Elliptical Proofs:

A few theorems

① **Spectral decomposition:**

Let Σ be a $k \times k$ positive-definite matrix
(Σ is symmetric and $\underset{\sim}{x}^T \Sigma \underset{\sim}{x} > 0 \;\; \forall \; \underset{\sim}{x} \in \mathbb{R}^k, \underset{\sim}{x} \neq \underset{\sim}{0}$)
then $\exists$ orthogonal $k \times k$ matrix Γ and a diagonal $k \times k$ matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$
$\ni \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, and

$$\Sigma = \Gamma \Lambda \Gamma'$$

(2) Cauchy-Schwartz

Let $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^k$, then

$$(\underset{\sim}{a}'\underset{\sim}{b})^2 \leq \underset{\sim}{a}'\underset{\sim}{a} \times \underset{\sim}{b}'\underset{\sim}{b}$$

with $=$ iff $\exists\, k_1, k_2 \in \mathbb{R}$, not both 0,

$\ni \quad k_1 \underset{\sim}{a} + k_2 \underset{\sim}{b} = 0$

(3) Extended Cauchy-Schwartz.

Let Σ be positive definite, then

$$(\underset{\sim}{a}'\underset{\sim}{b})^2 \leq \underset{\sim}{a}'\Sigma\underset{\sim}{a} \times \underset{\sim}{b}'\Sigma^{-1}\underset{\sim}{b}$$

with $=$ iff $\exists\, k_1, k_2$ not both 0

$\ni \; k_1 \Sigma \underset{\sim}{a} + k_2 \underset{\sim}{b} = \underset{\sim}{0}$

Proof: $(\underset{\sim}{a}'\underset{\sim}{b})^2 = (\underset{\sim}{a}'\Gamma\Lambda^{1/2}\Lambda^{-1/2}\Gamma'\underset{\sim}{b})^2$ with $\Sigma = \Gamma\Lambda\Gamma'$

$$\le (\underset{\sim}{a}'\Gamma\Lambda^{1/2}\Lambda^{1/2}\Gamma'\underset{\sim}{a})(\underset{\sim}{b}'\Gamma\Lambda^{-1/2}\Lambda^{-1/2}\Gamma'\underset{\sim}{b})$$

$$= \underset{\sim}{a}'\Sigma\underset{\sim}{a} \times \underset{\sim}{b}'\Sigma^{-1}\underset{\sim}{b}$$

with $=$ iff $\exists\, k_1, k_2$ not both 0

$$\ni k_1\Lambda^{1/2}\Gamma'\underset{\sim}{a} + k_2\Lambda^{-1/2}\Gamma'\underset{\sim}{b} = \underset{\sim}{0}$$

equivalently $k_1\Sigma\underset{\sim}{a} + k_2\underset{\sim}{b} = \underset{\sim}{0}$

by premultiplying by non-singular matrix $\Gamma\Lambda^{1/2}$.

Projections

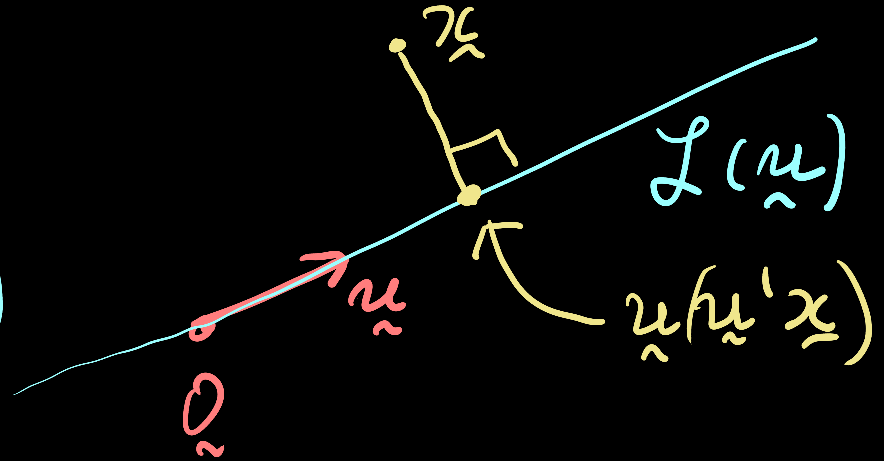
Orthogonal projections onto a line

Let $\underset{\sim}{x} \in \mathbb{R}^k$. Let $\underset{\sim}{u}$ be a unit vector in $\mathbb{R}^k$.

