

Tools for General Parametric Splines

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'spida' package in R developed for
the Summer Program In Data Analysis (SPIDA)
at York University

Contact: Georges Monette (georges@yorku.ca)

Flexible tools to explore general parametric splines:

- Splines are conceptually simple but generally awkward to use in statistical analyses
- Goal: make splines as easy to use in regression as categorical variables
- “General parametric spline:”
 - Polynomials of different degrees in each interval
 - Different degrees of smoothness at each knot – including possible discontinuity
- Natural parametrization with interpretable coefficients:
 - can use regular expression to test groups of coefficients
- Can estimate and test features of spline: slope, curvature, salti.
- Can interact with other variables (numerical or categorical), other splines, etc.
- Limitation: need to centre and scale variable

Installation

- 1) Install R <http://cran.r-project.org/>

Only once: Install **'car'** and **'spida'**. After starting R:

```
>1 install.packages('car')  
> install.packages('spida',  
  repos = 'http://r-forge.r-project.org')
```

- 2) Each time you run R: load spida
> library(spida)

Spida2

¹ The '>' sign is the prompt in R. Type only what comes after the '>'.

Main Functions:

gsp

Creates general parametric splines

wald

Uses regular expressions or hypothesis matrices to estimate coefficients and test hypotheses

Lmat

Generates hypothesis matrices

sc

Operates on spline functions to generate hypothesis matrices for spline derivatives: slope, curvature, salti at knots

smsp

Non-parametric splines using mixed models

Example:

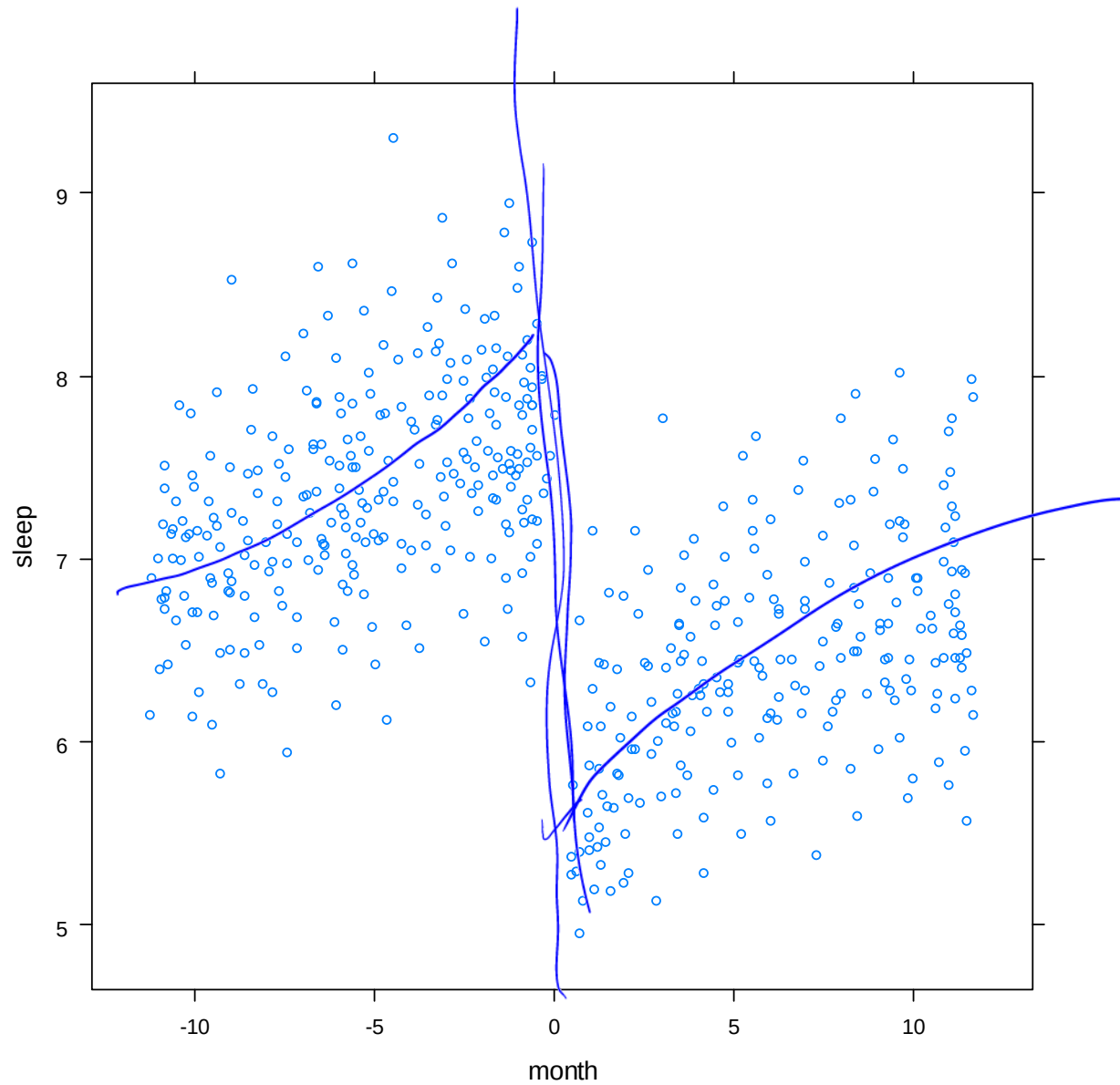
Simulated sleep data before and after birth.

Inspired by Kunkel, Reitav, Monette (2010)

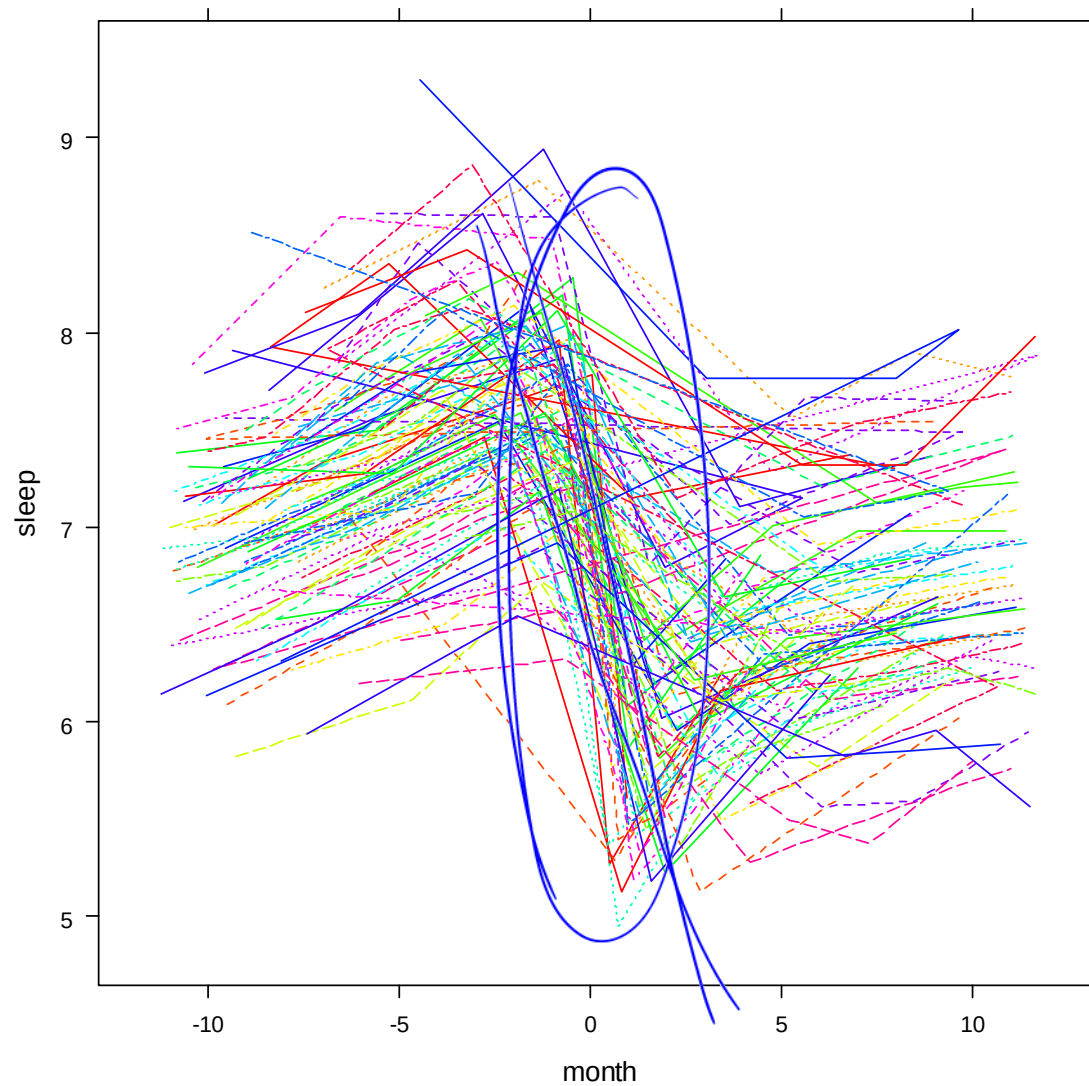
100 women each observed 5 times:

