# Where confidence ellipsoids come from

## Confidence Interval for $\mu$; $\sigma^2$ known

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\Pr\left(\left|\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right| \leq Z^{.975}\right) = .95$$

95% CI for $\mu$: $\bar{X} \pm Z^{.975}\frac{\sigma}{\sqrt{n}}$

Equivalently:

$$\Pr\left(\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \leq \chi_1^{2;.95}\right) = .95$$

degrees of freedom

So $\left\{ \mu : \left( \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \right)^2 \leq \chi_1^{2;.95} \right\}$

is a 95% confidence 1-dimensional ellipsoid

i.e an interval

## In higher dimensions:

Confidence region for $\underset{\sim}{\eta} = L \underset{\sim}{\beta}$

e.g if $\underset{\sim}{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$ and $L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We get $L\underset{\sim}{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$

$$\hat{\underset{\sim}{\beta}} \sim N\left(\underset{\sim}{\beta}, \sigma^2 (X'X)^{-1}\right)$$

$$\hat{\eta} = L\hat{\beta} \sim N\Big(\underbrace{L\underset{\sim}{\beta}}_{\eta}, \underbrace{\sigma^2 L (X^T X)^{-1} L^T}_{\text{Call this } \Sigma}\Big)$$

Then $(\hat{\eta}-\eta)^T \Sigma^{-1} (\hat{\eta}-\eta) \sim \chi^2_2$ degrees of freedom

So $\Pr\left((\hat{\eta}-\eta)^T \Sigma^{-1} (\hat{\eta}-\eta) \le \chi^{2;.95}_2\right) = .95$

note: $\hat{\eta}$ is a random variable,
$\eta$ is hypothetically fixed

WARNING

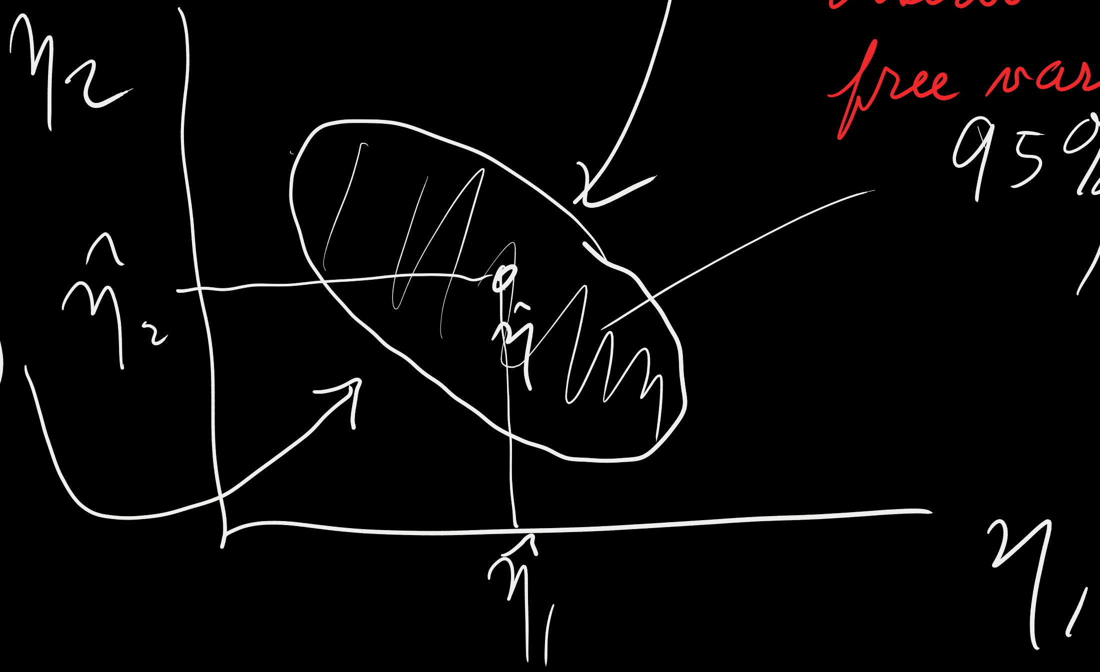
Big shift in perspective

So Confidence region:

$$\left\{ \underset{\sim}{\eta} : (\underset{\sim}{\eta}-\hat{\underset{\sim}{\eta}})^T \Sigma^{-1} (\underset{\sim}{\eta}-\hat{\underset{\sim}{\eta}}) \le \chi^2_{2;.95} \right\}$$

note: $\hat{\underset{\sim}{\eta}}$ is the value we observed and $\underset{\sim}{\eta}$ is a free variable.

$$\overbrace{\mathcal{E}(\underbrace{\hat{\underset{\sim}{\eta}}}_{\text{centre}},\ \underbrace{\chi^2_{2;.95}\times\Sigma}_{\text{shape}})}^{\text{notation for ellipsoids}}$$




95% Confidence region

When $\sigma^2$ is unknown and estimated with $s^2 = MSE$ independently of $\hat{\beta}$

we just replace $\chi_2^{2;.95}$ with $2F_{2,\nu}^{.95}$

where $\nu$ is the degrees of freedom for MSE.

In higher dimensions we replace 2 with $k$ where $\eta$ is a $k$-vector.

i.e $\chi_k^{2;.95}$ and $kF_{k,\nu}^{.95}$