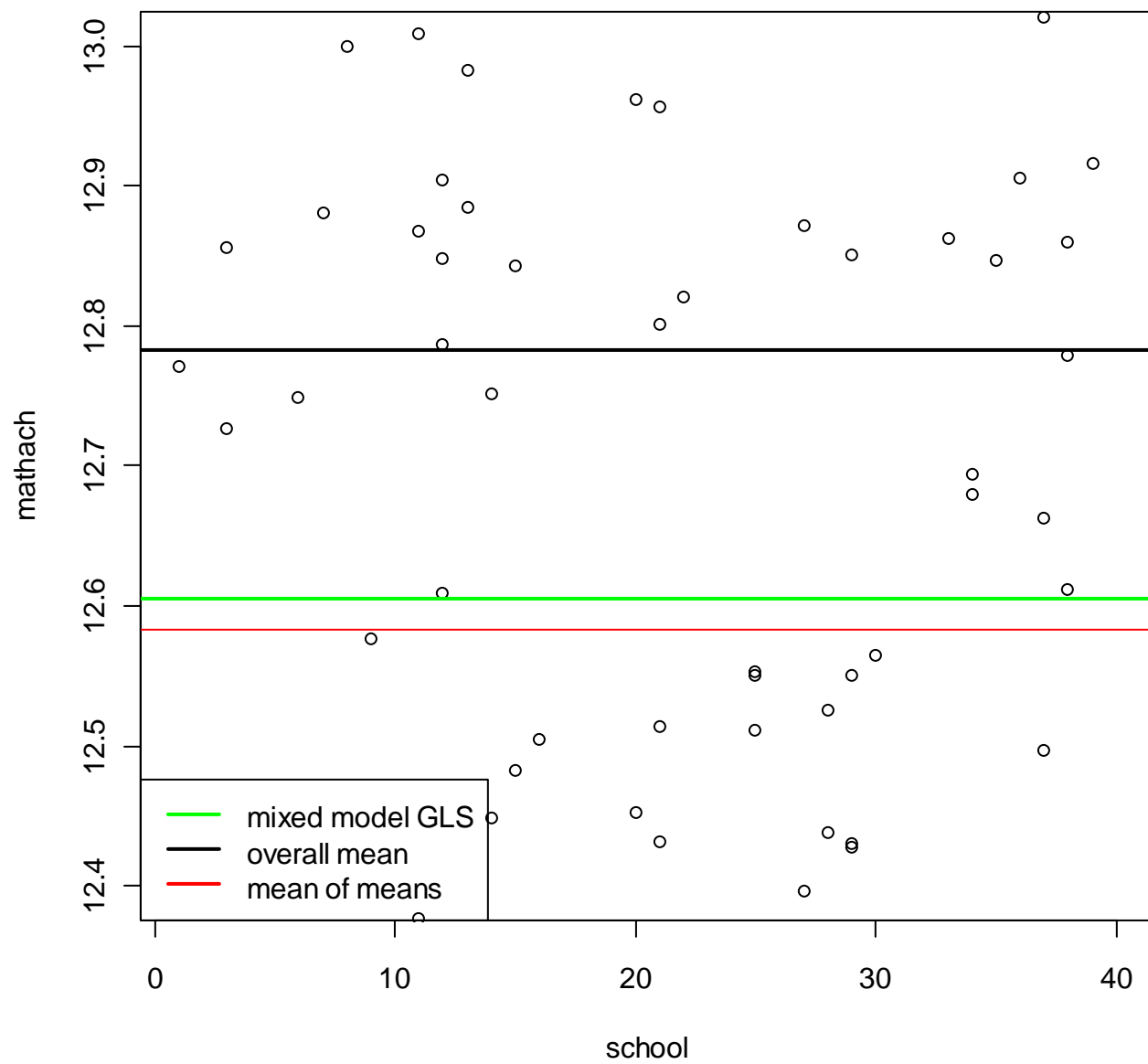
```
> coef( fit.oneway.re) == ( ranef(fit.oneway.re) + fixef( fit.oneway.re ))
      (Intercept)
```