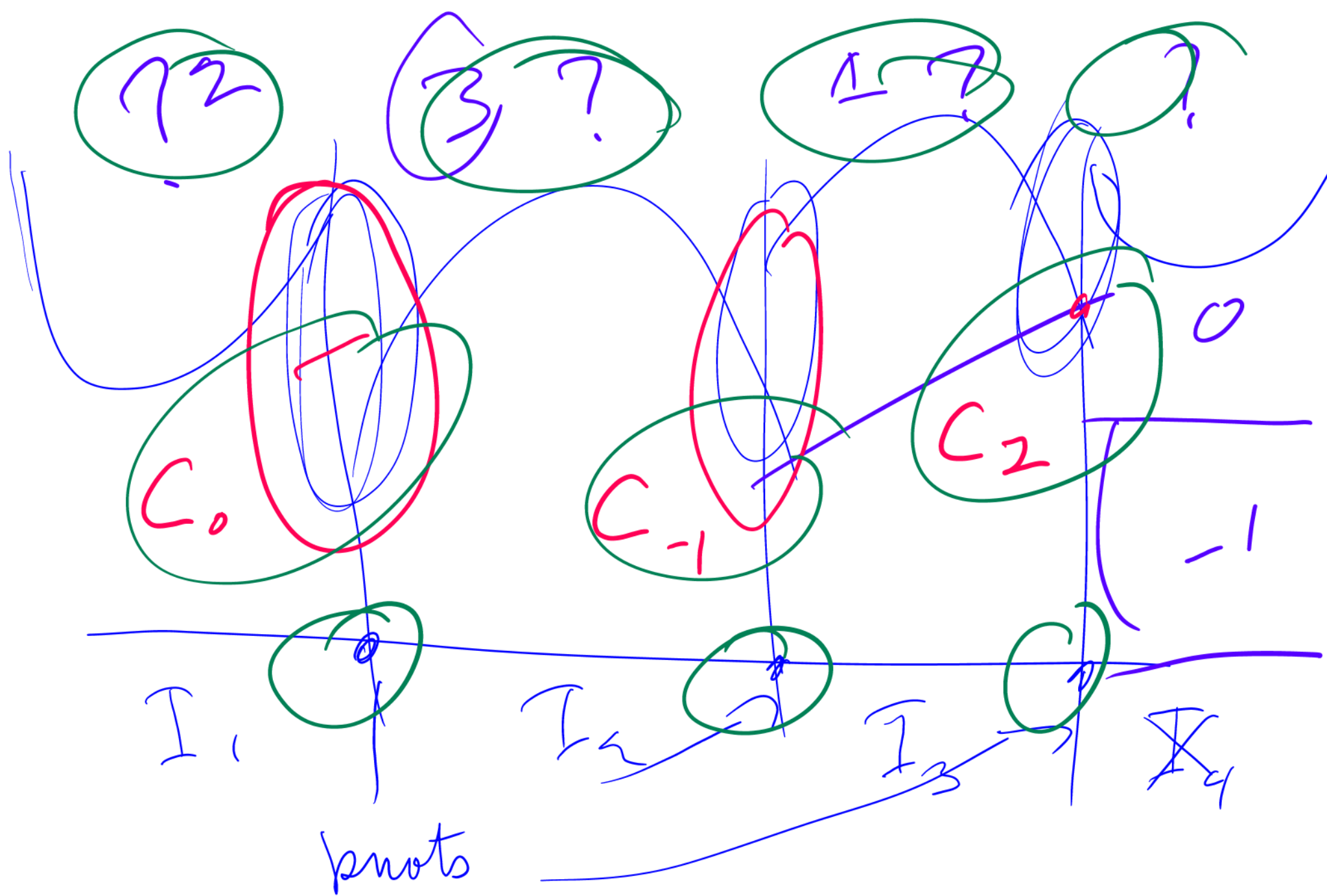
```
> sim.data <- read.csv(  
  'http://www.math.yorku.ca/~georges/Data/sim.data.csv')  
> dim(sim.data)  
[1] 500    4  
> head( sim.data)  
      X      month id      sleep  
1 1 -9.86370968   1 7.008448  
2 2 -6.95388563   1 7.336828  
3 3 -2.11407625   1 7.642227  
4 4  0.05344575   1 7.778133  
5 5  0.97390029   1 5.609873  
6 6 -7.66649560   2 7.184266
```

```
> xyplot( sleep ~ month, sim.data)
```



```
> td( col = rainbow(20))  
> xyplot( sleep ~ month, sim.data,  
          groups = id, type = 'l')
```

