Expectation - Maximization Algorithm

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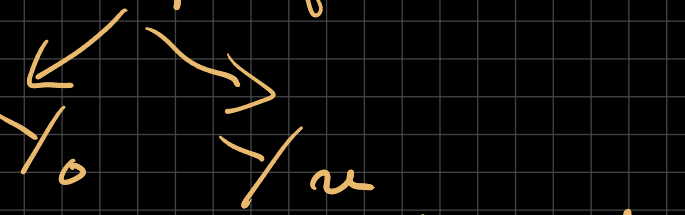
Applications:

- Missing data: Y_f full data



"Easy" model for $f(Y_f \mid \underset{\sim}{\theta})$

But complex for $f(Y_o \mid \underset{\sim}{\theta}) = \int f(Y_o, Y_u \mid \underset{\sim}{\theta})\, dY_u$

Each step has 2 sub-steps:

Given value $\underset{\sim}{\theta}_t$ from t^{th} step:

1) Expectation step:

Use $l(Y_f | \underset{\sim}{\theta}) = \log f(Y_f | \underset{\sim}{\theta})$

to work out

$$E\left(l(Y_f | \underset{\sim}{\theta}) \mid Y_0, \underset{\sim}{\theta}_t\right)$$

free parameter

parameter for E

Note: This is E of a function of $\underset{\sim}{\theta}$ given a value of $\underset{\sim}{\theta}_t$

$$= Q\left(\underset{\sim}{\theta} \mid \underset{\sim}{\theta}_t\right)$$

2) Maximize $Q(\underset{\sim}{\theta}, \underset{\sim}{\theta}_t)$ with respect to $\underset{\sim}{\theta}$

Let $\underset{\sim}{\theta}_{t+1} = \underset{\underset{\sim}{\theta}}{\operatorname{argmax}}\, Q(\underset{\sim}{\theta}, \underset{\sim}{\theta}_t)$

Repeat 2 steps until convergence.

1) $|\underset{\sim}{\theta}_{t+1} - \underset{\sim}{\theta}_t|$ small

2) $|Q(\theta_{t+1}, \theta_t) - Q(\theta_t, \theta_t)|$ small

- On many applications, Y_u is not "missing data" but "random unknown parameters" as in multilevel & longitudinal modeling
- On some MLM packages (e.g. nlme) the E-M algorithm is used "under the hood".
- It's a good method for many complex problems.
- It is generally not easy to apply but once you have an algorithm for a kind of problem then many people use it without knowing.

Example:

WARNING: LONG & TEDIOUS
BUT INTERESTING (I THINK)
AND GOOD PRACTICE

- Suppose you are sampling from

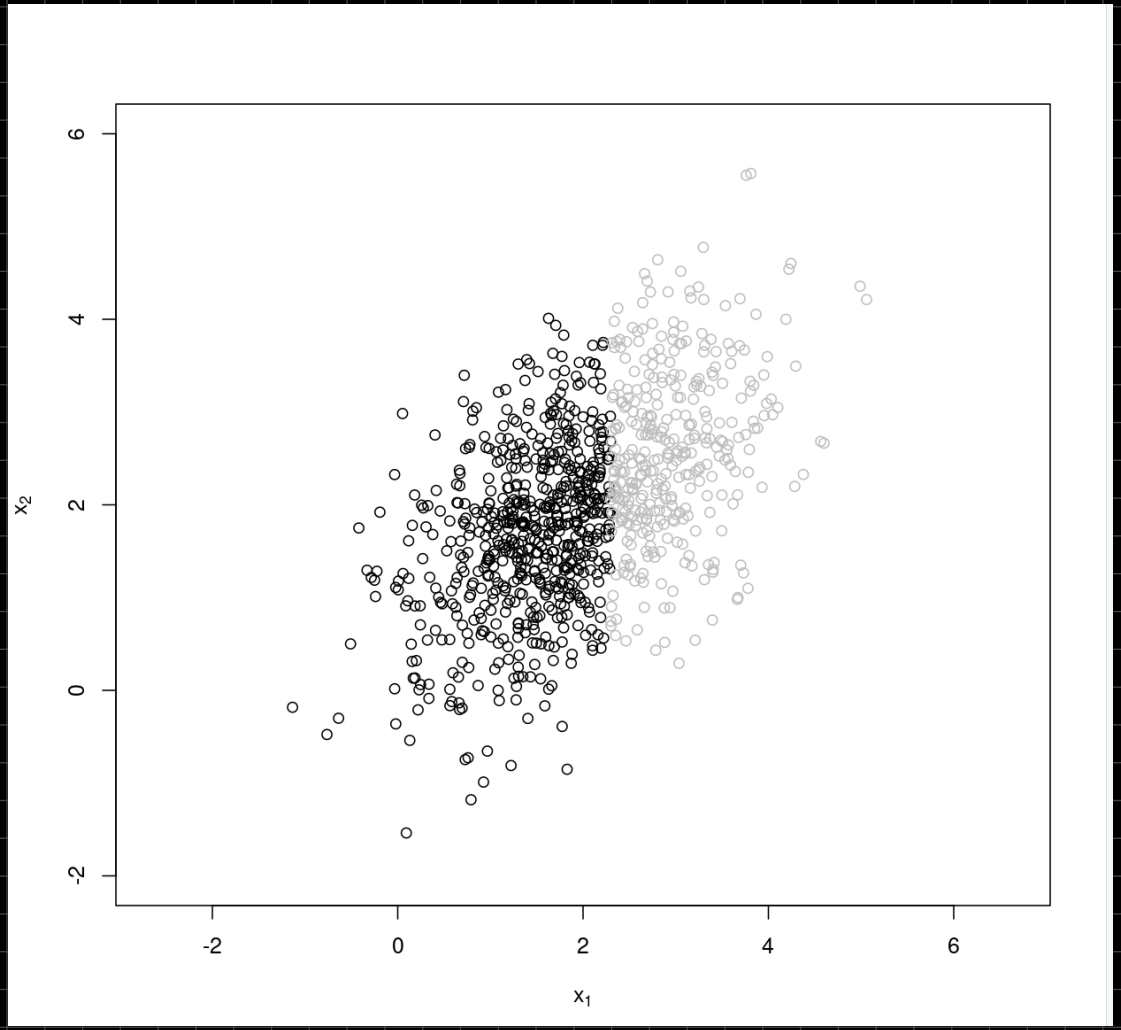
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N_2(\underset{\sim}{\mu}, \Sigma)$$

Simulation using $\underset{\sim}{\mu} = \begin{pmatrix} 2.1 \\ 1.9 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 1.1 & .9 \\ .9 & 1.2 \end{pmatrix}$