Then $\underset{\sim}{x} \mapsto \underset{\sim}{u}\,\underset{\sim}{u}'\underset{\sim}{x}$ projects $\underset{\sim}{x}$ orthogonally onto $\mathcal{L}(\underset{\sim}{u})$.

Proof:

Clearly $\underset{\sim}{u}(\underset{\sim}{u}'\underset{\sim}{x}) \in \mathcal{L}(\underset{\sim}{u})$



and $\left(\underset{\sim}{x} - \underset{\sim}{u}(\underset{\sim}{u}'\underset{\sim}{x})\right)^T \underset{\sim}{u} = \underset{\sim}{x}^T\underset{\sim}{u} - (\underset{\sim}{u}'\underset{\sim}{x})\underbrace{\underset{\sim}{u}'\underset{\sim}{u}}_{1} = 0$

Projections in general

Defn: $P_{k\times k}$ is a projection matrix iff $P^2 = P$

Lemma: If $P^2 = P$ and $P^T = P$ then P is the matrix of an orthogonal projection

i.e. $(\underset{\sim}{x} - P\underset{\sim}{x}) \perp P\underset{\sim}{x}$

Proof: Exercise

Note that the following are equivalent.

1) Σ is a non-singular variance matrix

i.e $\exists$ random vector $\underset{\sim}{X} \ni \Sigma = Var(\underset{\sim}{X})$
and Σ is non-singular

Exercise

2) Σ is positive definite

defn

3) $\underset{\sim}{x}' \Sigma \underset{\sim}{x} > 0 \quad \forall \underset{\sim}{x} \neq \underset{\sim}{0}$ and Σ is symmetric

4) $\exists \Gamma \ni \Gamma\Gamma' = I$, diag $\Lambda > 0$
$\ni \Sigma = \Gamma \Lambda \Gamma'$

The efficient to prove this is to find a cycle
e.g. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

5) $\exists$ non-singular matrix A

$\ni \ \Sigma = AA'$

Note: Σ is positive definite
iff Σ^{-1} is positive definite

Exercise

Shadow theorem:

Let Σ be a 2×2 positive definite matrix,

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}.$$

Let $\mathcal{E} = \{ \underset{\sim}{x} : \underset{\sim}{x}^T \Sigma^{-1} \underset{\sim}{x} = 1 \}$

Consider a unit vector $\underset{\sim}{u}$.

Fact Let $\underset{\sim}{x} \in \mathbb{R}^2$, then $\underset{\sim}{u}\,\underset{\sim}{u}'\underset{\sim}{x}$ is the orthogonal projection of $\underset{\sim}{x}$ onto $\mathcal{L}(\underset{\sim}{u})$ and $|\underset{\sim}{u}'\underset{\sim}{x}|$ is the length of the projection.

linear space spanned by $\underset{\sim}{u}$

The perpendicular shadow of $\mathcal{E}_c$ onto $\mathcal{L}(\underset{\sim}{u})$ is the line segment

$$(-\underset{\sim}{b}, \underset{\sim}{b})$$

where $\underset{\sim}{b} = (\underset{\sim}{u}^T \Sigma \underset{\sim}{u})^{1/2} \underset{\sim}{u}$

Proof: Use the extended C-S inequality

Let $\underset{\sim}{x} \in \mathcal{E}$. Then

$$(\underset{\sim}{u}^T \underset{\sim}{x})^2 \le (\underset{\sim}{u}^T \Sigma \underset{\sim}{u})(\underset{\sim}{x} \Sigma^{-1} \underset{\sim}{x})$$

$$\le \underset{\sim}{u}^T \Sigma \underset{\sim}{u} \quad \text{if } \underset{\sim}{x} \in \mathcal{E}$$