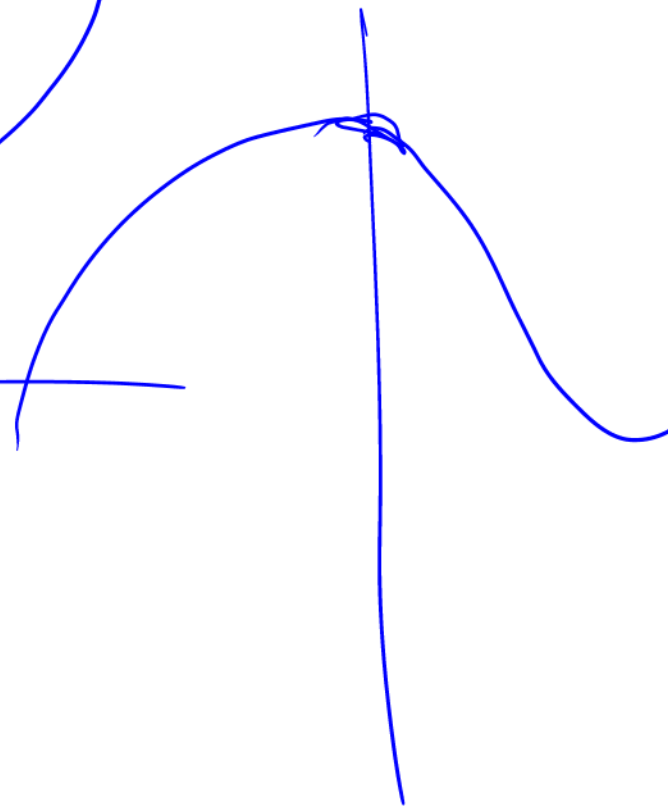
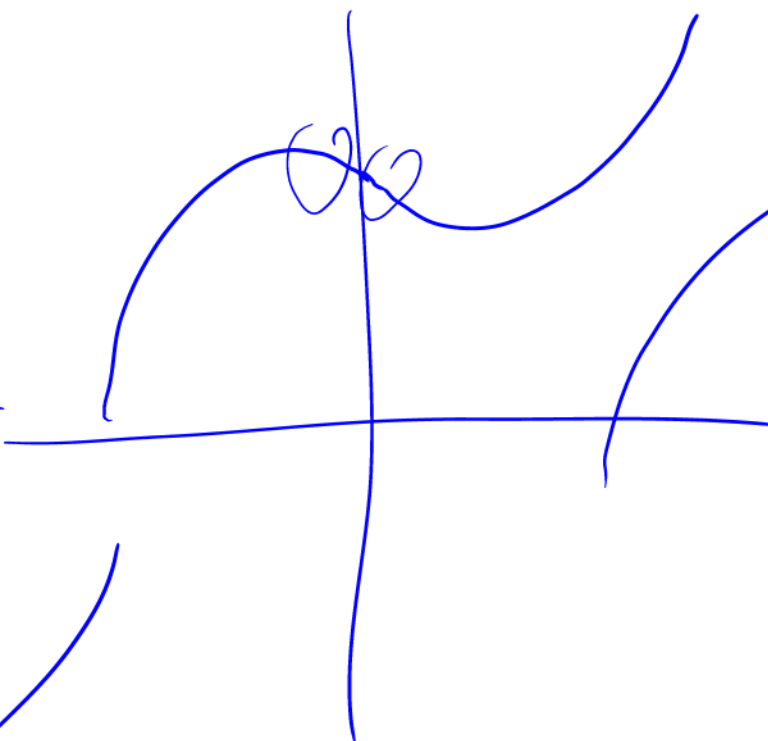
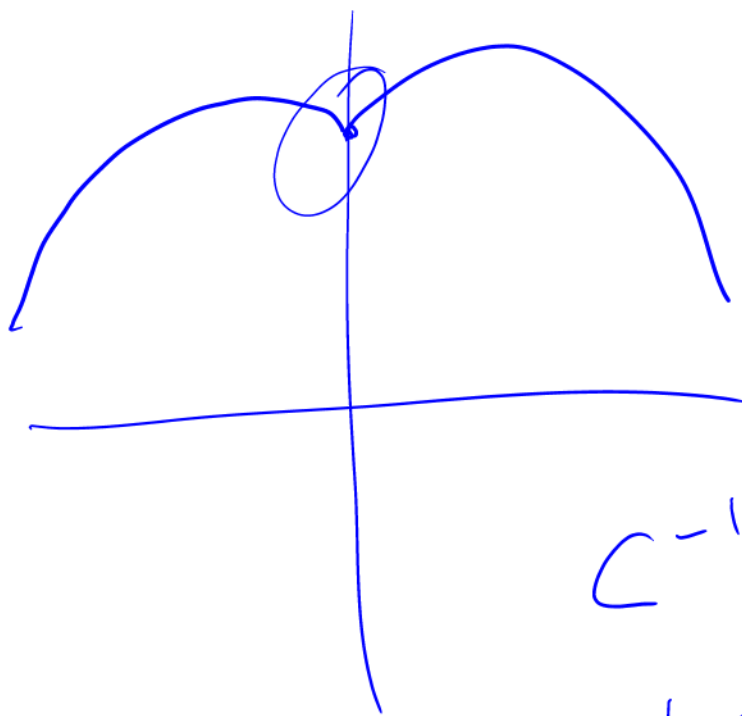




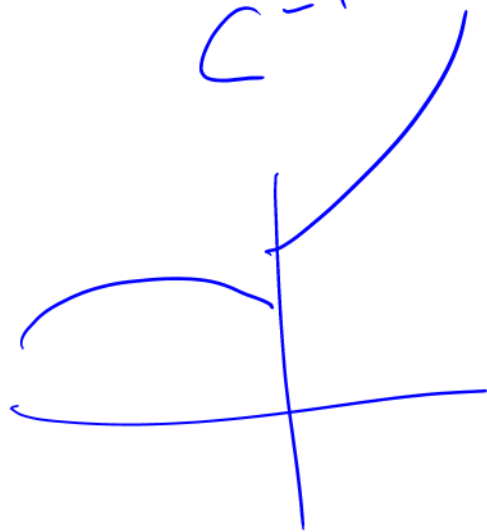
C^0

C^1

C^2



C^{-1}



Defining a spline function:

Create your own spline function

```
> sp <- function(x) gsp(x,
  knots = c(-5, 0, 5), # position of knots
  degree = c(2, 2, 2, 2), # quad in each interval
  smooth = c(1, -1, 1)) # smoothness at knot
```

degree parameter:

1,2,3,...	linear, quadratic, cubic, etc.
0	flat
-1	equal to 0

smooth parameter:

1	'smooth': same slope on both sides of knot	
2	same curvature on both sides	
0	continuous (no gap)	
-1	discontinuous	

What sp does:

Generates a portion of a model matrix:

```
> sp( c(-5, 0, 5) )
```

	D1(0)	D2(0)	C(0).0	C(0).1	C(0).2	C(5).2	C(-5).2
f(-5)	-5	12.5	0	0	0.0	0	0
f(0)	0	0.0	0	0	0.0	0	0
f(5)	5	12.5	1	5	12.5	0	0

linear
term

saltus
in level
at 0

change
in quad.
term at 5

Using your **sp** in a linear multilevel mixed model:

```
> library( nlme )  
> fit <- lme( sleep ~ sp(month), sim.data,  
             random = ~ 1 | id )
```

```
> summary( fit )
```

Linear mixed-effects model fit by REML

Data: sim.data

	AIC	BIC	logLik
	384.8794	426.8642	-182.4397

Random effects:

Formula: ~1 | id

	(Intercept)	Residual
StdDev:	0.4922676	0.2439248

. . . .

Fixed effects. sleep ~ sp(month)

	Value	Std.Error	DF	t-value	p-value
(Intercept)	7.770252	0.07562772	393	102.74344	0.0000
sp(month)D1(0)	0.007316	0.04231428	393	0.17289	0.8628
sp(month)D2(0)	-0.027195	0.01204080	393	-2.25853	0.0245
sp(month)C(0).0	-2.284334	0.10143225	393	-22.52078	0.0000
sp(month)C(0).1	0.295341	0.06867966	393	4.30027	0.0000
sp(month)C(0).2	-0.015195	0.01850084	393	-0.82130	0.4120
sp(month)C(5).2	0.027771	0.02041699	393	1.36018	0.1746
sp(month)C(-5).2	-0.048239	0.01943028	393	-2.48267	0.0135

Standardized Within-Group Residuals:

Min	Q1	Med	Q3	Max
-6.955982485	-0.326223723	-0.009829296	0.320549996	7.518591530

Number of Observations: 500

Number of Groups: 100


```
> round(model.matrix(fit) [1:5, ], 4)
```

	(Intercept)	sp(month)D1(0)	sp(month)D2(0)	sp(month)C(0).0	
1	1	-9.8637	48.6464	0	
2	1	-6.9539	24.1783	0	
3	1	-2.1141	2.2347	0	
4	1	0.0534	0.0014	1	
5	1	0.9739	0.4742	1	
	sp(month)C(0).1	sp(month)C(0).2	sp(month)C(5).2	sp(month)C(-5).2	
1	0.0000	0.0000	0	-11.8278	
2	0.0000	0.0000	0	-1.9088	
3	0.0000	0.0000	0	0.0000	
4	0.0534	0.0014	0	0.0000	
5	0.9739	0.4742	0	0.0000	