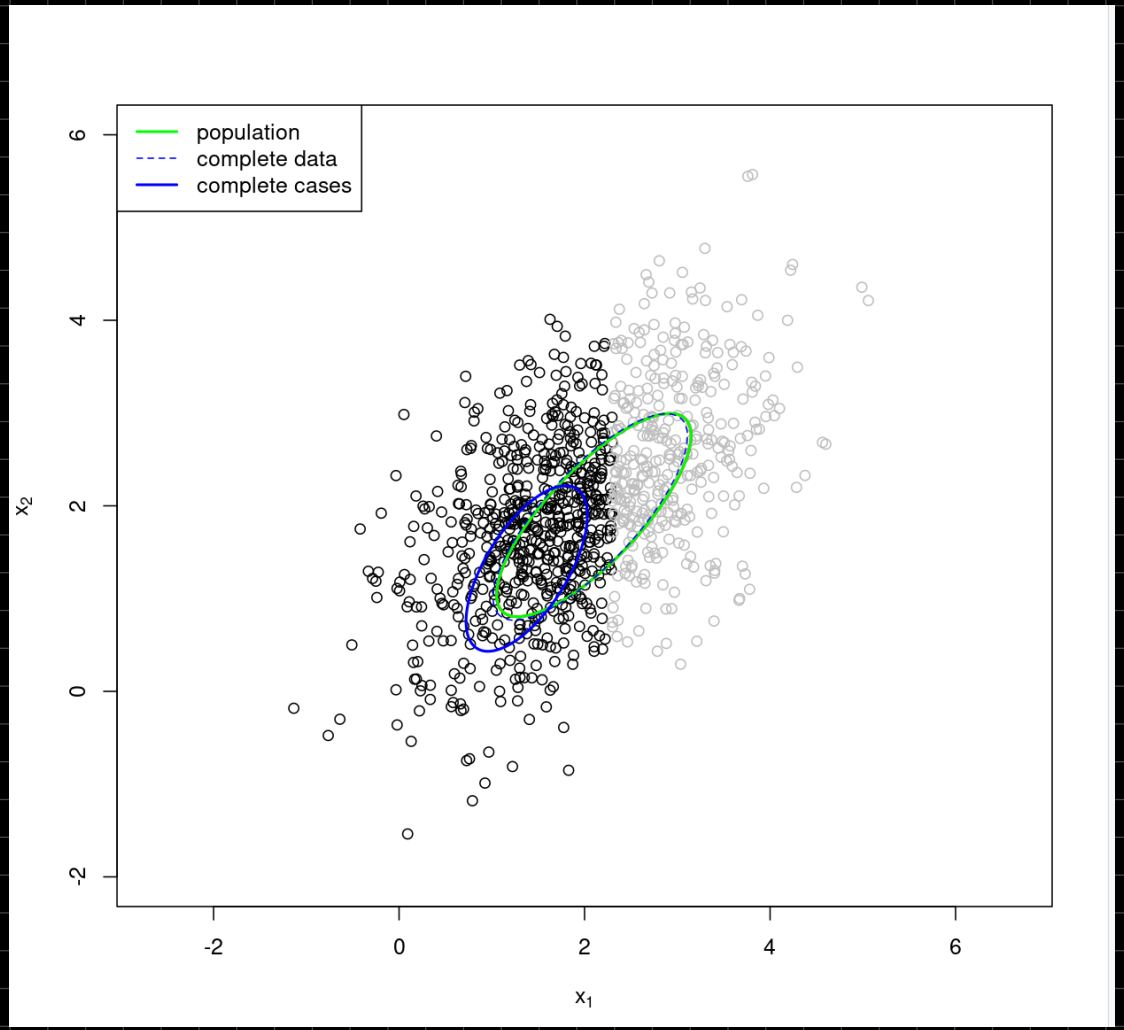
"true" but unknown

$n = 1,000$

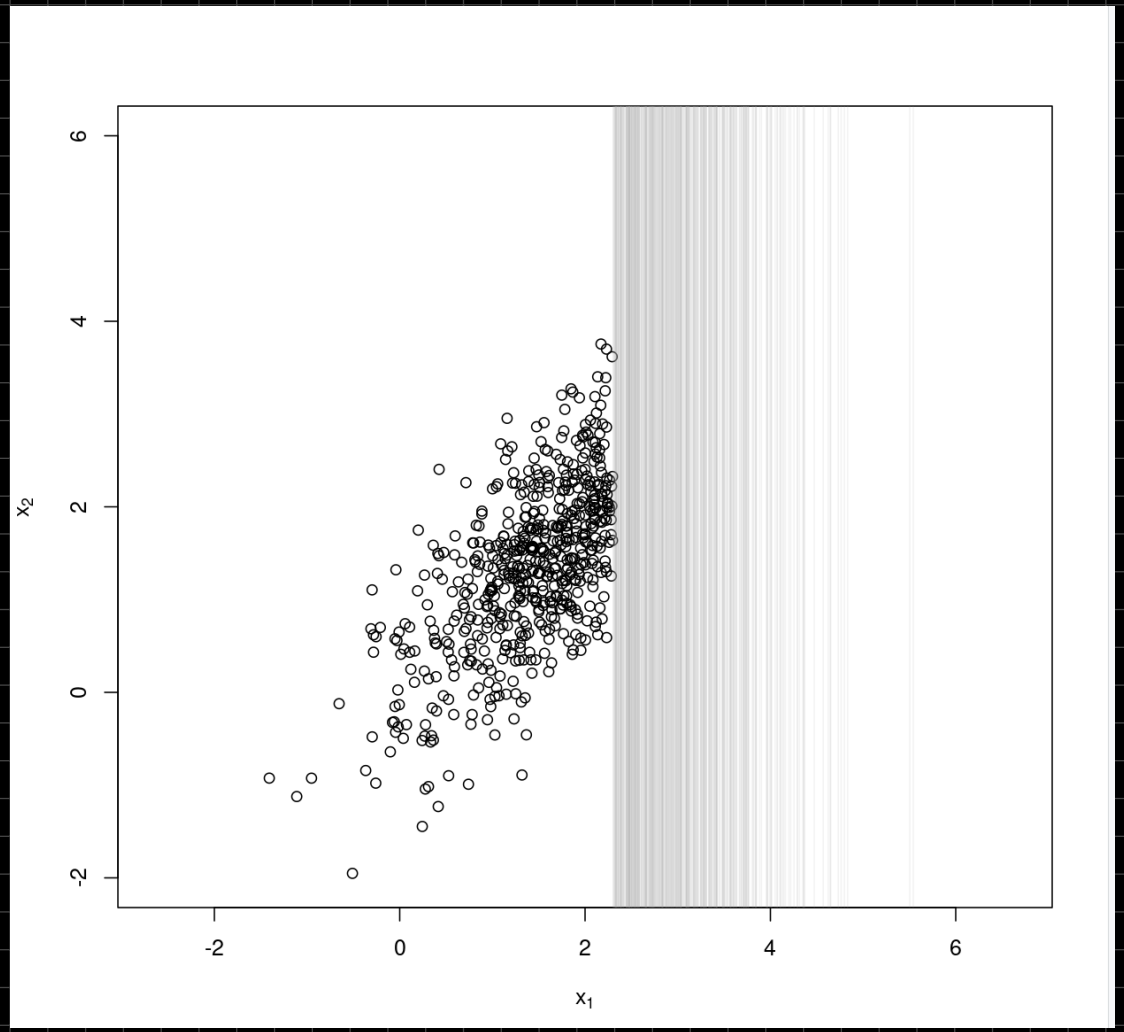
But x_2 not observed if $x_1 > 2.3$.



x_2 missing based on observed value of x_1


population
complete data
complete cases
6
4
2
0
-2
x_2
-2
0
2
4
6
x_1

The data we actually have:



Is there any hope for estimating μ_2?

μ_1 & $\sigma_1^2 = \sigma_{11}$ easy.

What about $\sigma_2^2 = \sigma_{22}$?

and $\rho_{12} = \sigma_{12} / \sqrt{\sigma_{11}\sigma_{22}}$

Data:

$$X = \begin{bmatrix} 2.1 & 3.2 \\ 1.0 & 1.2 \\ \vdots & \vdots \\ 2.2 & 0.9 \\ \hline 3.1 & NA \\ 4.2 & NA \\ \vdots & \vdots \\ 2.4 & NA \end{bmatrix} = \begin{bmatrix} X_c \\ \hline X_m \end{bmatrix}$$

X_c: complete cases

X_m: with missing data

$$= \begin{bmatrix} X_o \\ X_u \end{bmatrix}$$

X_o: observed

X_u: unobserved / missing data or random variable

If x_{ij} is missing, we think of x_{ij} as a random variable.

Since X_u is unknown we think of it as a random vector.

E-M analysis :

Complete data model : (detail to remind some neat methods)

If $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$

$$f(\underset{\sim}{x} \mid \underset{\sim}{\mu}, \Sigma_i) = \frac{1}{(2\pi)^{p/2}} \frac{1}{|\Sigma_i|^{1/2}} \exp\left\{-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu})\right\}$$

Sample of n: $f(\underset{\sim}{x_1}, \underset{\sim}{x_2}, \cdots, \underset{\sim}{x_n} \mid \underset{\sim}{\mu}, \Sigma)$

$$= \frac{1}{(2\pi)^{np/2}} \frac{1}{|\Sigma|^{n/2}} \exp\left\{-\frac{1}{2} \boxed{\sum_{i=1}^{n} (\underset{\sim}{x_i}-\underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x_i}-\underset{\sim}{\mu})}\right\}$$