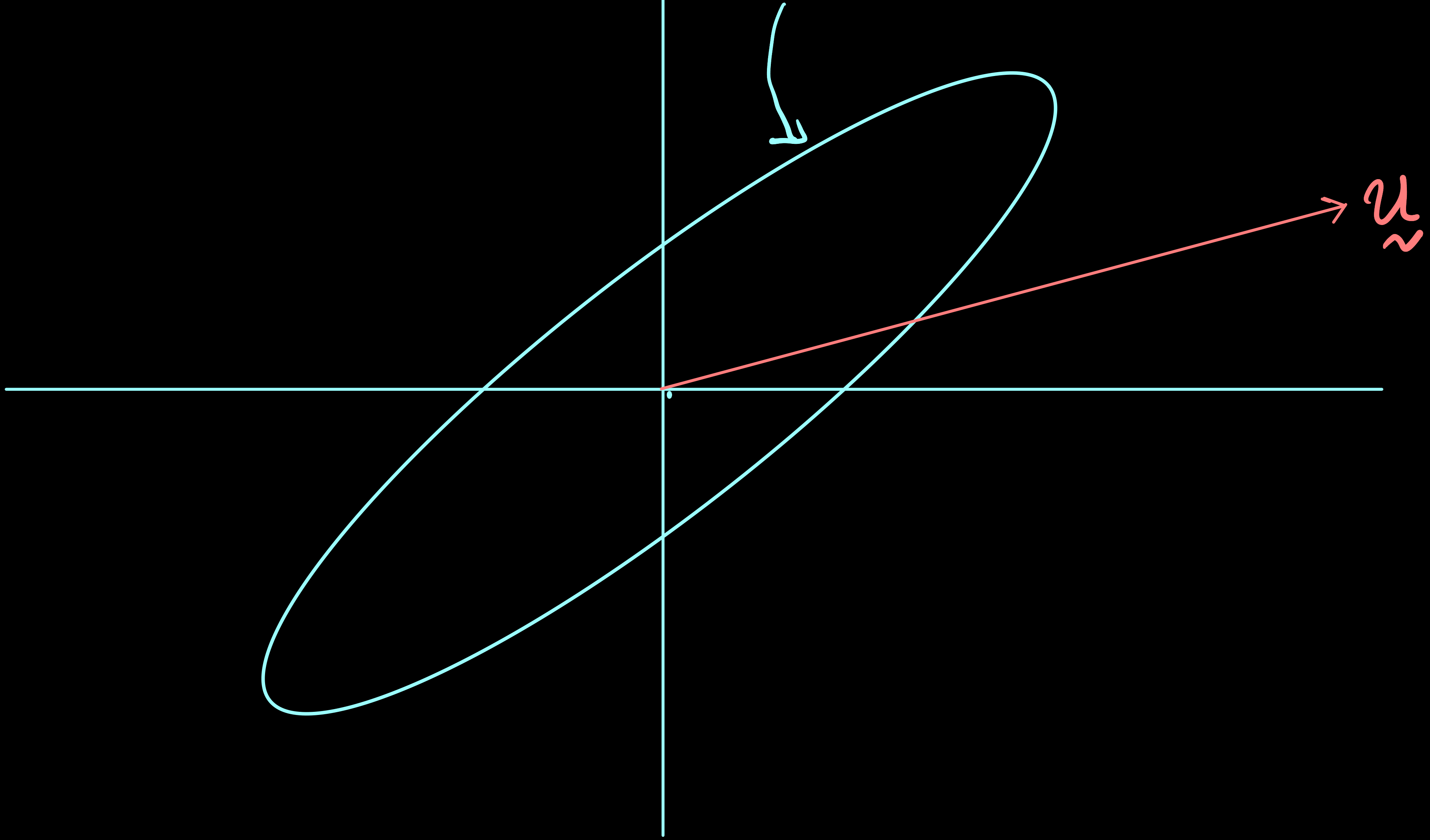
Let $\underset{\sim}{x}_0 = \dfrac{1}{\sqrt{\underset{\sim}{u}^T \Sigma \underset{\sim}{u}}} \Sigma \underset{\sim}{u}$ (pulled out of the blue because it works)

Then $\underset{\sim}{x}_0' \Sigma^{-1} \underset{\sim}{x}_0 = \dfrac{1}{\underset{\sim}{u}^T \Sigma \underset{\sim}{u}} u^T \Sigma \Sigma^{-1} \Sigma u = 1$

and $(\underset{\sim}{u}^T \underset{\sim}{x}_0)^2 = \left(\dfrac{u^T \Sigma u}{\sqrt{\underset{\sim}{u}^T \Sigma \underset{\sim}{u}}}\right)^2 = \underset{\sim}{u}^T \Sigma \underset{\sim}{u}$