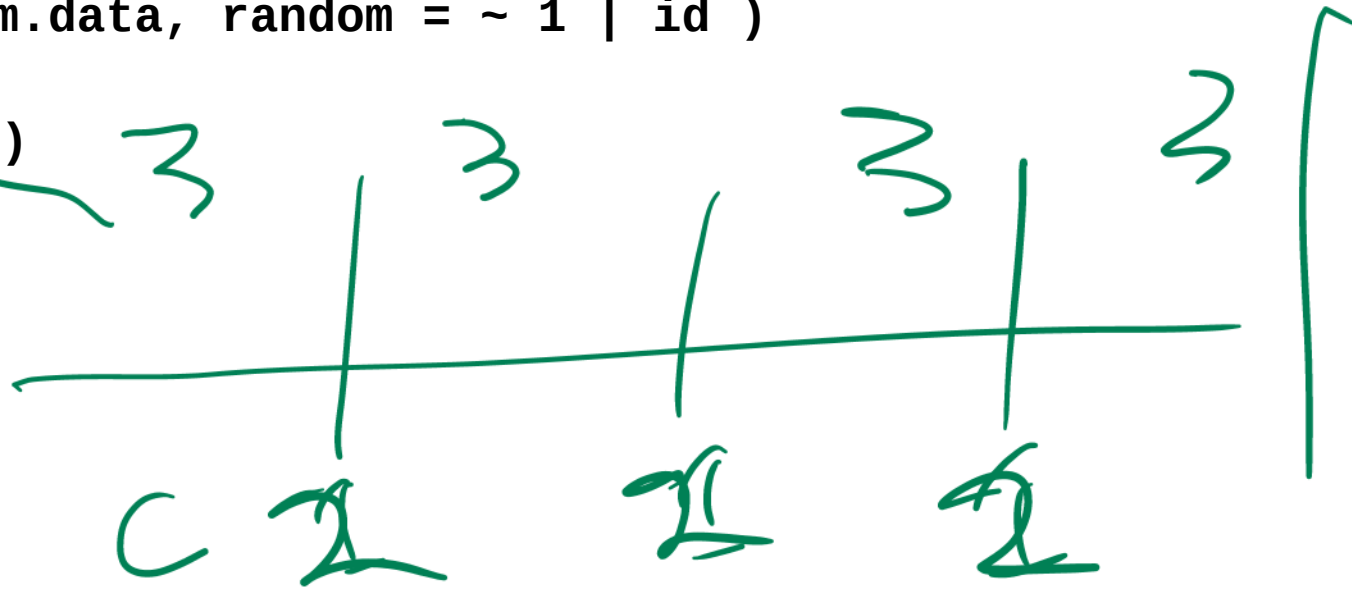
Natural spline

You can create almost every kind of spline used in regression

Linear in the extreme intervals

```
> spn <- function( x ) gsp( x,  
    knots    = c(-10, -5, 0, 5, 10),  
    degree   = c(1, 2, 2, 2, 2, 1),  
    smooth   = c( 1, 1, -1, 1, 1))  
  
> fitn <- lme( sleep ~ spn(month),  
    sim.data, random = ~ 1 | id )  
  
> summary(fitn)
```

...



Fixed effects: sleep ~ spn(month)

	Value	Std.Error	DF	t-value	p-value
(Intercept)	7.770115	0.07572201	393	102.61369	0.0000
spn(month)D1(0)	0.007163	0.04256467	393	0.16829	0.8664
spn(month)D2(0)	-0.027267	0.01214689	393	-2.24475	0.0253
spn(month)C(0).0	-2.278971	0.10190952	393	-22.36269	0.0000
spn(month)C(0).1	0.290233	0.06946431	393	4.17816	0.0000
spn(month)C(0).2	-0.013372	0.01882470	393	-0.71035	0.4779
spn(month)C(5).2	0.024200	0.02135186	393	1.13338	0.2577
spn(month)C(-5).2	-0.048604	0.01975742	393	-2.46003	0.0143

Note: Although the estimable coefficients have the same interpretation for both splines, the natural spline matrix is necessarily different. There are no estimable coefficients attached to the boundary knots.

Viewing the fitted model:

Create a data frame of values of functional predictors – here just months:

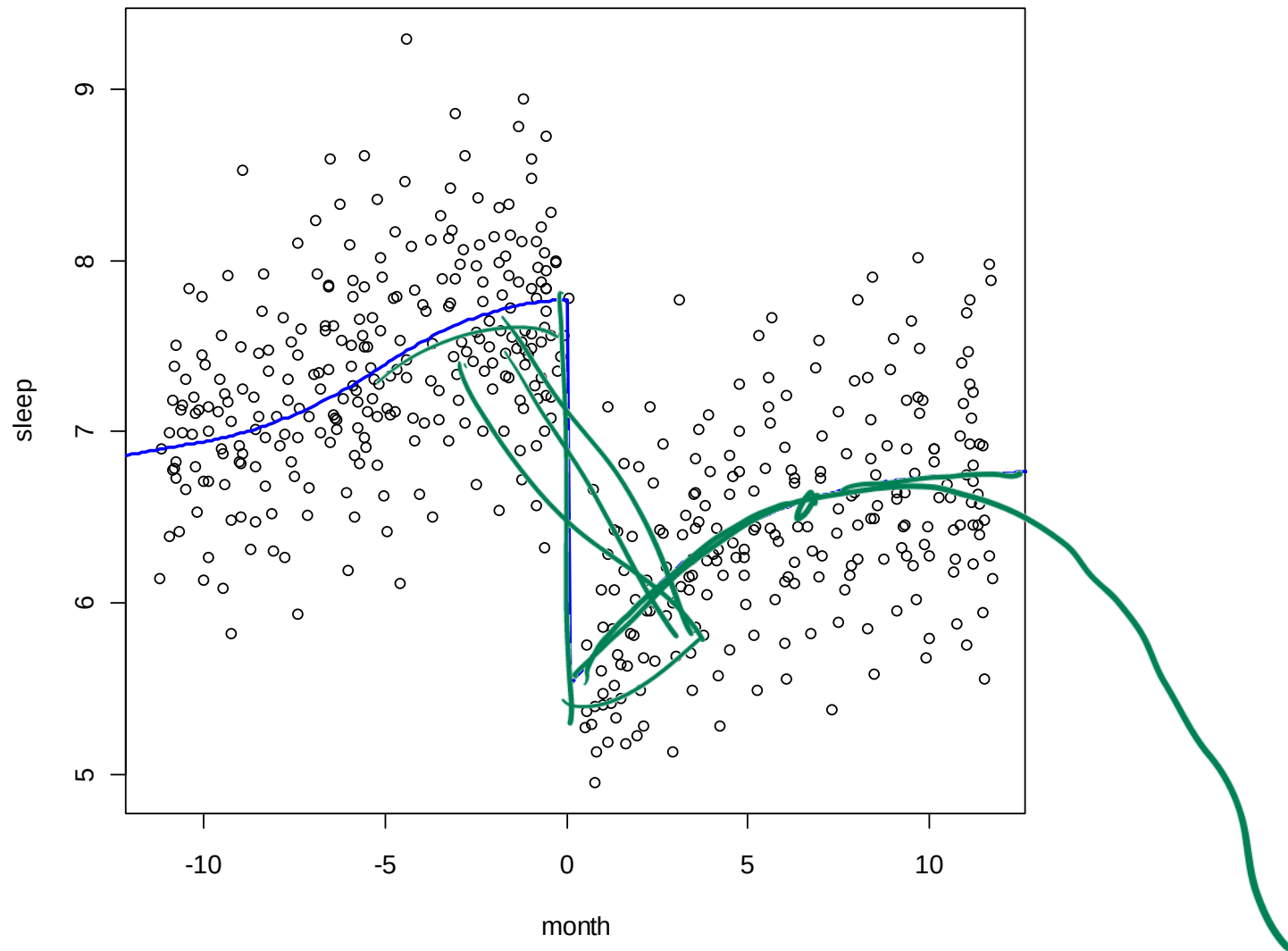
```
> pred <- expand.grid( month = seq(-13,13,.1))2  
> pred$sleep <- predict( fitn, pred, level = 0 )  
# Add predicted sleep to the 'pred' data frame  
> head( pred )
```

	month	sleep
1	-13.0	6.832263
2	-12.9	6.835944
3	-12.8	6.839625
4	-12.7	6.843306
5	-12.6	6.846987
6	-12.5	6.850669

*adding predicted
sleep in the the data
frame '**pred**'*

² I used expand.grid although it is really needed only for multiple functional predictors

```
> plot( sleep ~ month, sim.data)  
> lines( sleep ~ month, pred, col = 'blue', lwd = 2)
```



Estimation and testing:

Overall test for effect of time:

```
> wald( fitn, 'month')
```

	numDF	denDF	F-value	p-value
month	7	393	366.2914	<.00001

'month' is a "regular expression" matching names of coefficients

Coefficients	Estimate	Std. Error	DF	t-value	p-value
spn(month)D1(0)	0.007163	0.042565	393	0.168293	0.86644
spn(month)D2(0)	-0.027267	0.012147	393	-2.244749	0.02534
spn(month)C(0).0	-2.278971	0.101910	393	-22.362687	<.00001
spn(month)C(0).1	0.290233	0.069464	393	4.178156	0.00004
spn(month)C(0).2	-0.013372	0.018825	393	-0.710346	0.47791
spn(month)C(5).2	0.024200	0.021352	393	1.133384	0.25774
spn(month)C(-5).2	-0.048604	0.019757	393	-2.460034	0.01432

Can we get rid of knots and just use a quadratic?

```
> wald( fitn, 'C')
```

numDF	denDF	F.value	p-value
-------	-------	---------	---------

C	5	393	402.4999	<.00001
---	---	-----	----------	---------

No

Coefficients	Estimate	Std.Error	DF	t-value	p-value
spn(month)C(0).0	-2.278971	0.101910	393	-22.362687	<.00001
spn(month)C(0).1	0.290233	0.069464	393	4.178156	0.00004
spn(month)C(0).2	-0.013372	0.018825	393	-0.710346	0.47791
spn(month)C(5).2	0.024200	0.021352	393	1.133384	0.25774
spn(month)C(-5).2	-0.048604	0.019757	393	-2.460034	0.01432

Can we get rid of non-zero knots?

```
> wald( fitn, 'C\\(5|C\\(-5')
```

```
# '(' is a special character and needs to be 'escaped' to
```

```
# have its literal meaning
```

```
# '|' means 'or'
```

	numDF	denDF	F.value	p.value
--	-------	-------	---------	---------

C\\(5 C\\(-5	2	393	3.646731	0.02696
--------------	---	-----	----------	---------

Coefficients	Estimate	Std.Error	DF	t-value	p-value
--------------	----------	-----------	----	---------	---------

spn(month)C(5).2	0.024200	0.021352	393	1.133384	0.25774
------------------	----------	----------	-----	----------	---------

spn(month)C(-5).2	-0.048604	0.019757	393	-2.460034	0.01432
-------------------	-----------	----------	-----	-----------	---------

Using a hypothesis matrix:

```
> Lm <- rbind("at 5 mos" = c(0,0,0,0,0,0,1,0),
              "at -5 mos" = c(0,0,0,0,0,0,0,1))
> Lm
```

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]
at 5 mos	0	0	0	0	0	0	1	0
at -5 mos	0	0	0	0	0	0	0	1

```
> wald( fitn, Lm)
      numDF denDF  F.value p.value
1         2   393 3.646731 0.02696
```

	Estimate	Std.Error	DF	t-value	p-value
at 5 mos	0.024200	0.021352	393	1.133384	0.25774
at -5 mos	-0.048604	0.019757	393	-2.460034	0.01432

Generating Standard Errors

Wald tests generate estimated standard errors for coefficients, we need to capture them for plotting

Example: plotting the estimated response

L matrix as model matrix evaluated over a range of predictors

```
> Lest <- with( pred, cbind( 1, spn(month)))  
> dim(Lest)  
[1] 261    8  
> head(Lest)
```

		D1(0)	D2(0)	C(0).0	C(0).1	C(0).2	C(5).2	C(-5).2
f(-13)	1	-13.0	80	0	0	0	0	-27.5
f(-12.9)	1	-12.9	79	0	0	0	0	-27.0
f(-12.8)	1	-12.8	78	0	0	0	0	-26.5
f(-12.7)	1	-12.7	77	0	0	0	0	-26.0
f(-12.6)	1	-12.6	76	0	0	0	0	-25.5
f(-12.5)	1	-12.5	75	0	0	0	0	-25.0


```

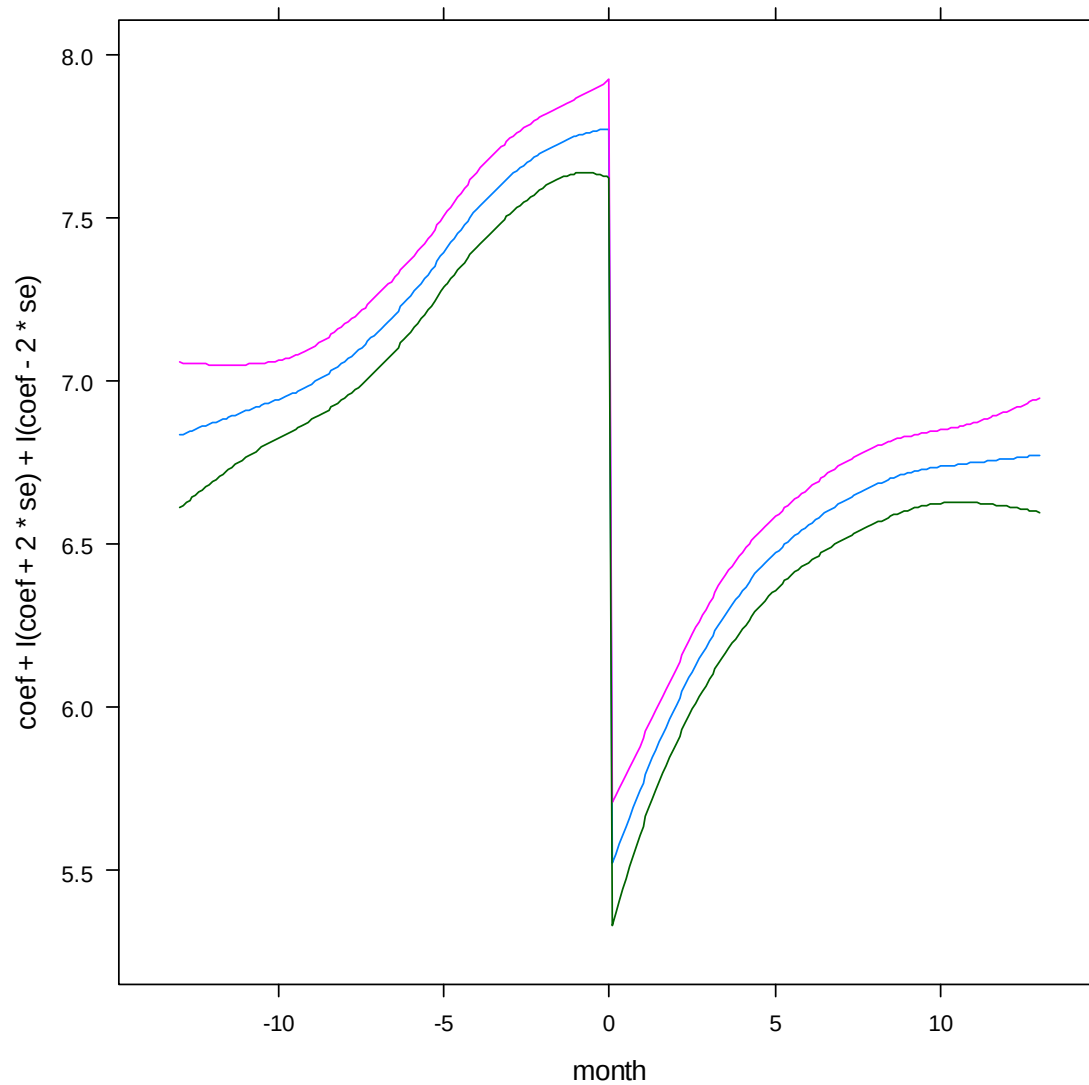
> head( pred )
      month      sleep
1 -13.0 6.832263
2 -12.9 6.835944
3 -12.8 6.839625
4 -12.7 6.843306
5 -12.6 6.846987
6 -12.5 6.850669

> ww <- as.data.frame(wald(fitn, Lest))
      # capture estimates and SEs
> ww <- cbind( ww, pred)
      # combine back with pred
> head( ww )
      coef      se month      sleep
f(-13) 6.832263 0.1111460 -13.0 6.832263
f(-12.9) 6.835944 0.1089242 -12.9 6.835944
f(-12.8) 6.839625 0.1067182 -12.8 6.839625
f(-12.7) 6.843306 0.1045293 -12.7 6.843306
f(-12.6) 6.846987 0.1023583 -12.6 6.846987
f(-12.5) 6.850669 0.1002066 -12.5 6.850669