Playing with $\sum_{i=1}^{n} (\underset{\sim}{x}_i - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x}_i - \underset{\sim}{\mu})$

$$= \sum_{i=1}^{n} \operatorname{tr}\left[(\underset{\sim}{x}_i - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x}_i - \underset{\sim}{\mu}) \right]$$

1×1

$$= \sum_{i=1}^{n} \operatorname{tr} \underset{p \times p}{\Sigma^{-1}} \underset{p \times 1}{(\underset{\sim}{x}_i - \underset{\sim}{\mu})} \underset{1 \times p}{(\underset{\sim}{x}_i - \underset{\sim}{\mu})^T}$$

$$= \operatorname{tr} \Sigma^{-1} \sum_{i=1}^{n} (\underset{\sim}{x}_i - \underset{\sim}{\mu})(\underset{\sim}{x}_i - \underset{\sim}{\mu})^T$$

$$= \operatorname{tr} \Sigma^{-1} \sum_{i=1}^{n} (\underset{\sim}{x}_i \underset{\sim}{x}_i^T - \underset{\sim}{x}_i \underset{\sim}{\mu}^T - \underset{\sim}{\mu} \underset{\sim}{x}_i^T + \underset{\sim}{\mu}\underset{\sim}{\mu}^T)$$

TRACE

- tr is sum of diagonal elts
- $\operatorname{tr}(AB) = \operatorname{tr}(BA)$
- $\operatorname{tr}(A) = \operatorname{tr}(A^T)$
- $\operatorname{tr}(A) = a$ if $A = [a]$ is 1×1
- $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$

by linearity

$$= \operatorname{tr}\Sigma^{-1}\left(X^T X - n\bar{x}\mu^T - n\mu\bar{x}^T + n\mu\mu^T \right)$$

$$= n\mu^T\Sigma^{-1}\mu + \operatorname{tr}\,\Sigma^{-1}\underbrace{\left(X^T X - \left(\sum_{i=1}^{n} x_i\right)\mu^T - \mu\left(\sum_{i=1}^{n} x_i\right)^T \right)}_{\text{focus on this}}$$

split X_c & X_m
complete cases / cases with missing X_2

$$X_c' X_c - \left(\sum_{i \in X_c} x_i\right)\mu^T - \mu\left(\sum_{i \in X_c} x_i\right)^T$$

$$+ X_m' X_m - \sum_{i \in X_m} x_i \mu^T - \mu\left(\sum_{i \in X_m} x_i\right)^T$$

(*)

We need to find $E(* \mid X_{observed}, \mu_t, \Sigma_t)$

Why? Because $\underset{\sim}{l}(X \mid \mu, \Sigma)$ is linear in $(*)$

So $Q(\underset{\sim}{\theta} \mid \underset{\sim}{\theta}_t) = E(l(X \mid \underset{\sim}{\theta}) \mid X_0, \underset{\sim}{\theta}_t)$

$= l(E((*) \mid \underset{\sim}{\theta}) \mid X_0, \underset{\sim}{\theta}_t)$

i.e. $E(l(*)) = l(E(*))$

which works if l is linear in $(*)$
and $(*)$ includes all random variables.

Plan of attack:

note that $\underset{\sim}{\mu}$ is an argument of l
Not a parameter for E

$$X_c' X_c - \left(\sum_{i \in X_c} \underset{\sim}{x}_i\right) \underset{\sim}{\mu}^T - \underset{\sim}{\mu} \left(\sum_{i \in X_c} \underset{\sim}{x}_i\right)^T$$

This part is constant given X_0 ... or is it?

$$+ X_m' X_m - \left[\sum_{i \in X_m} \underset{\sim}{x}_i\right] \underset{\sim}{\mu}^T - \underset{\sim}{\mu} \left[\sum_{i \in X_m} \underset{\sim}{x}_i\right]^T$$

ditto

This part has both constants (x_i's) and random variables (X_2's) given X_0.