P5762C	TRUE
P2639C	TRUE
P8854C	TRUE
. . .	
C3992C	TRUE
C5619C	TRUE
C2208C	TRUE
C2658C	TRUE
C1906C	TRUE
C9586G	TRUE

```
> plot( mathach ~ as.numeric(sid) , hs)
> abline( h = fixef( fit.oneway.re), col = 'green',
lwd = 2)
> abline( h = mean( hs$mathach), col = 'black')
> abline( h = mean( c(tapply( hs$mathach, hs$sid,
mean))), col = 'red')
>
> ?legend
> legend( 'bottomright', c('mixed model GLS',
'overall mean', 'mean of means'),
+               col =
c('green','black','red'), lty = 1, lwd = 2)
```



```
plot( mathach ~ as.numeric(sid) , hs, ylim = c(12.4,13), xlab = 'school')
abline( h = fixef( fit.oneway.re), col = 'green', lwd = 2)
abline( h = mean( hs$mathach), col = 'black', lwd = 2)
abline( h = mean( c(tapply( hs$mathach, hs$sid, mean))), col = 'red')
legend( 'bottomleft', c('mixed model GLS', 'overall mean', 'mean of means'),
       col = c('green','black','red'), lty = 1, lwd = 2)
```



```

> hs$n <- capply( hs$sid, hs$sid, length) # sample size in each school
> hs$mathach.ols <- capply( hs$mathach, hs$sid, mean)
>
> hs1.sid <- up(hs, ~ sid)
> rownames(hs1.sid) == rownames( coef( fit.oneway.re)) # check that they
match
[1] TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE
TRUE
[18] TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE
TRUE
[35] TRUE TRUE TRUE TRUE TRUE TRUE
>
> hs1.sid$blup <- coef( fit.oneway.re ) [,1]
> some( hs1.sid)

```

	school	sector	Size	Sector	PRACAD	DISCLIM	HIMINTY	Sex.comp	Sex.cat	sid	what
P8854C	8854	0	745	Public	0.18	-0.228	0	0.5312500	Coed	P8854C	p
P2771C	2771	0	415	Public	0.24	1.048	0	0.5090909	Coed	P2771C	p
P5640C	5640	0	1152	Public	0.41	0.256	0	0.4210526	Coed	P5640C	p
P7345C	7345	0	978	Public	0.64	0.336	1	0.5178571	Coed	P7345C	p
P6897C	6897	0	1415	Public	0.55	-0.361	0	0.5918367	Coed	P6897C	p
C4530G	4530	1	435	Catholic	0.60	-0.245	1	1.0000000	Girls	C4530G	p
C7342B	7342	1	1220	Catholic	0.46	0.380	1	0.0000000	Boys	C7342B	p
C5720C	5720	1	381	Catholic	0.65	-0.352	0	0.4528302	Coed	C5720C	p
C7688B	7688	1	1410	Catholic	0.65	-0.575	0	0.0000000	Boys	C7688B	p
C1906C	1906	1	400	Catholic	0.87	-0.939	0	0.5094340	Coed	C1906C	p

	ses.sch	mathach.sch	n	mathach.ols	blup
P8854C	-0.75675000	4.239781	32	4.239781	5.356214
P2771C	-0.33945455	11.844109	55	11.844109	11.906661
P5640C	-0.17659649	13.160105	57	13.160105	13.115900
P7345C	0.03325000	11.338554	56	11.338554	11.440975

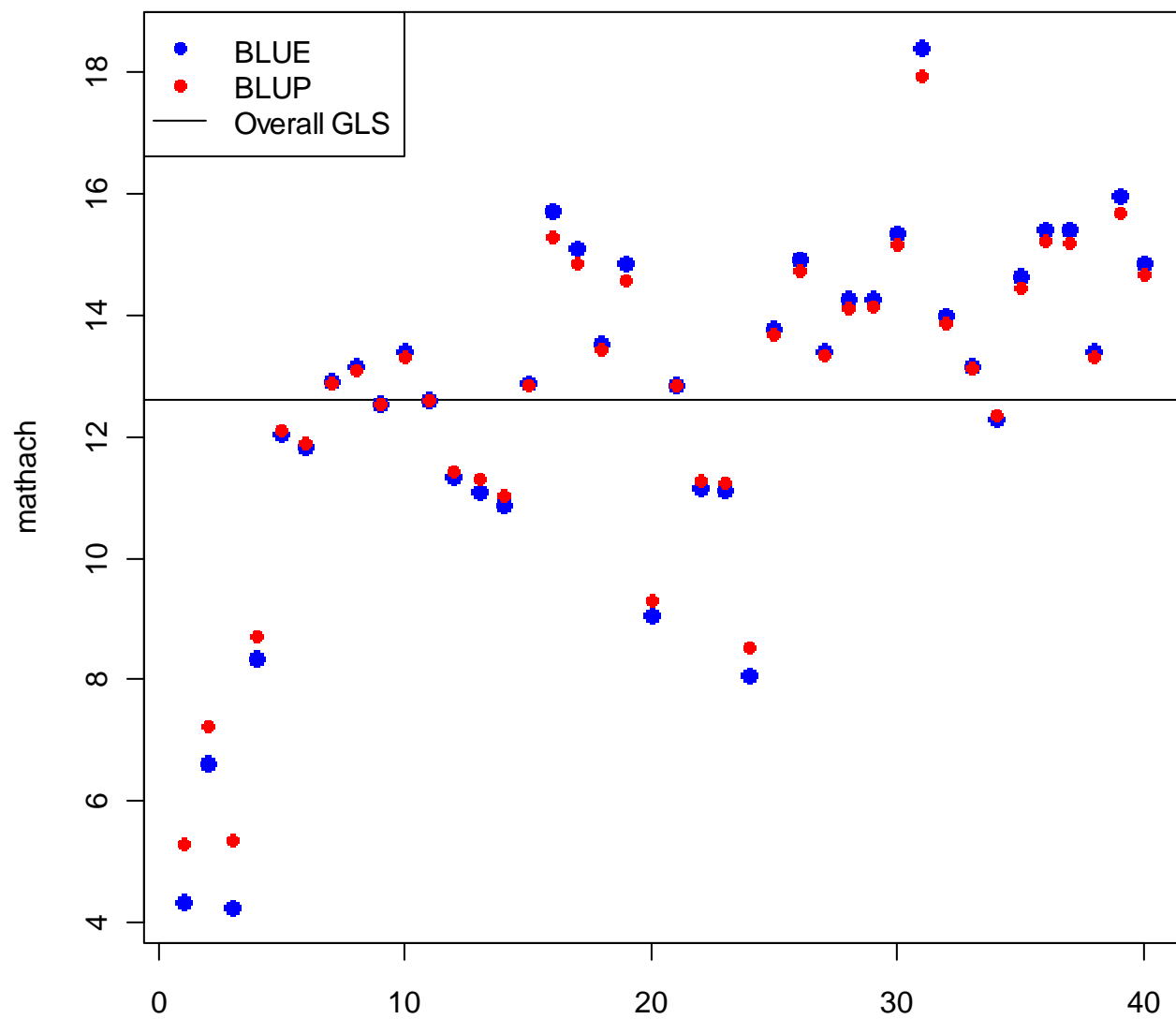
P6897C	0.34955102	15.097633	49	15.097633	14.869792
C4530G	-0.59688889	9.055698	63	9.055698	9.313204
C7342B	-0.44782759	11.166414	58	11.166414	11.279062
C5720C	0.03256604	14.282302	53	14.282302	14.139565
C7688B	0.18588889	18.422315	54	18.422315	17.935733
C1906C	0.51162264	15.983170	53	15.983170	15.695718