```

Easy plotting:

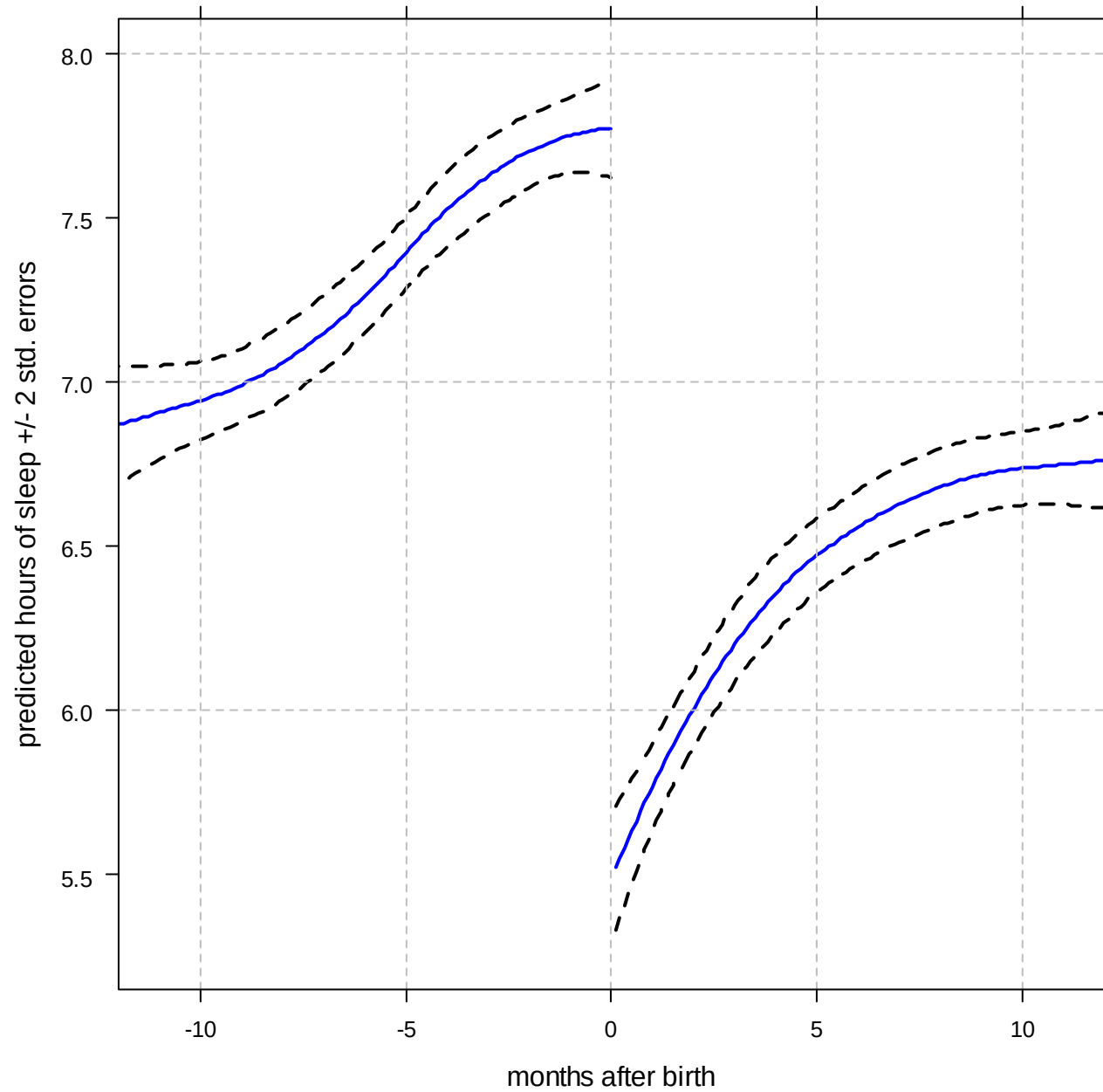
```
> xyplot(coef + I(coef + 2*se) + I(coef - 2*se) ~  
month, ww, type = 'l')
```



Fancier plotting:

```
> td( col = c('blue','black','black'),
      lty = c( 1,2,2), lwd = 2)

> xyplot( coef + I(coef + 2*se) + I(coef - 2*se) ~
          month,
          rbind(ww[1:131,],NA,ww[132:261,]),
          type = 'l',
          panel = function(x,y,...) {
            panel.xyplot( x, y, ...)
            panel.abline( v = c(-10,-5,0,5,10),
                          col = 'gray', lty = 2)
            panel.abline( h = seq(0,10,1),
                          col = 'gray', lty = 2)
          },
          xlim = c(-12,12),
          ylab =
            'predicted hours of sleep +/- 2 std. errors',
          xlab = "months after birth")
```



Inference concerning a spline

All fixed effects coefficients:

```
> wald( fitn )
```

```
numDF denDF F.value p.value  
8 393 2688.578 <.00001
```

Coefficients	Estimate	Std.Error	DF	t-value	p-value
(Intercept)	7.770115	0.075722	393	102.613693	<.00001
spn(month)D1(0)	0.007163	0.042565	393	0.168293	0.86644
spn(month)D2(0)	-0.027257	0.012147	393	-2.244749	0.02534
spn(month)C(0).0	-2.278971	0.101910	393	-22.362687	<.00001
spn(month)C(0).1	0.290233	0.069464	393	4.178156	0.00004
spn(month)C(0).2	-0.013372	0.018825	393	-0.710346	0.47791
spn(month)C(5).2	0.024200	0.021352	393	1.133384	0.25774
spn(month)C(-5).2	-0.048604	0.019757	393	-2.460034	0.01432

Test saltus at birth (month = 0):

```
> wald( fitn , "C\\(0.*0$")
```

```
numDF denDF F.value p.value  
C\\(0.*0$ 1 393 500.0898 <.00001
```

Coefficients	Estimate	Std.Error	DF	t-value	p-value
spn(month)C(0).0	-2.278971	0.10191	393	-22.36269	<.00001

Better labelled:

```
> wald( fitn , list("discontinuity at 0" = "C\\(0.*0$"))
              numDF denDF  F.value p.value
discontinuity at 0      1    393 500.0898 <.00001
```

Operating on splines: **sc** function

sc(spn, x, D, type)

generates a hypothesis matrix to estimate the D^{th} power coefficient of the spline **spn** at the points given by **x**. '**type**' only has an effect when the value of **x** corresponds to a knot. In that case, **type** determines whether the evaluation is to the left, to the right or across the knot.

sc parameters:		
D	0	value of spline
	1	first derivative
	2,...	quadratic,... terms
type	0	limit from the left of knot
	1	limit from the right of knot
	2	saltus: limit from the right minus limit from the left

Example:

Estimate value, slope and quadratic component at -1 months:

```
> sc(spn, c(-1, -1, 1), D = c(1, 2, 3) )
```

	D1(0)	D2(0)	C(0).0	C(0).1	C(0).2	C(5).2	C(-5).2
D1(-1)	1	-1	0	0	0	0	0
D2(-1)	0	1	0	0	0	0	0
D3(-1)	0	0	0	0	0	0	0

```
> wald( fitn, cbind(c(1,0,0,0),  
+ sc(spn, c(-1, -1, -1, -1), D = c(0,1,2,3) )))
```

	numDF	denDF	F.value	p.value
	1	3	393	6937.729 <.00001

	Estimate	Std.Error	DF	t-value	p-value
g(-1)	7.749318	0.057332	393	135.165234	<.00001
D1(-1)	0.034430	0.030741	393	1.120008	0.26339
D2(-1)	-0.027267	0.012147	393	-2.244749	0.02534
D3(-1)	0.000000	0.000000	Inf	NaN	NaN

Using the `sc` function to test a saltus (type = 2):

```
> Lm <- cbind( 0, sc( spn, 0, D = 0, type = 2))
> Lm
```