We need $E(\quad | X_0, \mu^{(t)}, \Sigma^{(t)})$

Start with:

① $E\left(\sum_{i \in X_m} x_i \mid X_0, \mu^{(t)}, \Sigma^{(t)}\right)$

$$= \sum_{i \in X_m} E\left(x_i \mid X_0, \mu^{(t)}, \Sigma^{(t)}\right)$$

$$= \sum_{i \in X_m} \begin{pmatrix} x_{1i} \\ \mu_2^{(t)} + \frac{\sigma_{12}^{(t)}}{\sigma_{11}^{(t)}}\left(x_{1i} - \mu_1^{(t)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i \in X_m} x_{1i} \\ n_m \mu_2^{(t)} + \frac{\sigma_{12}^{(t)}}{\sigma_{11}^{(t)}}\left(\sum_{i \in X_m} x_{1i} - n_m \mu_1^{(t)}\right) \end{pmatrix}$$

$$= \left(S_{1m} \right.$$

$$\left. = x_m \mu_2^{(t)} + \frac{\sigma_{12}^{(t)}}{\sigma_{11}^{(t)}} \left(S_{1m} - x_m \mu_1^{(t)} \right) \right)$$

② $$E\left(\sum_{i \in X_m} x_{1i} x_{2i} \mid X_0, \mu^{(t)}, \Sigma^{(t)} \right)$$

$$= \sum_{i \in X_m} x_{1i} \, E\left(x_{2i} \mid X_0, \mu^{(t)}, \Sigma^{(t)} \right)$$

$$= \sum_{i \in X_m} x_{1i} \left(\mu_2^{(t)} + \frac{\sigma_{12}^{(t)}}{\sigma_{11}^{(t)}} \left(x_{1i} - \mu_1^{(t)} \right) \right)$$

$$= \mu_2^{(t)} S_{1m} + \frac{\sigma_{12}^{(t)}}{\sigma_{11}^{(t)}} \left(\sum x_{1i}^2 - \mu_1^{(t)} S_{1m} \right)$$

③ $$E\left(\sum x_{2i}^2 \mid X_0, \mu^{(t)}, \Sigma^{(t)} \right)$$

$$= \sum_{i=1}^{n} \left\{ \left[E\left(x_{2i} \mid X_0, \mu^{(t)}, \Sigma^{(t)} \right) \right]^2 + Var\left(x_{2i} \mid X_0, \mu^{(t)}, \Sigma^{(t)} \right) \right\}$$

$$= \sum_{i=1}^{n} \left\{ \left(\mu_2^{(t)} + \frac{\sigma_{12}^{(t)}}{\sigma_{11}^{(t)}} \left(x_{1i} - \mu_1^{(t)} \right) \right)^2 + \sigma_{22}^{(t)} - \frac{\left(\sigma_{12}^{(t)} \right)^2}{\sigma_{11}^{(t)}} \right\}$$

As we get ready to write an algorithm to do this, we try to identify intermediate quantities to compute.

An important one is the vector

$$\underset{i \in X_m}{E}\left(\underset{\sim}{x}_2 \mid \underset{\sim}{x}_1\right) = \mu_2^t + \frac{\sigma_{12}^t}{\sigma_{11}^t}\left(\underset{\sim}{x}_1 - \mu_1^t\right) = \underset{\sim}{v}$$

"R notation"

$= \underset{\sim}{e}_{2\cdot 1}$

Next $\underset{\sim}{x_1} \cdot \underset{\sim}{v} = \underset{\sim}{xv}$

$\underset{\sim}{v2} = \underset{\sim}{v} \times \underset{\sim}{v}$

↑ element-wise

$Cvar = \sigma_{22}^{t} - \frac{(\sigma_{12}^{t})^2}{\sigma_{11}^{t}}$

So: $E(X_m^T X_m \mid X_o, \mu^{(t)}, \Sigma^{(t)})$

$$= \begin{bmatrix} sum(\underset{\sim}{x_1} \times \underset{\sim}{x_1}) & sum(\underset{\sim}{xv}) \\ sum(\underset{\sim}{xv}) & sum(\underset{\sim}{v2}) + n_m \times Cvar \end{bmatrix}$$

