Chapter 6 : Monte Carlo - Generating random #'s

Why bother?

Suppose $X \sim$ pdf f

and you need to know $P(X \in A)$

But $\int_A f(x)\,dx$ is hard to evaluate.

Use LLN : Sample N X_i iid pdf f

$$p_N = \frac{\#(X \in A)}{N} \longrightarrow \int_A f(x)\,dx \quad \text{by LLN}$$

$$SE(p_N) \approx \sqrt{\frac{p_N(1-p_N)}{N}}$$

If f is a density wrt μ $\Big\}$ $\qquad \dfrac{\sum_{i=1}^{N} h(\underset{\sim}{X}_i)}{N} \xrightarrow{?} \int h(\underset{\sim}{x}) f(\underset{\sim}{x})\, d\mu(\underset{\sim}{x})$

? = in probability in general

= a.s. if $\int |h(\underset{\sim}{x})| f(\underset{\sim}{x})\, d\mu(\underset{\sim}{x}) < \infty$

If $\int (h(\underset{\sim}{x}))^2 f(\underset{\sim}{x})\, d\mu(\underset{\sim}{x}) < \infty$ then

$$\mu = \int h(x) f(x)\, d\mu x < \infty$$

and $\sigma^2 = \int (h(\underset{\sim}{x}) - \mu)^2 f(\underset{\sim}{x})\, d\mu(\underset{\sim}{x}) < \infty$

and $\dfrac{\sqrt{N}}{\sigma}\left(\dfrac{\sum h(X_i)}{N} - \mu\right) \xrightarrow{\mathcal{D}} N(0,1)$

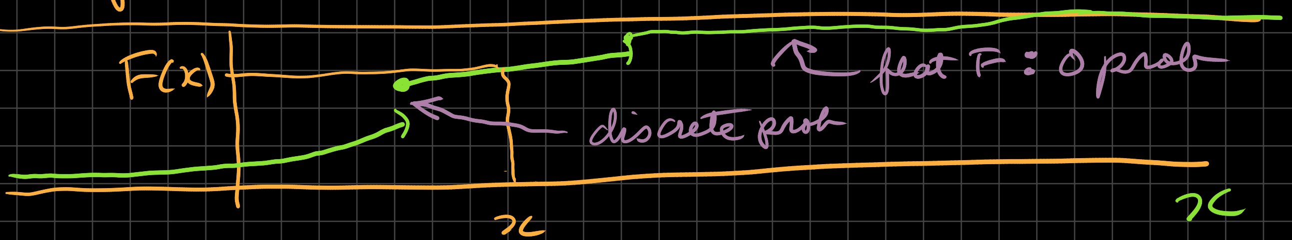
Generating random numbers:

$U(0,1)$:

- Big topic in the mid to late 1900s.
- We'll take it for granted.

pdf (density) or pmf (mass) function f on $\mathbb{R}$

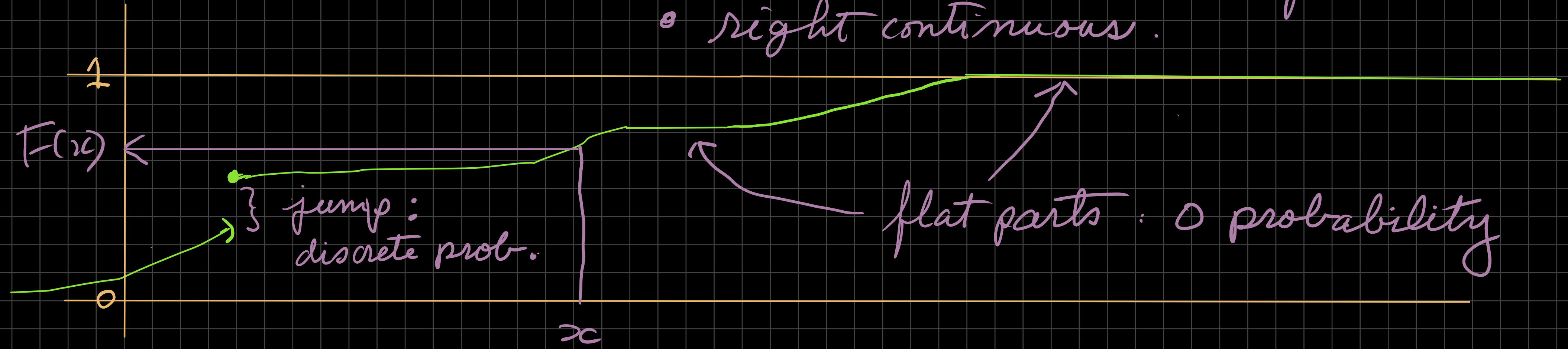
If you can evaluate F^{-1} where $F(x) = \Pr(X \le x)$