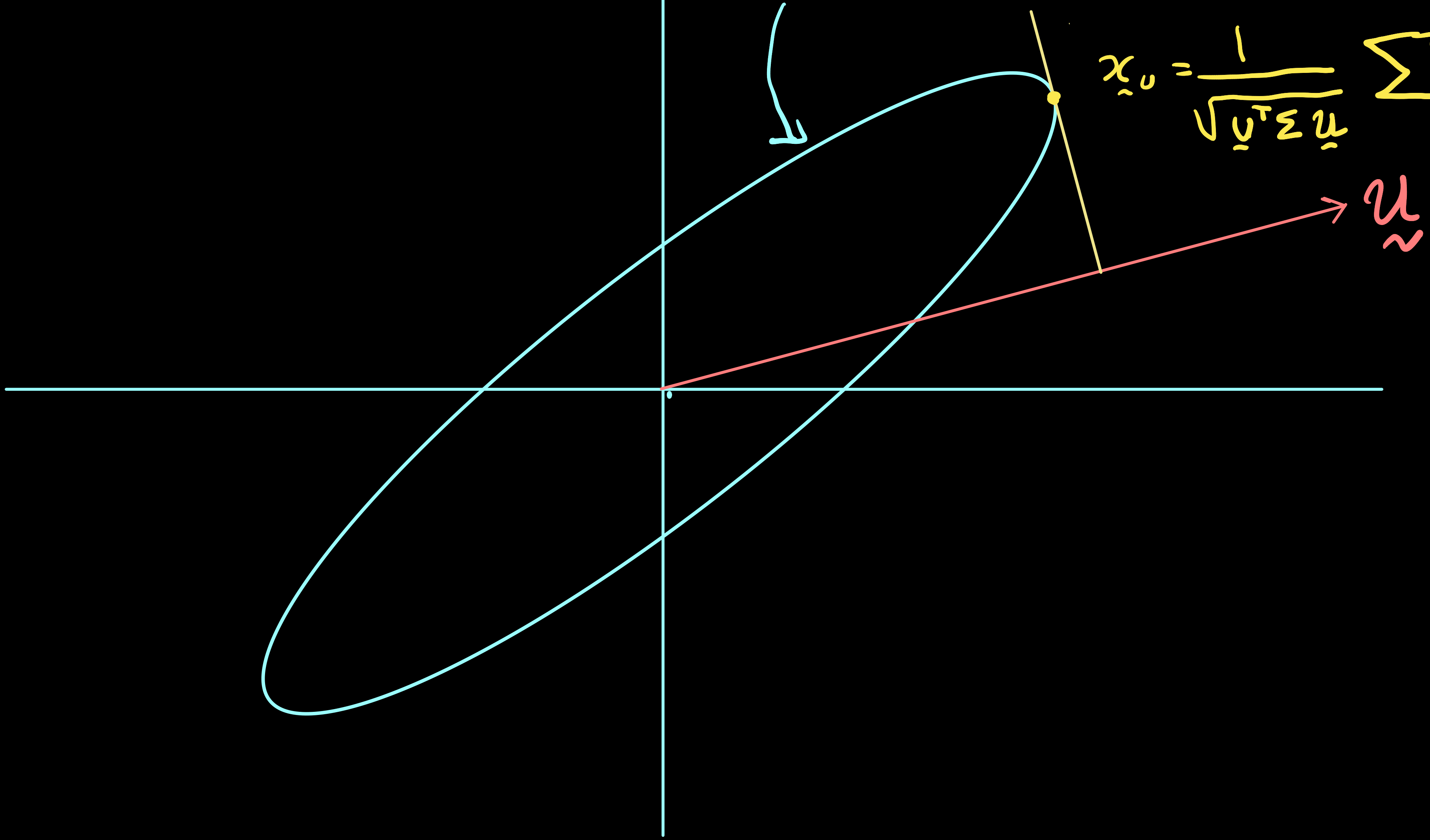
So $\underset{\sim}{x}_0 \in \mathcal{E}$ and $|u^T x_0|$ achieves its maximum value.

$$\mathcal{E} = \{ \underset{\sim}{x} : \underset{\sim}{x}^T \Sigma^{-1} \underset{\sim}{x} = 1 \}$$