		D1(0)	D2(0)	C(0).0	C(0).1	C(0).2	C(5).2	C(-5).2
g(0+)-g(0-)	0	0	0	1	0	0	0	0

```
> wald( fitn, Lm )
      numDF denDF  F.value p.value
1         1    393 500.0898 <.00001
```

	Estimate	Std.Error	DF	t-value	p-value
g(0+)-g(0-)	-2.278971	0.10191	393	-22.36269	<.00001

```
> head( model.matrix( fitn))
(Intercept) spn(month)D1(0) spn(month)D2(0) spn(month)C(0).0
1           1      -9.86370968      48.646384300             0
2           1      -6.95388563      24.178262681             0
3           1      -2.11407625       2.234659188             0
4           1       0.05344575       0.001428224             1
5           1       0.97390029       0.474240891             1
6           1      -7.66649560      29.387577401             0
```


	spn(month)C(0).1	spn(month)C(0).2	spn(month)C(5).2	spn(month)C(-5).2
1	0.00000000	0.00000000	0.000000e+00	-11.827836
2	0.00000000	0.00000000	0.000000e+00	-1.908835
3	0.00000000	0.00000000	0.000000e+00	0.000000
4	0.05344575	0.001428224	-1.441111e-18	0.000000
5	0.97390029	0.474240891	1.717766e-16	0.000000
6	0.00000000	0.00000000	0.000000e+00	-3.555099

```
> L.level <- cbind( 1, spn( c(-10, 10)))
```

```
> L.level
```

		D1(0)	D2(0)	C(0).0	C(0).1	C(0).2	C(5).2	C(-5).2
f(-10)	1	-10	50	0	0	0	0.0	-12.5
f(10)	1	10	50	1	10	50	12.5	0.0

```
> wald( fitn, L.level )
```

	numDF	denDF	F.value	p.value
1	2	393	8216.843	<.00001

	Estimate	Std.Error	DF	t-value	p-value	Lower 0.95	Upper 0.95
f(-10)	6.942695	0.058820	393	118.0330	<.00001	6.827054	7.058337
f(10)	6.735666	0.055838	393	120.6283	<.00001	6.625887	6.845445

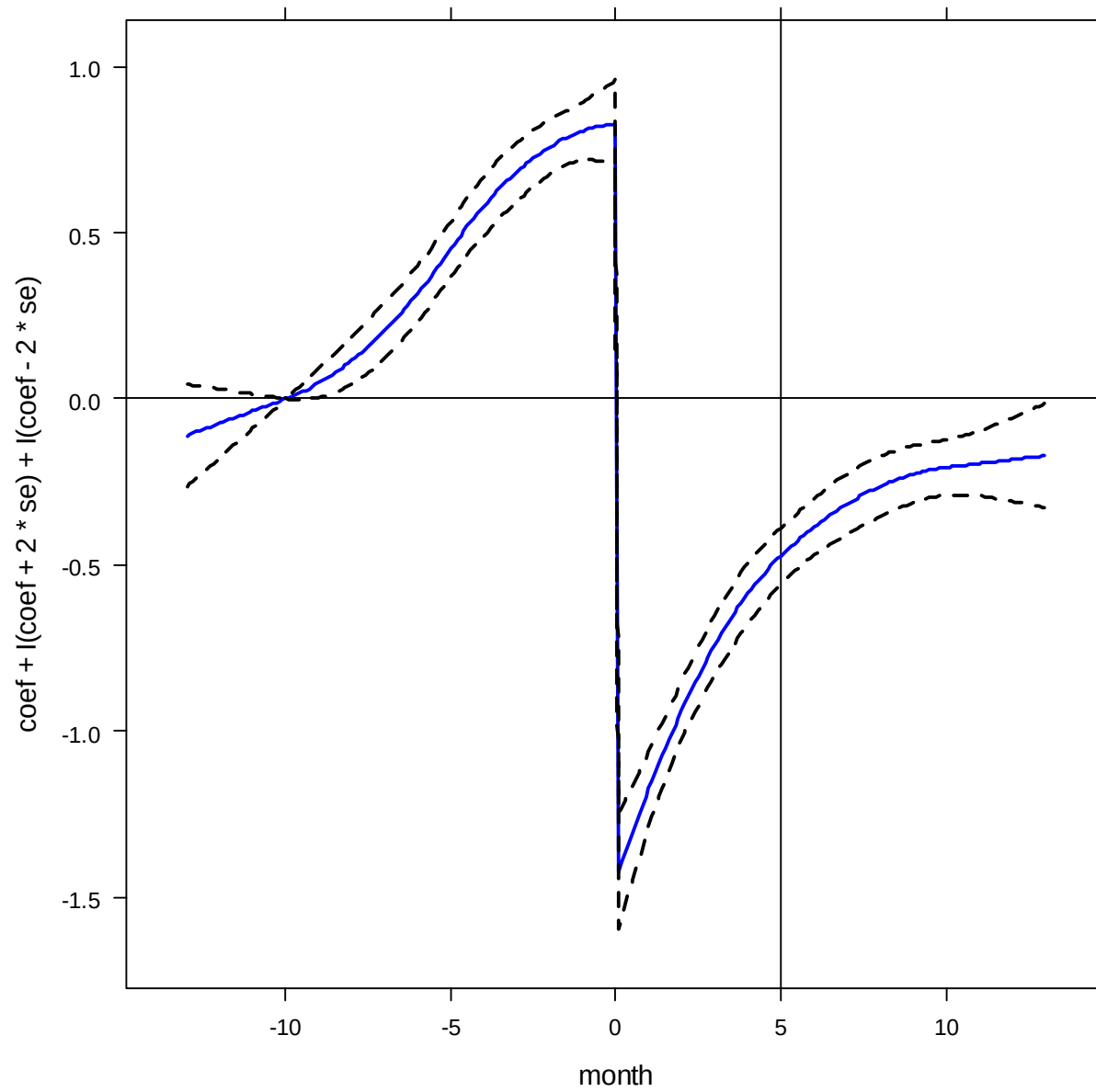
```
> wald( fitn, L.level[2,] - L.level[1,] )
      numDF denDF F.value p.value
1         1   393 24.4368 <.00001
```

```
      Estimate Std.Error   DF   t-value p-value Lower 0.95 Upper 0.95
Larg -0.20703    0.04188 393  -4.94359 <.00001  -0.289367  -0.124692
```

```
> ?sc
> Lslope <- with( pred, cbind( 0 , sc( spn, month, D = 1)))
> ws <- as.data.frame(wald(fitn,Lslope))
> head(ws)
```

```
      coef      se
D1( -13) 0.03681075 0.02576518
D1( -12.9) 0.03681075 0.02576518
D1( -12.8) 0.03681075 0.02576518
D1( -12.7) 0.03681075 0.02576518
D1( -12.6) 0.03681075 0.02576518
D1( -12.5) 0.03681075 0.02576518
> ws <- cbind(ws, pred)
```

```
> xyplot( coef + I(coef + 2*se)+I(coef - 2*se) ~ month,  
          ws, type = 'l')
```