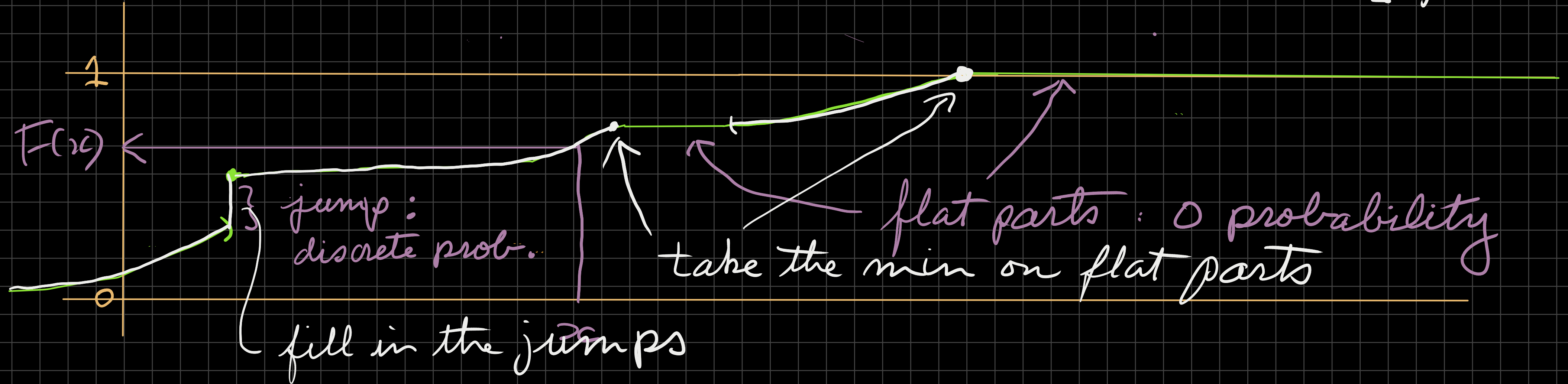
$F(x)$

e.g. mixed distribution discrete and continuous

$F(x) = P(X \le x)$:
- monotone increasing on $\mathbb{R}$
- right continuous.



Then define $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $u \in [0,1]$



If $U \sim U(0,1)$
$X = F^{-1}(U)$
then $X \sim$ cdf F

formal proof in 3131?

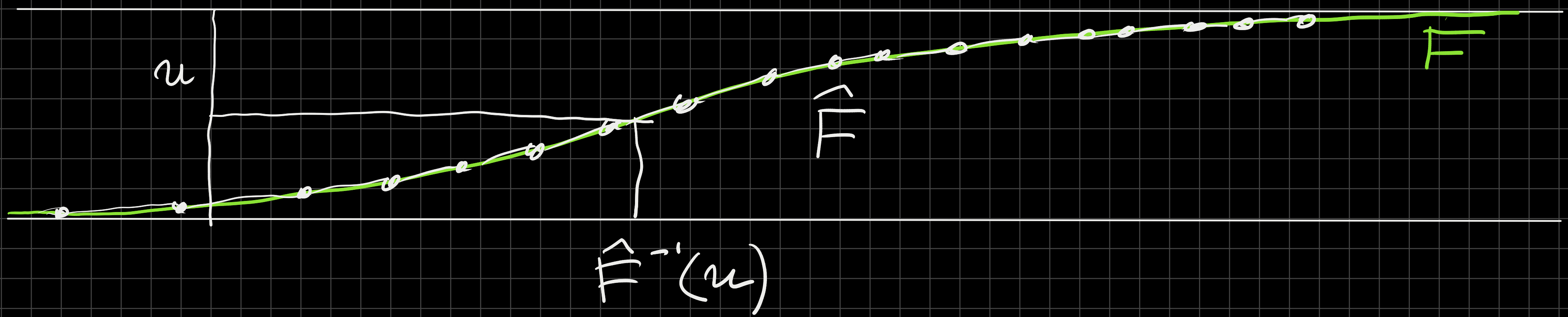
cdf = cumulative distribution function

Recap: If 1) X is a random variable
2) F^{-1} easy to compute
Then $X = F^{-1}(U)$ is easy way to generate random Xs.