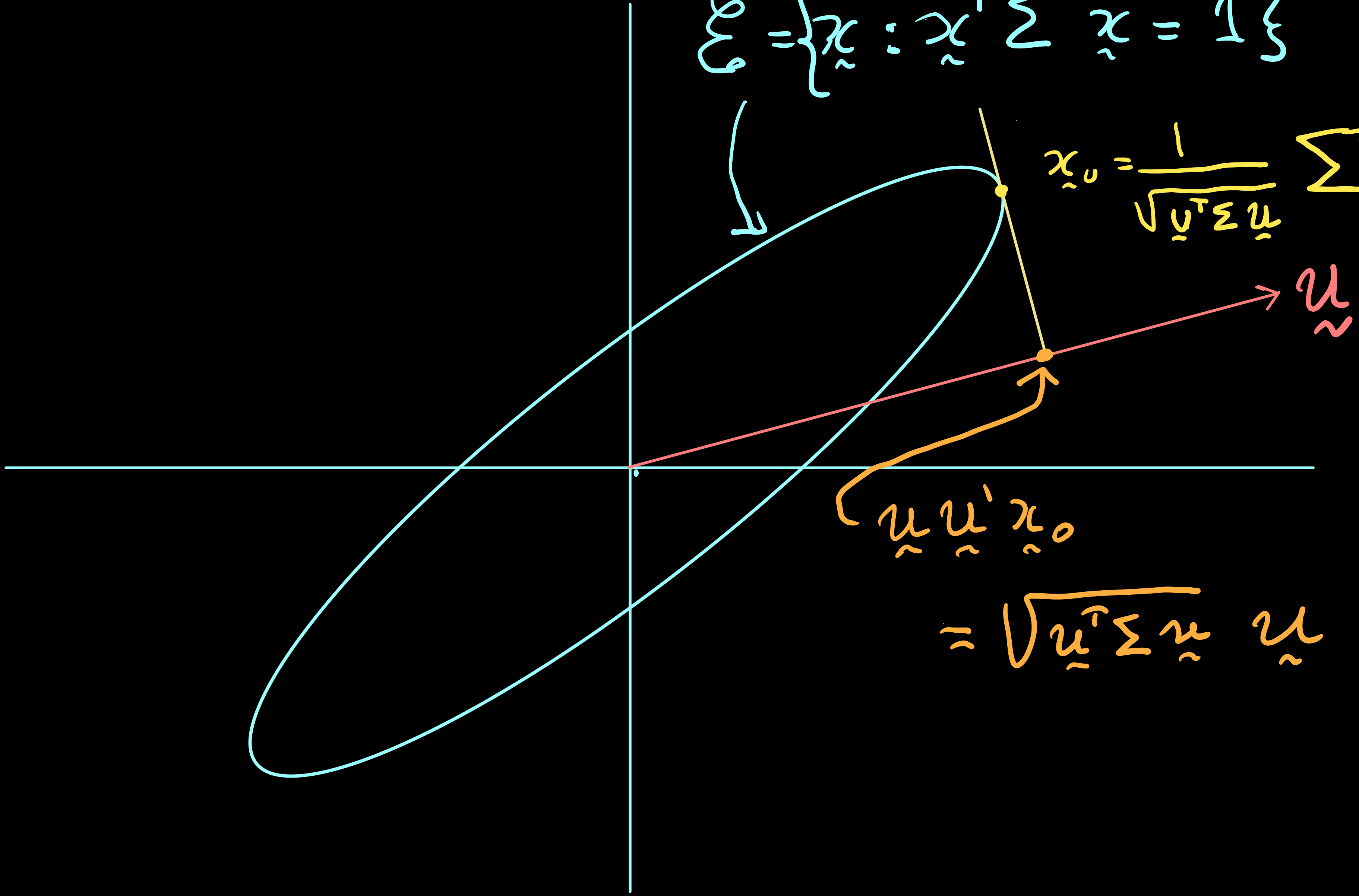


$$\mathcal{E} = \{\underset{\sim}{x} : \underset{\sim}{x}^T \Sigma^{-1} \underset{\sim}{x} = 1\}$$

$$\underset{\sim}{x}_u = \frac{1}{\sqrt{\underset{\sim}{u}^T \Sigma \underset{\sim}{u}}} \Sigma \underset{\sim}{u}$$