>

>

>

```
> plot( c(1,40), range( hs1.sid$mathach.ols), xlab = '', ylab = 'mathach',
type = 'n')
> abline( h = fixef( fit.oneway.re ), col = 'black', lwd = 1.5)
> points( 1:40, hs1.sid$mathach.ols, col = 'blue', pch = 16, cex = 1.2)
> points( 1:40, hs1.sid$blup, col = 'red', pch = 16)
> legend( 'topleft', c('BLUE', 'BLUP', 'Overall GLS'),
+       col = c('blue','red','black'),
+       pch = c(16,16,NA),
+       lty = c(NA,NA, 1))
>
```



```

> ## by sample size + a few more plotting bells and whistles
>
> plot( range( hs1.sid$n), range( hs1.sid$mathach.ols),
+       xlab = 'sample size', ylab = 'mathach', type = 'n',
+       main = 'Shrinking from the BLUE to the BLUP -- relationship with n')
> abline( h = fixef( fit.oneway.re ), col = 'green3', lwd = 2)
> points( hs1.sid$n, hs1.sid$mathach.ols, col = 'blue', pch = 16, cex = 1.2)
> points( hs1.sid$n, hs1.sid$blup, col = 'red', pch = 16)
> arrows( hs1.sid$n, hs1.sid$mathach.ols, hs1.sid$n, hs1.sid$blup, length=
.1)