What if F^{-1} is hard to compute

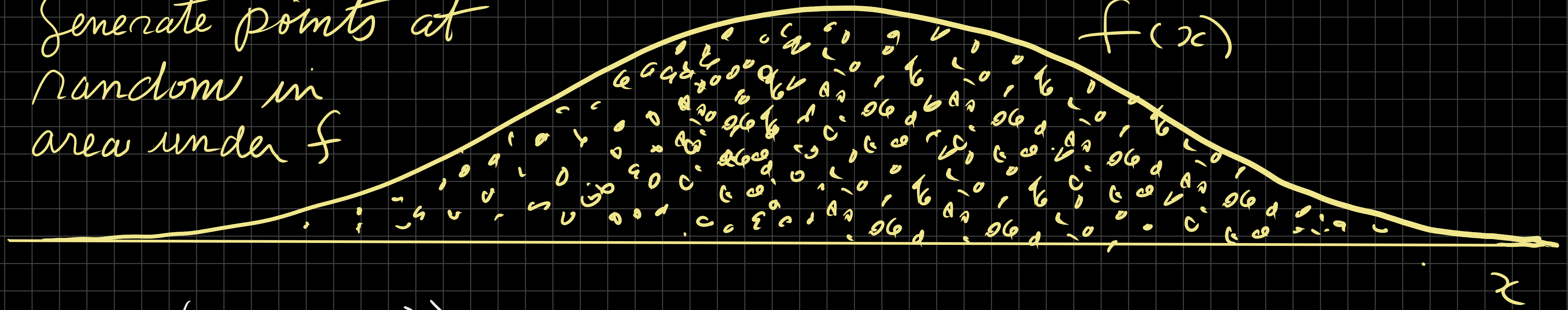
If F easy and continuous see 6.2.2

Idea: Evaluate F at many points and use linear interpolation

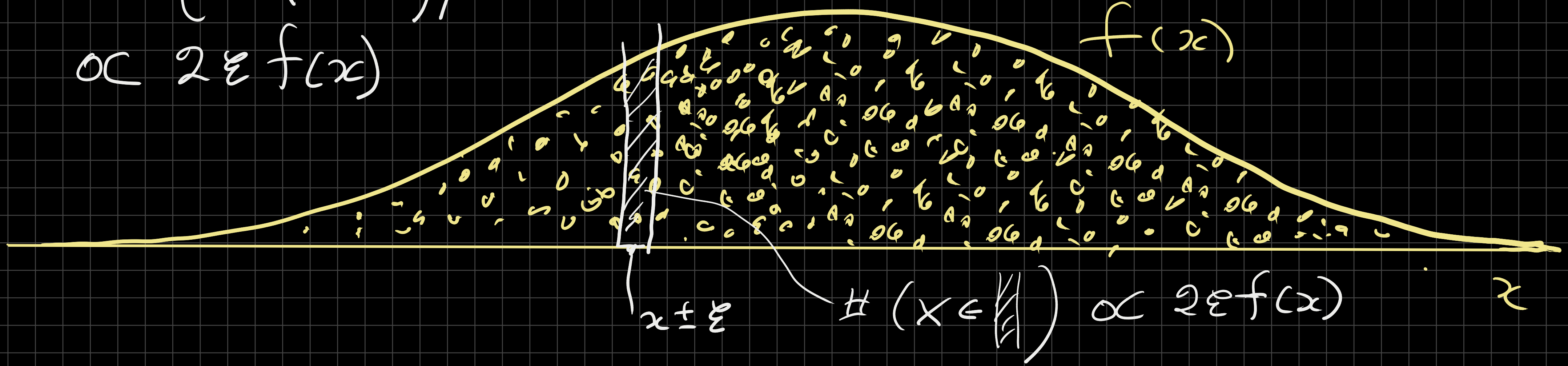


Rejection sampling : Density f

Basic idea:
Generate points at random in area under f



Then $\#\big(X \in (x + \varepsilon)\big)$
$\propto 2\varepsilon f(x)$



So $X_i \sim$ pdf f

Basis of rejection sampling

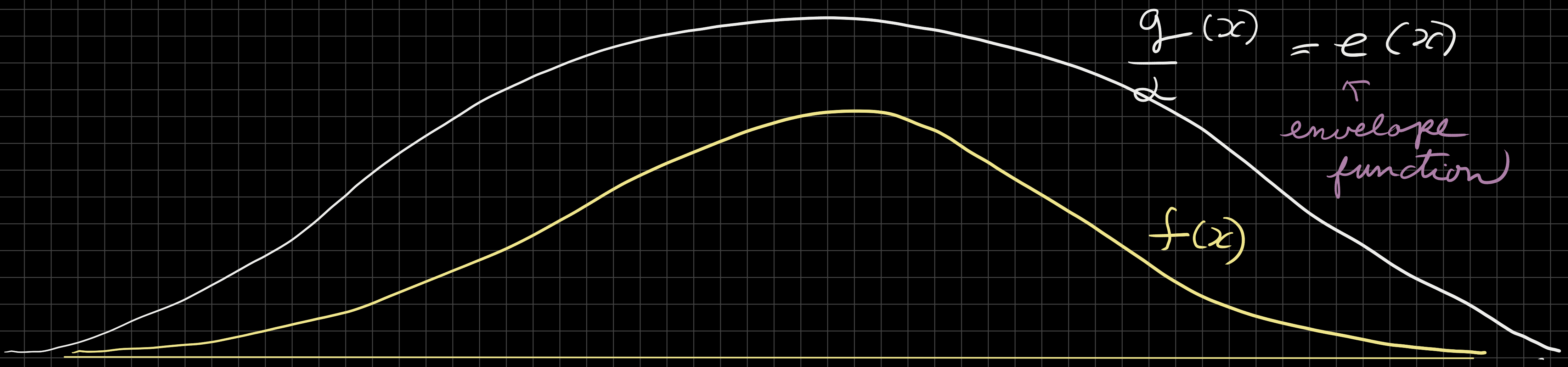
If F^{-1} hard but f easy

and there is a pdf g with easy G^{-1}

such that g can be scaled up to $\frac{g}{\alpha}$

so $$\frac{g(x)}{\alpha} \geq f(x)$$

then:



Procedure for sample of $N \sim$ pdf f

1) Generate $Y \sim$ pdf g
2) generate $U \sim U(0,1)$
3) Keep $X_i = Y$ iff $U > \frac{f(Y)}{e(Y)}$
4) Repeat until you have kept N X_i's.

Alternative description

3) Let $V = \frac{g(Y)}{\alpha} \times U$

So (X, V) is uniformly distributed in area under $\frac{g(x)}{\alpha}$

Keep X iff $V < f(X)$

so...