$$\mathcal{E} = \{ x : x^T \Sigma^{-1} x = 1 \}$$

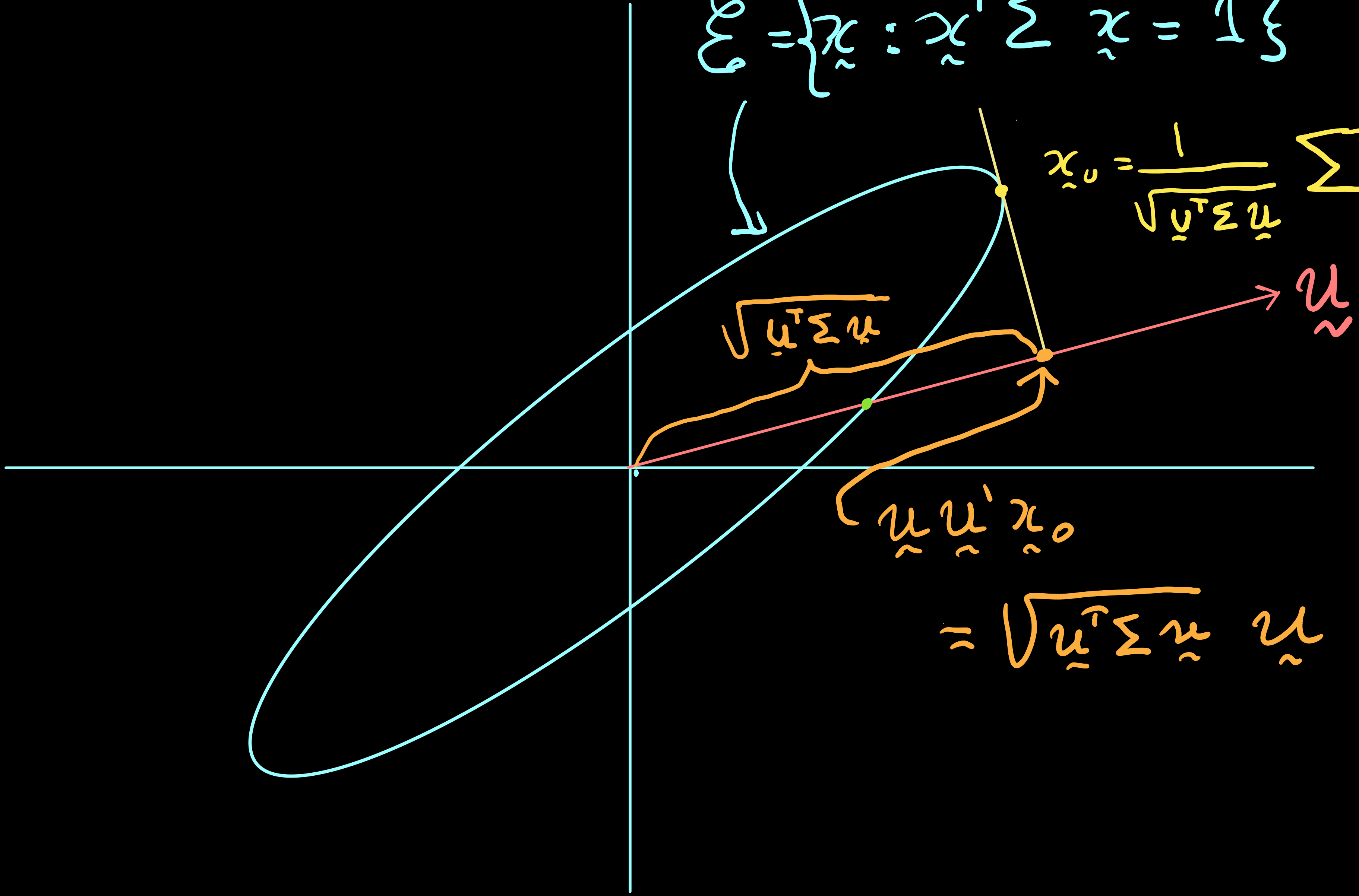


$$x_0 = \frac{1}{\sqrt{u^T \Sigma u}} \Sigma u$$