```

Warning message:

```

In arrows(hs1.sid$n, hs1.sid$mathach.ols, hs1.sid$n, hs1.sid$blup,  :
  zero-length arrow is of indeterminate angle and so skipped

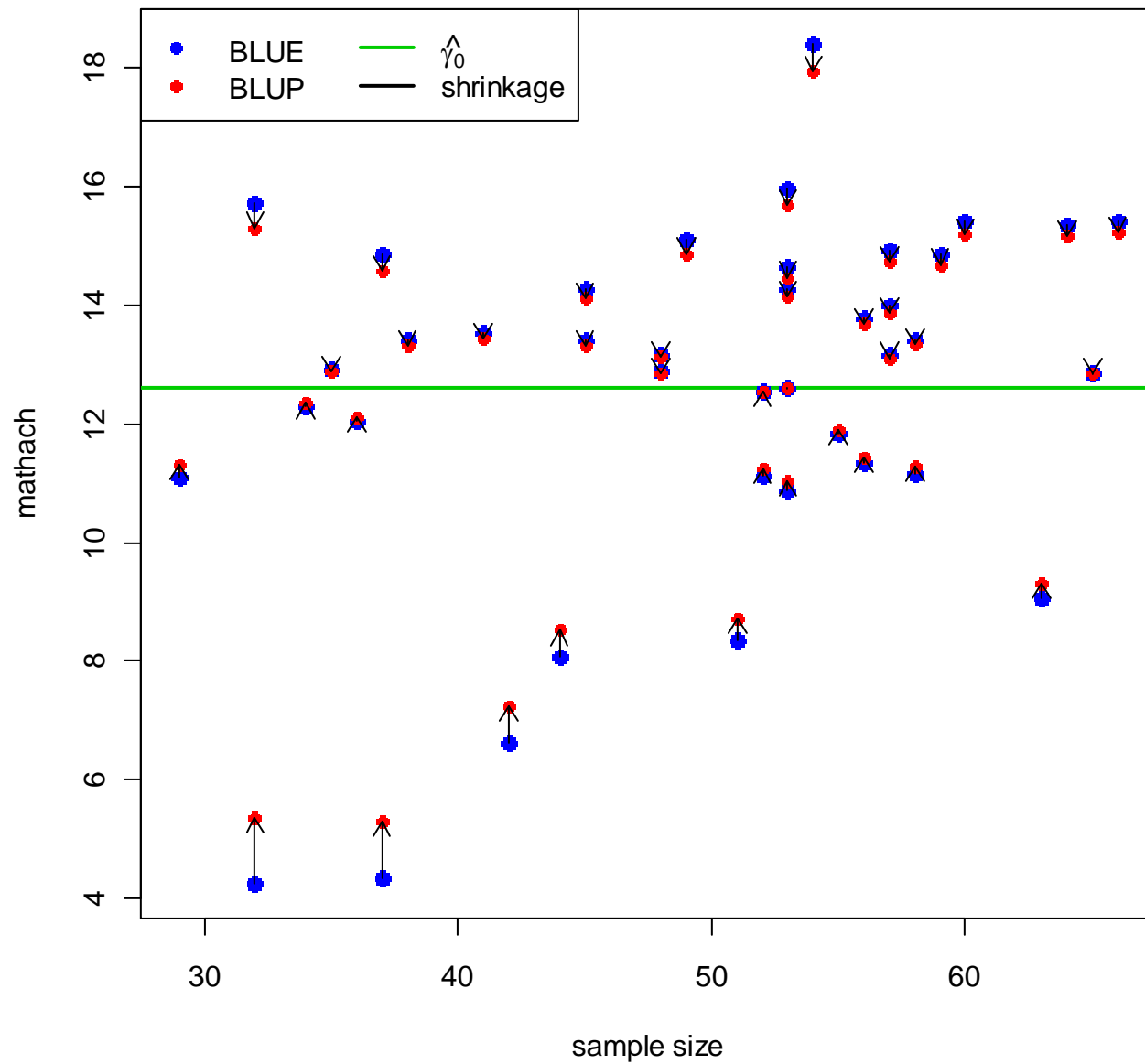
```

```

> legend( 'topleft',
+       # c('BLUE', 'BLUP', 'Overall GLS', 'shrinkage'),
+       expression(BLUE, BLUP, hat(gamma[0]), shrinkage),
+       ncol = 2,
+       col = c('blue', 'red', 'green3', 'black'),
+       lwd = c(NA, NA, 2, 2),
+       pch = c(16, 16, NA, NA),
+       lty = c(NA, NA, 1, 1))
>

```

## Shrinking from the BLUE to the BLUP -- relationship with n



Note how shrinkage is roughly proportional to the distance of the BLUE from the overall GLS estimate (green line) and smaller as  $n$  gets larger. Note also that the spread of the BLUE is greater with smaller  $n$ , illustrating the notion that the BLUE is not as good an estimate in this case.