Test question: Describe rejection sampling algorithm and prove it work.

Proportion accepted : $\dfrac{\int f(y)\,dy}{\int e(y)\,dy} = \alpha$

Note: To make α close to 1 you need g proportionally close to f.

$$\frac{g(x)}{\alpha} \geq f(x)$$

$$\alpha \leq \frac{g(x)}{f(x)}$$

$$\alpha \leq \sup\left\{\frac{g(x)}{f(x)}\right\}$$

$$\log \alpha \leq \sup \{ \log g(x) - \log f(x) \}$$

Rejection sampling also works if f is known only up to a proportional constant, i.e.

$$\int f(x)\, dx = c \quad \text{unknown}$$

Then $X_i \sim$ pdf $\frac{f}{c}$

But proportion accepted is α / c

Note: Idea works for $\underset{\sim}{X} \in \mathbb{R}^p$

Example 6.2 Bayesian posterior using prior as a sampling distribution

Sample of $N=10$ from Poisson(λ)

8, 3, 4, 3, 1, 7, 2, 6, 2, 7

Prior $\pi(\lambda)$ $\quad \lambda \sim \text{lognormal}(4, 0.5^2)$

Let $L(\lambda \mid \underset{\sim}{x})$ be likelihood $\displaystyle\prod_{i=1}^{10} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$