$$u u' x_0 = \sqrt{u^T \Sigma u}\; u$$

$$\mathcal{E} = \{x : x^T \Sigma^{-1} x = 1\}$$

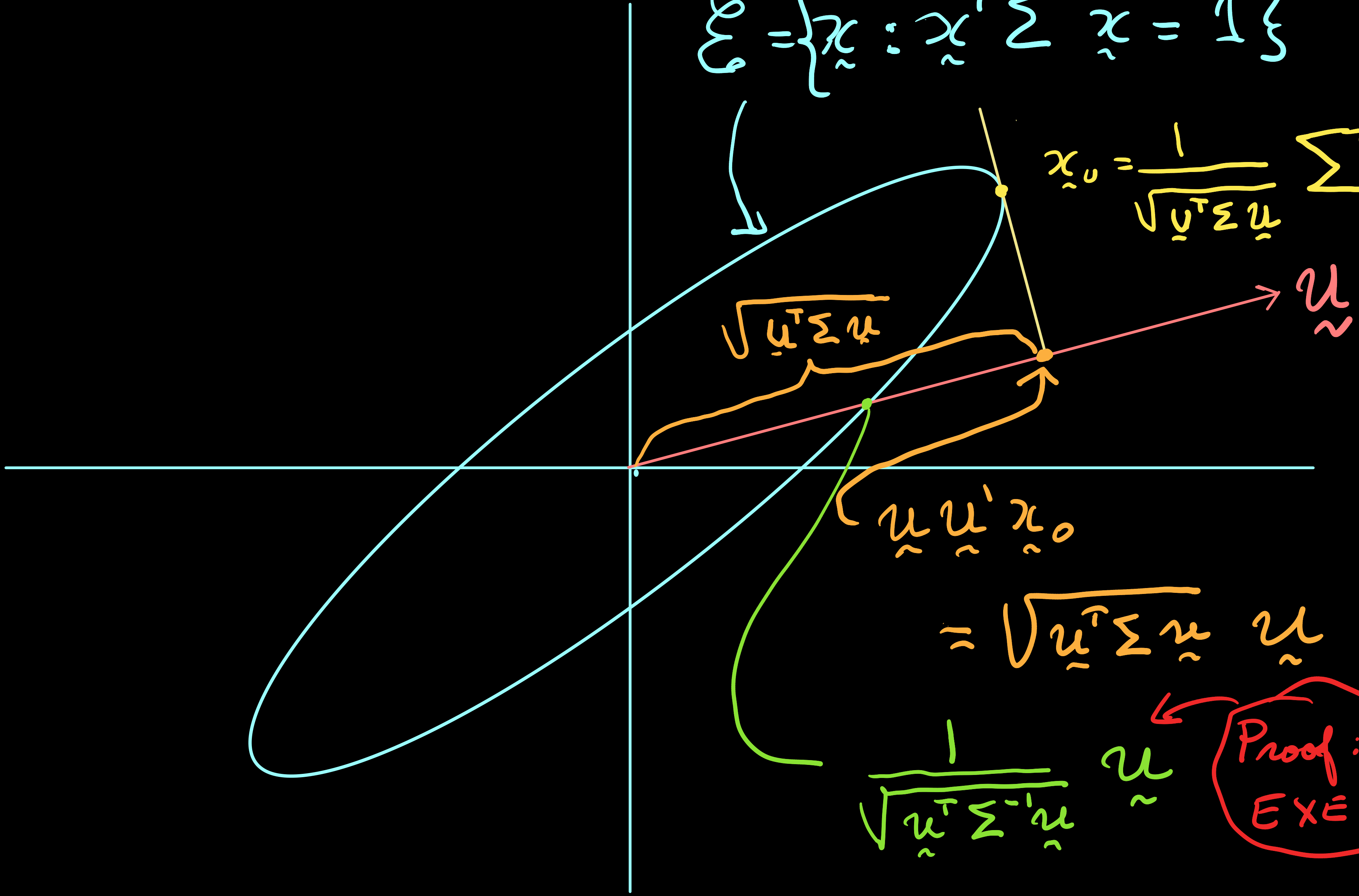
$$x_0 = \frac{1}{\sqrt{u^T \Sigma u}} \Sigma u$$