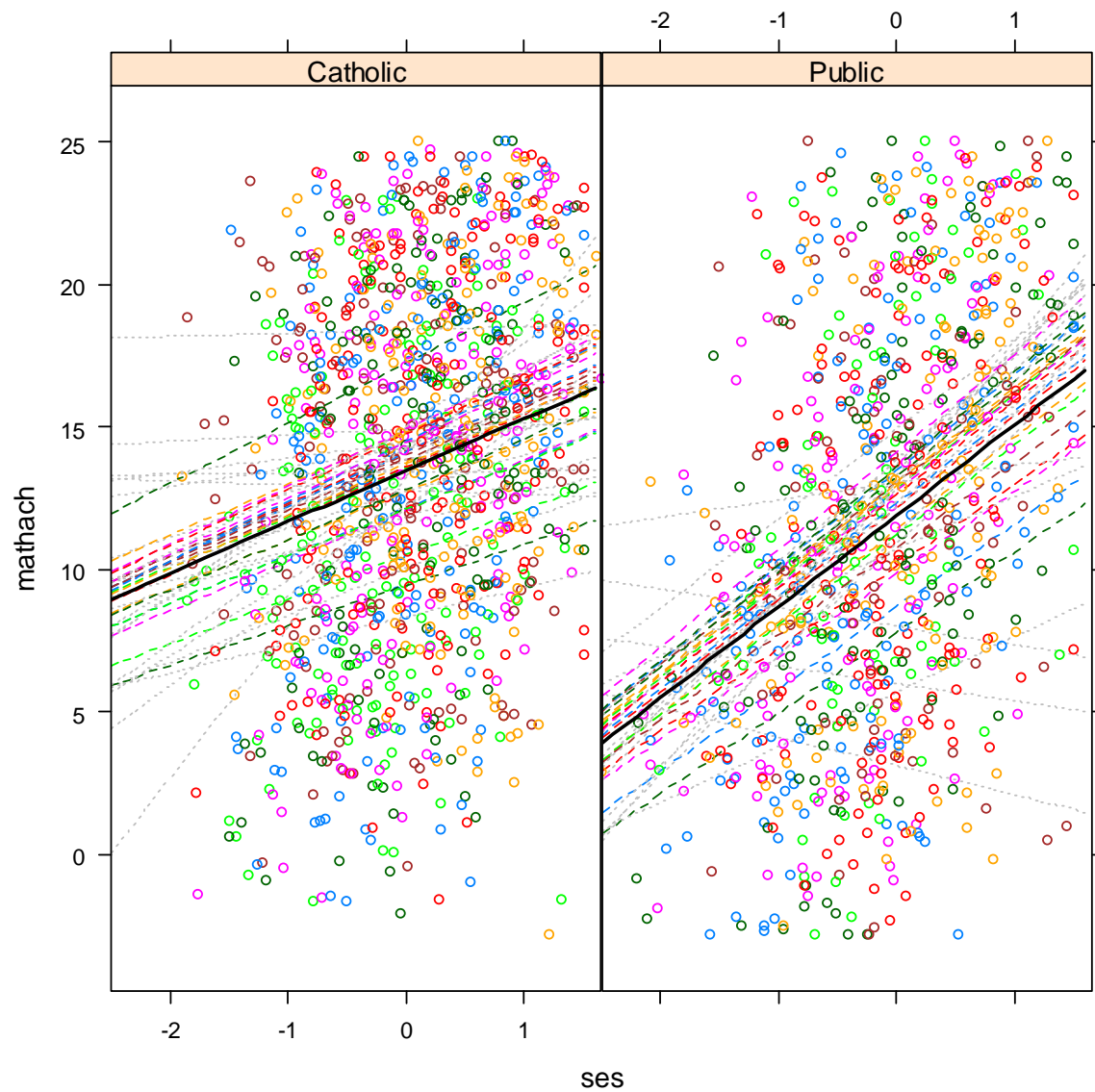
The GLS estimate is an 'optimal' estimate that takes all these issues into account. What is being estimated is the overall mean of the population from which schools are drawn. This *mean* (as a *parameter* of the population of schools) is defined to give the same weight to all schools, regardless of sample size.

The GLS mixed model *estimator* gives less weight to schools with smaller  $n$  but only because their data gives an estimate with larger variance.

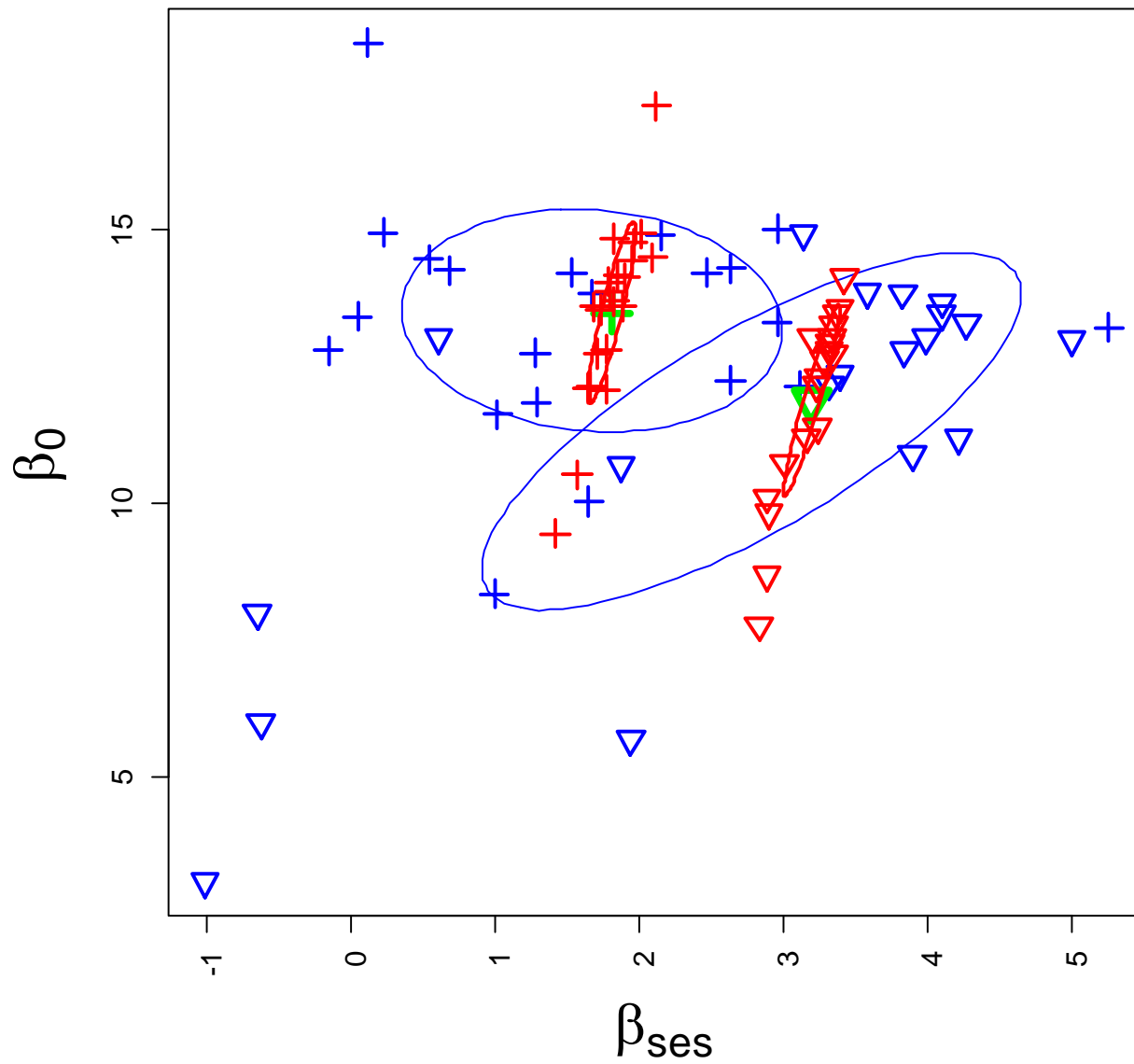
The *BLUP* is a reasonable estimator for a particular school as long as the information from other schools deserves the weight it gets in shrinking the *BLUE*. If a school is not '*exchangeable*' in the sample with other schools, i.e. if some known characteristic distinguishes it so that it can't be thought

