Metropolis-Hastings algorithm

Given $f > 0$ on Ω, $\int f(\underset{\sim}{x})\, d\underset{\sim}{x} < \infty$
but possibly unknown

- Start with $\underset{\sim}{x}_0 \ni f(\underset{\sim}{x}_0) > 0$
- Step from $\underset{\sim}{x}_t$ to $\underset{\sim}{x}_{t+1}$:

  Use proposal distribution $g(\cdot \mid \underset{\sim}{x})$
  and generate random $\underset{\sim}{x}^*$ from $g(\cdot \mid \underset{\sim}{x}_t)$

e.g. $g(\cdot \mid \underset{\sim}{x})$ obtained by taking random
step $\underset{\sim}{\varepsilon} \sim h(\cdot)$ and setting $\underset{\sim}{x}^* = \underset{\sim}{x}_t + \underset{\sim}{\varepsilon}$

M-H Ratio

$$R(\underset{\sim}{u}, \underset{\sim}{v}) = \underbrace{\frac{f(\underset{\sim}{v})}{f(\underset{\sim}{u})}}_{\text{Did } \underset{\sim}{u} \to \underset{\sim}{v} \text{ move up/ or down?}} \times \frac{g(u|v)}{g(v|u)}$$

"$\underset{\sim}{u}$ to $\underset{\sim}{v}$"

If h symmetric then this = 1

otherwise: bias adjustment for g

Compute $R(\underset{\sim}{x}_t, \underset{\sim}{x}^*) =$ "R"

<u>If $R \geq 1$</u> then $\underset{\sim}{x}_{t+1} = x^*$

<u>If $R < 1$</u> then $U \sim U(0,1)$

$$\underset{\sim}{x}_{t+1} = \begin{cases} \underset{\sim}{x}^* & \text{if } U \leq R \\ \underset{\sim}{x}_t & \text{otherwise} \end{cases}$$

Proof that $f(\underset{\sim}{x})$ is a stationary distribution of the M-H process.

Suppose $\underset{\sim}{X}_t \sim$ pdf f Suppose f is actual pdf

See bottom of p. 202

Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2$ be 2 pts $\ni f(\underset{\sim}{x}_i) > 0$ watch out

and $f(\underset{\sim}{x}_2) \geq f(\underset{\sim}{x}_1)$ wlog.

So joint density of $f_{X_t, X_{t+1}}(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$

$$= f(\underset{\sim}{x}_1)\, g(x_2 | x_1)$$

since $R(x_1, x_2) \geq 1$

Reversing $f_{X_t, X_{t+1}}(x_2, x_1)$

$$f(x_2)\, g(x_1 \mid x_2) \left(\frac{f(x_1)}{f(x_2)}\right) \times \frac{g(x_2 \mid x_1)}{g(x_1 \mid x_2)}$$

$$= f(\underset{\sim}{x}_1)\, g(\underset{\sim}{x}_2 \mid \underset{\sim}{x}_1)$$

$$\overset{!!}{=} f_{X_t, X_{t+1}}(x_1, x_2)$$

Joint distribution is "mixed" with a possibly singular component on the diagonal $P(\underset{\sim}{X}_t = \underset{\sim}{X}_{t+1}) > 0$

But this component is also symmetric

So marginal distributions of

$\underset{\sim}{X}_t$ and $\underset{\sim}{X}_{t+1}$ are the same.

$\therefore\ \underset{\sim}{X}_{t+1} \sim f$ if $\underset{\sim}{X}_t \sim f$

Note:
- This does not guarantee that M-H converges to f.
- If it does, it might not converge in "reasonable" time.