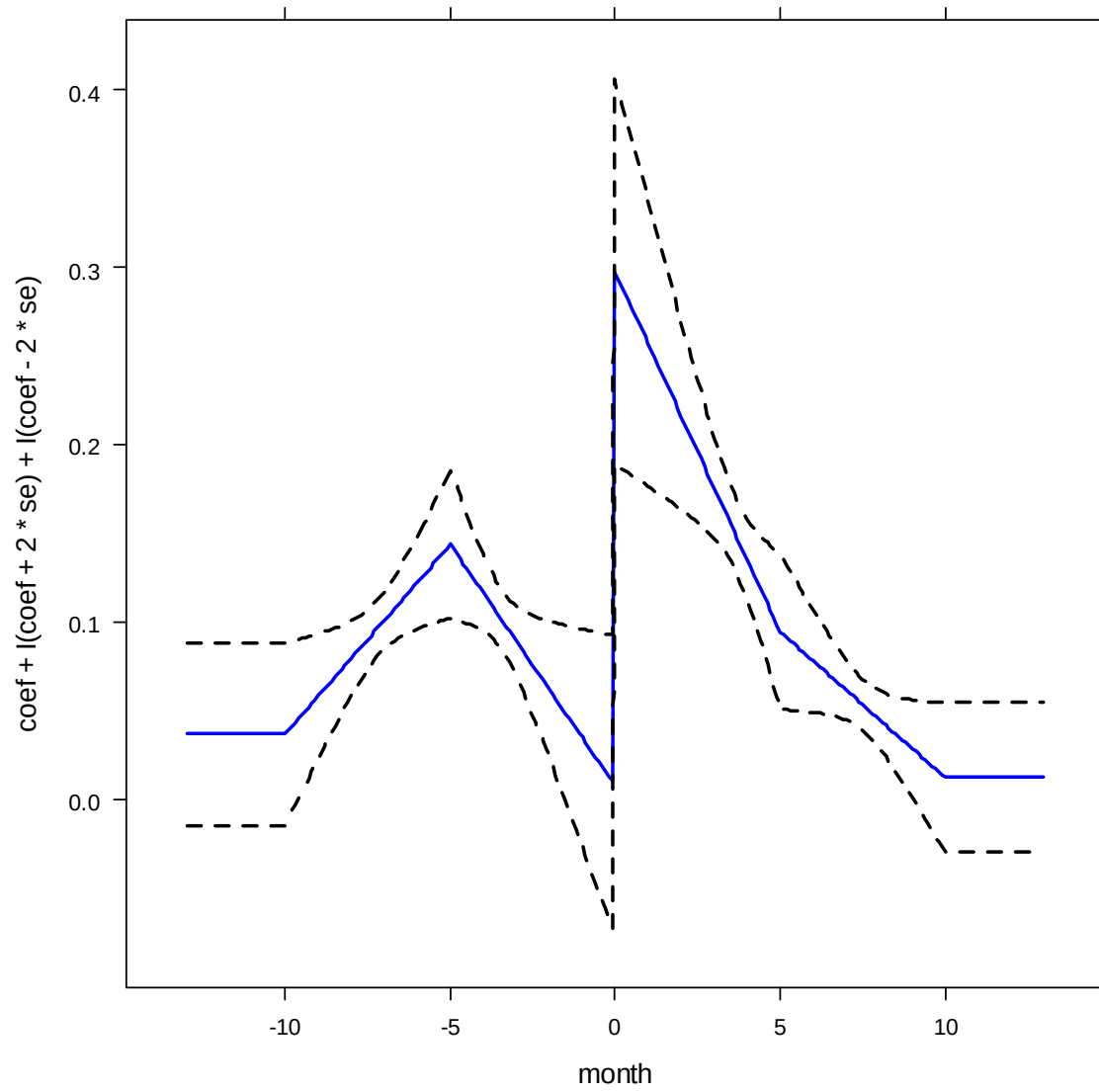


```

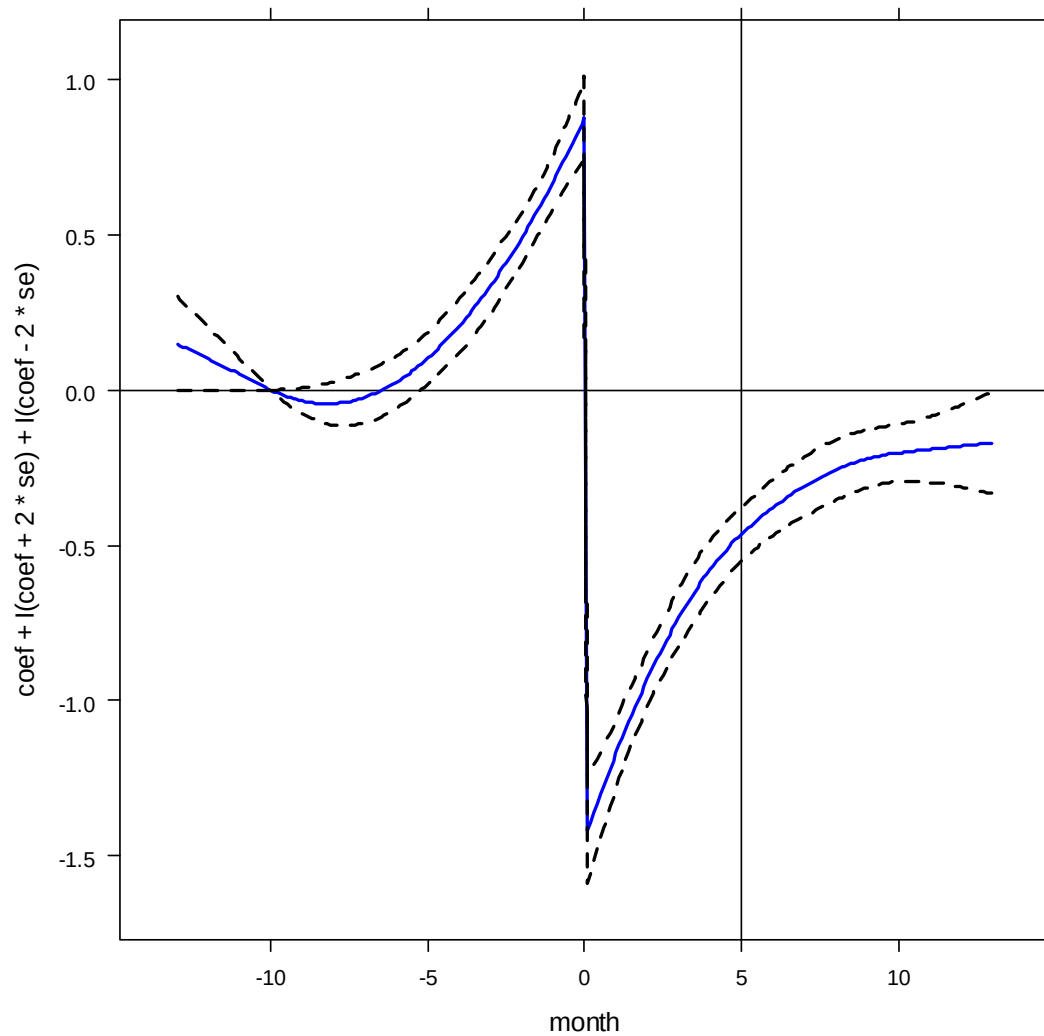
> head( Lest )
      D1(0) D2(0) C(0).0 C(0).1 C(0).2 C(5).2 C(-5).2
f(-13)  1 -13.0   80      0      0      0      0   -27.5
f(-12.9) 1 -12.9   79      0      0      0      0   -27.0
f(-12.8) 1 -12.8   78      0      0      0      0   -26.5
f(-12.7) 1 -12.7   77      0      0      0      0   -26.0
f(-12.6) 1 -12.6   76      0      0      0      0   -25.5
f(-12.5) 1 -12.5   75      0      0      0      0   -25.0
> L.m10 <- cbind( 1, spn(-10))
> L.m10
      D1(0) D2(0) C(0).0 C(0).1 C(0).2 C(5).2 C(-5).2
f(-10)  1  -10   50      0      0      0      0   -12.5
> Ldiff.m10 <- Lest - L.m10[rep(1,nrow(Lpred)),]
> wd <- as.data.frame( wald( fitn, Ldiff.m10))
> wd <- cbind(wd, pred)
> wdr <- as.data.frame( wald( fitnr, Ldiff.m10))
> wdr <- cbind(wdr, pred)

```

```
> xyplot(  coef + I(coef + 2*se)+I(coef - 2*se)
~ month, wd, type = 'l',
abline = list(h=0,v=5))
```



```
> xyplot( coef + I(coef + 2*se)+I(coef - 2*se) ~  
          month, wdr, type = 'l',  
          abline = list(h=0, v=5))
```



Caution:

Since gsp uses raw parametrization (in contrasts with b-splines, for example), you should center and rescale the predictor. A range within -10 to 10 for a spline with no power higher than cubic should be adequate.