of as 'just another school in the sample' then the *BLUE* should be preferred to the *BLUP*.

## *Intercepts and slopes as outcomes*

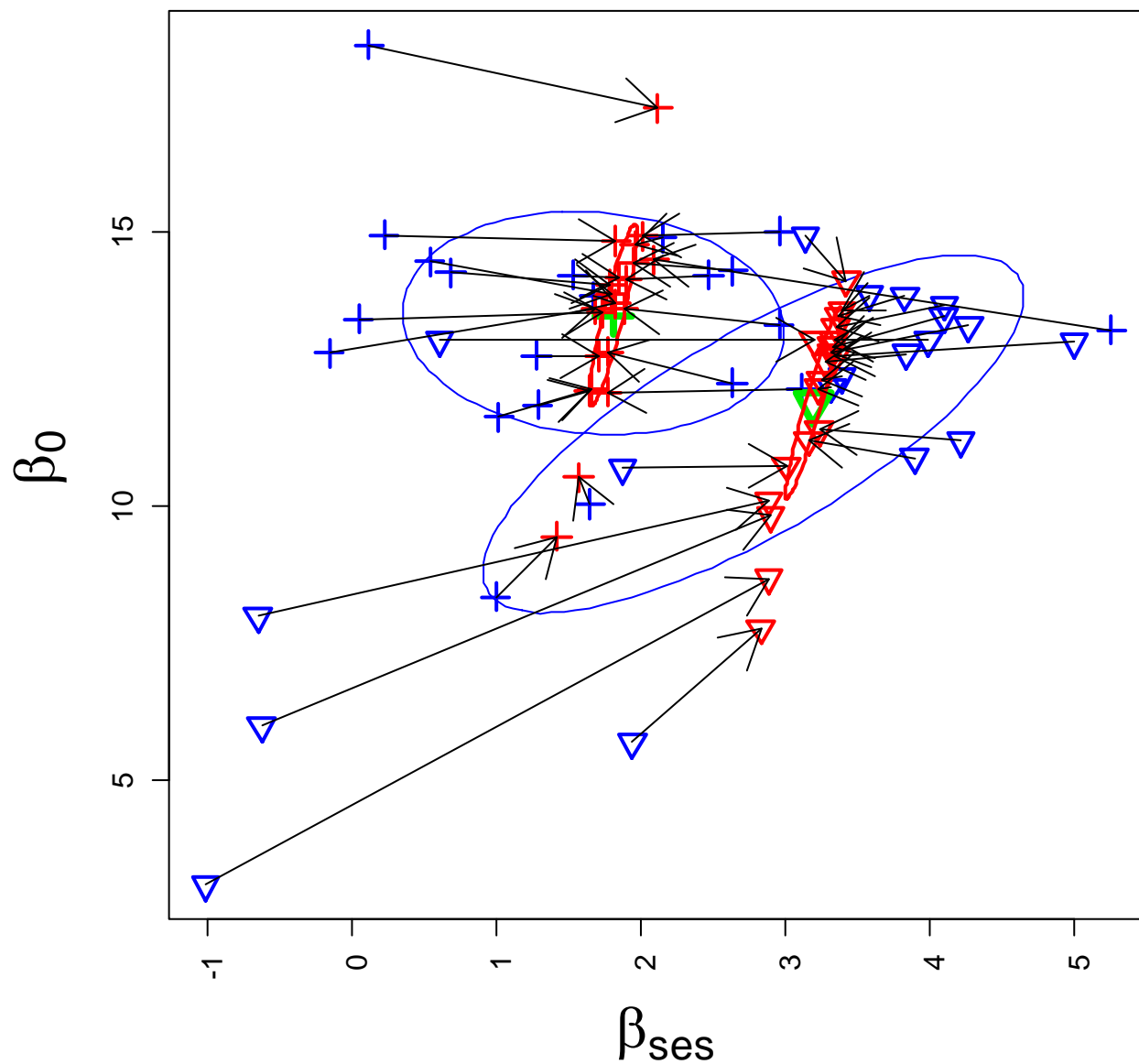


**Figure 18: BLUPS from a model with random slopes**

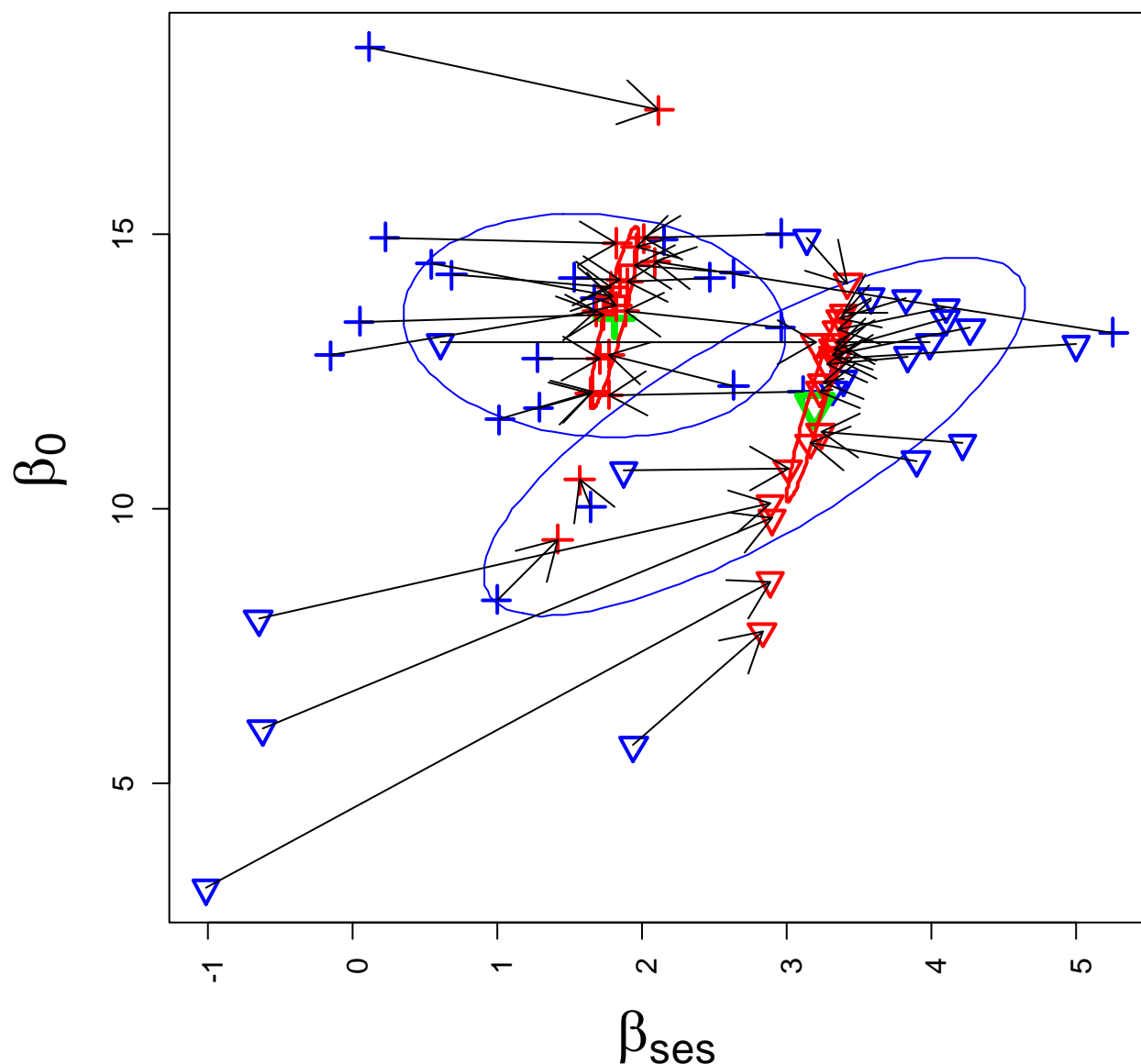


**Figure 19: BLUEs in blue and BLUPs in red. Mean value in green.**

The BLUPS show much less variability wrt  $\beta_{\text{ses}}$  than the BLUES. This is because the BLUPS recognize that much of the variability in  $\beta_{\text{ses}}$  is explicable by the large variability in  $\hat{\beta}_{\text{ses}}$  due to the samples. It does not interpret that variability as indicative of a variability in the slopes of the 'true' lines. The variability in intercepts, on the other hand, IS preserved in the BLUPS.