$$u u' x_0$$

$$= \sqrt{u^T \Sigma u}\; u$$

$$\mathcal{E} = \{ x : x^T \Sigma^{-1} x = 1 \}$$



$$x_0 = \frac{1}{\sqrt{u^T \Sigma u}} \Sigma u$$

$$u u' x_0 = \sqrt{u^T \Sigma u}\; u$$

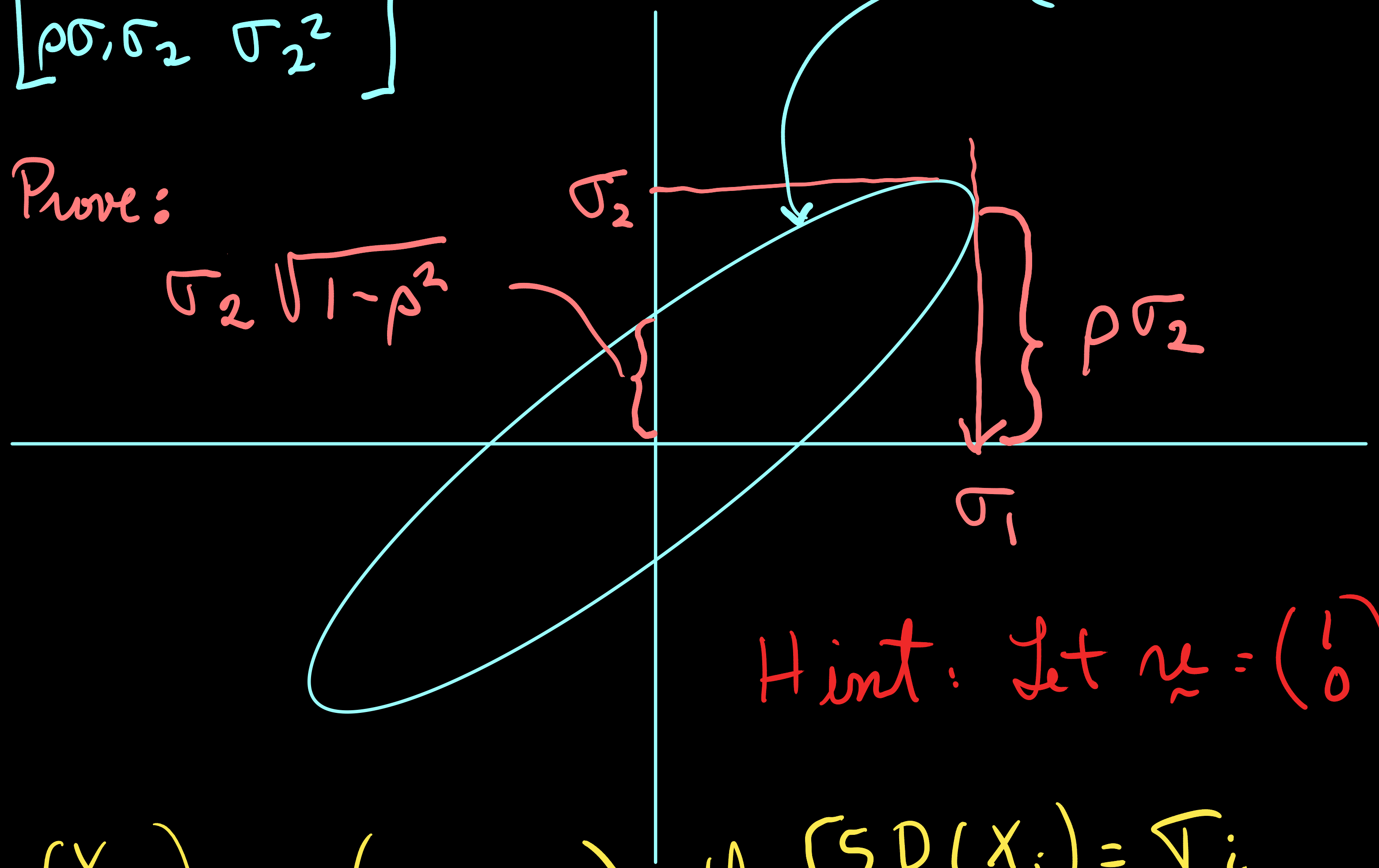
$$\frac{1}{\sqrt{u^T \Sigma^{-1} u}}\, u$$

Proof: EXER

If $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

$\{ \underset{\sim}{x} : \underset{\sim}{x}^T \Sigma^{-1} \underset{\sim}{x} = 1 \}$

EXER: Prove:



Hint: Let $\underset{\sim}{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Review: $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N(\underset{\sim}{0}, \Sigma)$ then $\begin{cases} SD(X_i) = \sigma_i \\ SD(X_2 \mid X_1) = \sigma_2\sqrt{1-\rho^2} \end{cases}$