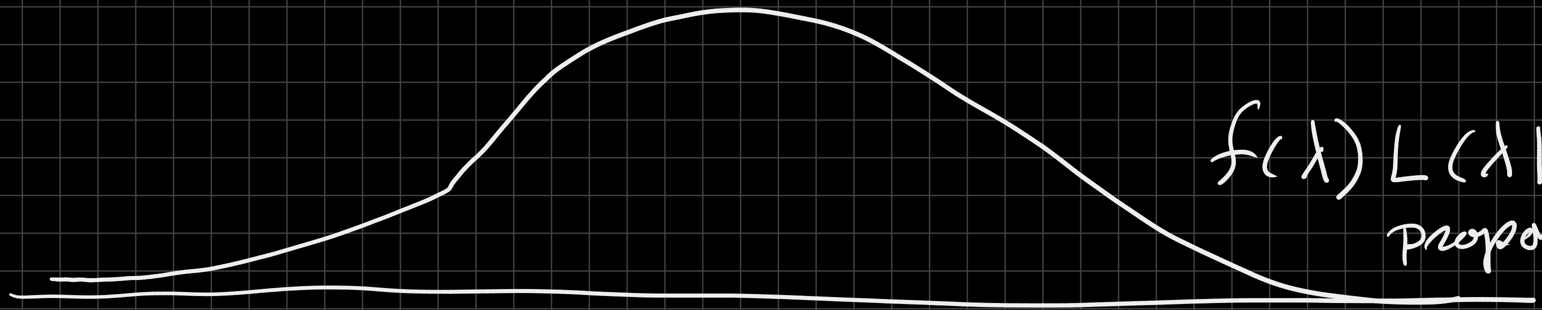
$f(\lambda)$ = prior = use dlnorm

$L(\lambda|x)$ = use dpois

Posterior is $f(\lambda)L(\lambda|\underset{\sim}{x})$

need α so

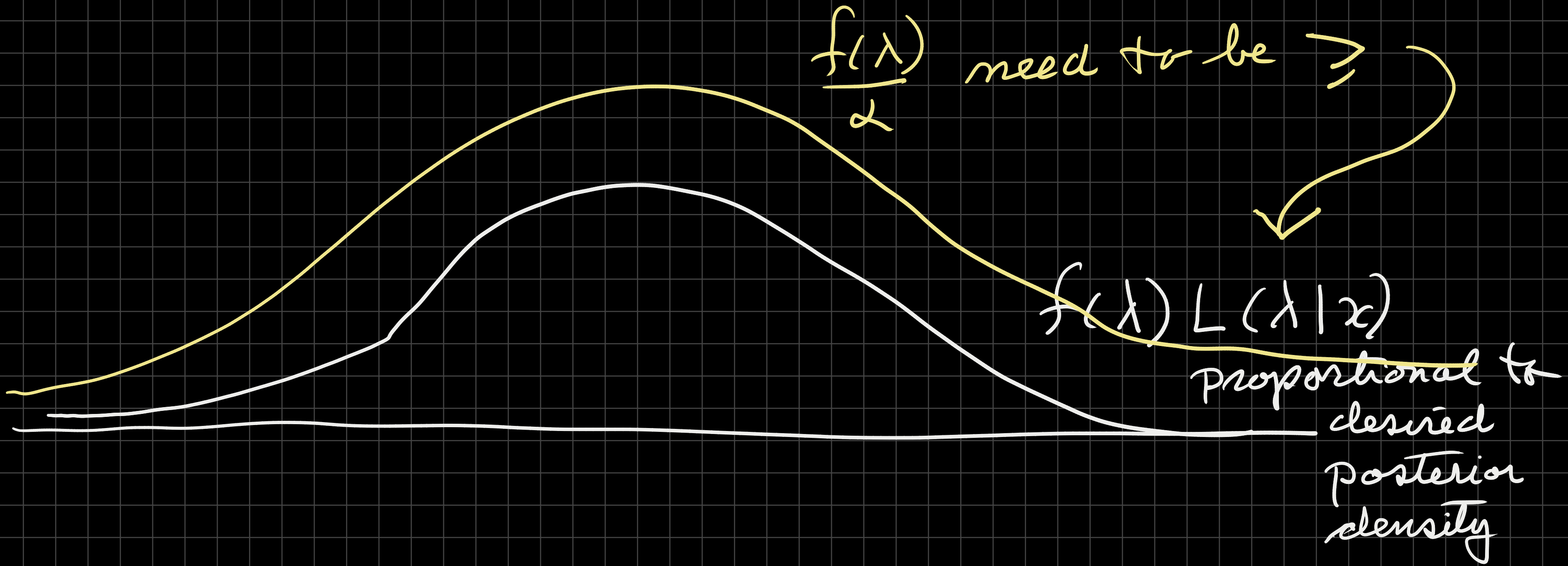
$$e(\lambda) = \frac{f(\lambda)}{\alpha} \geq f(\lambda)L(\lambda|\underset{\sim}{x})$$