Why BLUPs are called 'shrinkage' estimators. It is an inverse variance weighted combination of the BLUE and of the population estimate.



If we knew the population mean line  $\gamma$ , the between cluster variance,  $G$  and the within-cluster variance,  $\sigma^2$ , the best predictor of  $\beta_j$ , the line for school  $j$ , combines  $\gamma$  and the BLUE,  $\hat{\beta}_j$ :

$$\tilde{\beta}_j = (\dots)^{-1} \times$$

$$\left\{ G^{-1} \gamma + \left[ \sigma^2 (X_j' X_j)^{-1} \right]^{-1} \hat{\beta}_j \right.$$

where

$$(\dots) = G^{-1} + \left[ \sigma^2 (X_j' X_j)^{-1} \right]^{-1}$$

Note that

$$\text{Var}(\beta_j) = G$$

$$\text{Var}(\hat{\beta}_j | \beta_j) = \sigma^2 (X_j' X_j)^{-1}$$

$$\text{Var}(\hat{\beta}_j) = G + \sigma^2 (X_j' X_j)^{-1}$$

$$\text{Var}(\tilde{\beta}_j) = \left\{ G^{-1} + \left[ \sigma^2 (X_j' X_j)^{-1} \right]^{-1} \right\}^{-1}$$

Note: the BLUPS vary less than G and the BLUES vary more than G.

$$\text{Var}(\tilde{\beta}_j) \leq \text{Var}(\beta_j) = G \leq \text{Var}(\hat{\beta}_j)$$

Note the estimated population lines for each sector are much closer to the centre of the BLUP ellipse than to the BLUE ellipse. Why?

The estimated population lines can be expressed as weighted combination of either the BLUES or of the BLUPS. However the weights VARY LESS when using the BLUPS than the BLUES.

*How can both BLUEs and BLUPs be 'best'?*

How can that be?

They are best for different things.

Recall the regression paradox: the best prediction of son's heights are best individually but they don't look like the distribution of son's heights. Best locally is not necessarily best at reproducing overall criteria.

BLUE is best for resampling from the same school over and over again. The BLUP is **best on average** for resampling from the population of schools and students.

If I'm a heartless bureaucrat and I want to be close on average I'll use the BLUP.

It's a bit like the basis of discrimination. If I don't have much information about you, I might use what I think I know about the group you seem to come from (here Catholic or Public) and I'll combine the two sources of information in an 'optimal' way.

If I really care to get a particular special school right, I would use the BLUE. The BLUP is justified only if the school is *exchangeable* with other schools in the sample and population conditional on the contextual variables.

# Lab 1

Lab 1, which will probably take almost 2 days to complete, covers the implementation of concepts seen in these slides as well as many complementary concepts that seem to be better presented in the context of an actual analysis. Some of the ideas covered in Lab 1

Lab 1:

- First example: Between Sector gap in Math Achievement
  - Randomly selecting a subsample of clusters (schools)
  - Having a first look at multilevel data
  - Creating new Level 2 variables from Level 1 data
  - Seeing data in 3d
  - A second look at multilevel data: targeted to a model
  - Seeing fitted lines in beta space
  - Between and within cluster effects
  - Fitting a mixed model
  - Handling NAs (simplest considerations)

- Non-convergence
- First diagnostics: Hausman test
- Contextual variables to the rescue
- Interpretation of models with contextual effects
- Estimating the compositional (= between) effect
- Alternative equivalent parametrizations for the FE (fixed effects) model.
- Alternative non-equivalent parametrizations for the RE (random effects) model
- Diagnostics based on Level 1 residuals
- Diagnostics based on Level 2 residuals (REs)
- Influence diagnostics
- Plotting the fitted model: hand-made effect plots
- Linking the picture and the numbers
- Formulating and testing linear hypotheses
- Graphs to show confidence bounds for hypotheses
- Second example: Minority status and Math Achievement
  - Preliminary diagnostics using Level 1 OLS model

- OLS influence diagnostics
- Scaling Level 1 variables
- Fitting a mixed model
- Dealing with non-convergence
- Building the RE model with a forward stepwise approach
- Simulation to adjust p-values
- Test for contextual effects II
- Simplifying the model
- Using regular expression for easy tests of complex hypotheses
- Some Level 2 diagnostics
- Near-singularity: a pancake in 3D
- Visualizing the model: hand-made effect plots II
- The minority-majority gap
- Comparing different RE models
- More diagnostics
- Marginal and conditional models
- Refining the FE model
- Multilevel R Squared

- Visualizing the model to construct hypotheses