Posterior is $f(\lambda) L(\lambda | \underset{\sim}{x})$

need α so

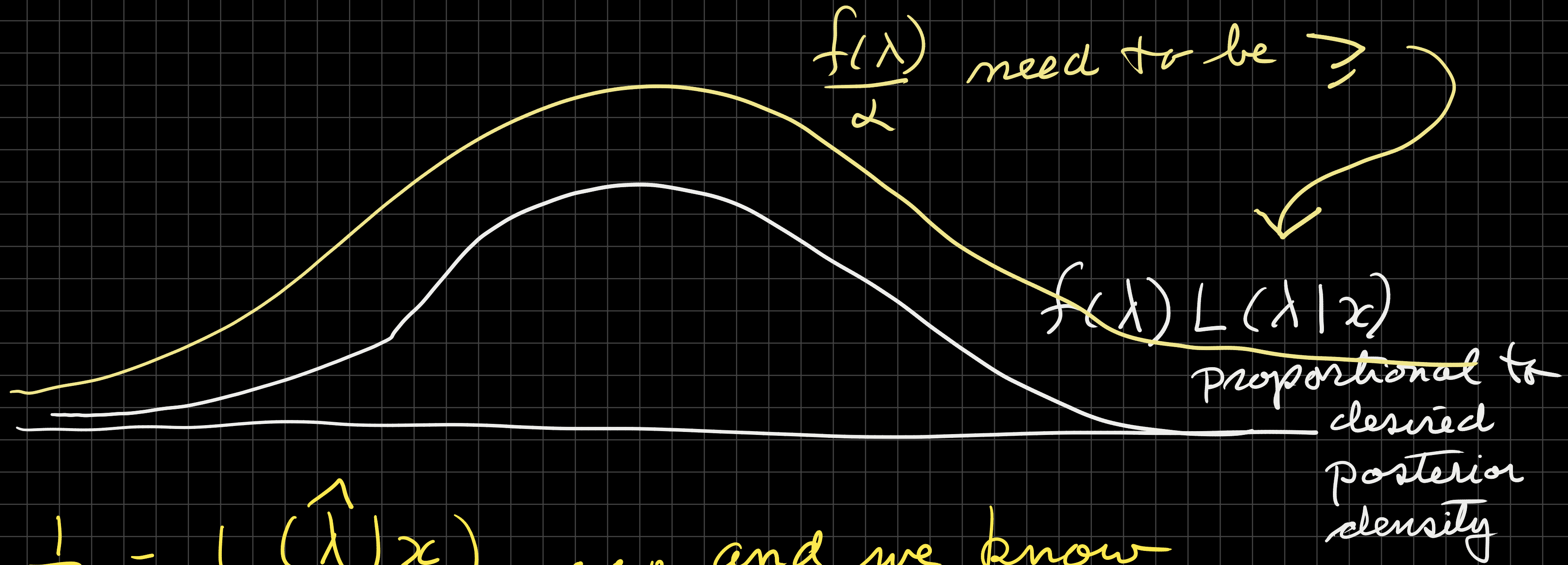
$$e(\lambda) = \frac{f(\lambda)}{\alpha} \geq f(\lambda) L(\lambda | \underset{\sim}{x})$$



Posterior is $f(\lambda)L(\lambda|\underset{\sim}{x})$

need α so

$$e(\lambda) = \frac{f(\lambda)}{\alpha} \geq f(\lambda)L(\lambda|\underset{\sim}{x})$$



Easy: $\frac{1}{\alpha} = L(\hat{\lambda}|\underset{\sim}{x})$ ... and we know $\hat{\lambda} = \bar{x}$ for Poisson(λ)

i.e. $\quad \dfrac{f(\lambda)}{\alpha} = f(\lambda) \times L(\hat{\lambda} \mid x)$

$$\geq f(\lambda) \times L(\lambda \mid x) = g(x)$$

which is proportional to target density