So back to the original

$$X_c' X_c - \left(\sum_{i \in X_c} \tilde{x}_i\right) \underset{\sim}{\mu}^T - \underset{\sim}{\mu} \left(\sum_{i \in X_c} \tilde{x}_i\right)^T$$

$$+ E\left(X_m' X_m - \sum_{i \in X_m} \tilde{x}_i \underset{\sim}{\mu}^T - \underset{\sim}{\mu} \left(\sum_{i \in X_m} \tilde{x}_i\right)^T \middle| X_o, \mu^t, \Sigma^t\right)$$

$$= X_c' X_c + S_m - (S_c + S_m)\mu^T - \mu (S_c + S_m)^T$$

where $S_m = E(X_c' X_c \mid X_o, \mu^t, \Sigma^t)$

$$S_c = \sum_{i \in X_c} \tilde{x}_i \qquad S_m = E\left(\sum_{i \in X_m} \tilde{x}_i\right)$$

$$S'_m = \begin{bmatrix} \sum_{i \in X_m} x_{1i}^2 & \sum_{i \in X_m} x_{1i} e_{2.1,i} \\ = & \sum_{i \in X_m} e_{2.1,i}^2 + n_m \left(\sigma_{22}^{(t)} - \frac{(\sigma_{12}^{(t)})^2}{\sigma_{11}^{(t)}} \right) \end{bmatrix}$$

$$\underset{\sim}{S}_m = \sum_{i \in X_m} \begin{pmatrix} x_{1i} \\ \mu_2^t + \frac{\sigma_{12}}{\sigma_{11}}^t (x_{1i} - \mu_1^t) \end{pmatrix}$$

Let $P = X_c^T X_c + S_m$

$$\underset{\sim}{S} = \underset{\sim}{S}_c + \underset{\sim}{S}_m$$

Then $Q(\theta, \theta^t)$ has the form:

$$\ldots \operatorname{tr} \Sigma^{-1}\left(P - \underset{\sim}{S}\underset{\sim}{\mu}^T - \underset{\sim}{\mu}\underset{\sim}{S}^T\right) \ldots$$

In full:

$$Q(\underset{\sim}{\theta}, \underset{\sim}{\theta}^t) = k - \frac{n}{2}\log|\Sigma| - \frac{n}{2}\underset{\sim}{\mu}^T\Sigma^{-1}\underset{\sim}{\mu}$$

$$- \frac{1}{2}\operatorname{tr}\Sigma^{-1}\left(P - \underset{\sim}{S}\mu^T - \mu\underset{\sim}{S}^T\right)$$

– This is the log-likelihood for a $N_2(\mu, \Sigma)$ if P is the SSCP matrix and $\underset{\sim}{S}$ is the sum of $\underset{\sim}{x}$'s.

– Which we know is maximized if

$$\underset{\sim}{\mu} = \hat{\underset{\sim}{\mu}} = \frac{1}{n}\underset{\sim}{S}$$

$$\Sigma_1 = \hat{\Sigma} = \left(P - \frac{1}{n}\underset{\sim}{S}\underset{\sim}{S}^T\right)/n$$

So the M-step is easy!

$$\mu^{t+1} = \hat{\mu} = \frac{1}{n}\underset{\sim}{S} \text{ and } \Sigma^{t+1} = \hat{\Sigma}$$

$$f(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n \mid \mu, \Sigma) = \frac{1}{(2\pi)^{n/2}} \frac{1}{|\Sigma|^{n/2}} \left\{ -\frac{1}{2} \sum_{i=1}^{n} (x_i - \mu)^T \Sigma^{-1} (x_i - \mu) \right\}$$

$$\sum (x_i - \mu)^T \Sigma^{-1} (x_i - \mu)$$

$$\sum (x_i - \bar{x} + \bar{x} - \mu)^T \Sigma^{-1} (x_i - \bar{x} + \bar{x} - \mu)$$

$$= \sum (x_i - \bar{x})^T \Sigma^{-1} (x_i - \bar{x}) + \sum (x_i - \bar{x})^T \Sigma^{-1} (\bar{x} - \mu)$$

$$+ \sum (\bar{x} - \mu)^T \Sigma^{-1} (x_i - \bar{x}) + \sum (\bar{x} - \mu)^T \Sigma^{-1} (\bar{x} - \mu)$$

$$= \text{tr} \sum_{i=1}^{n} \Sigma^{-1} (x_i - \bar{x})(x_i - \bar{x})^T + \underset{\sim}{0} + \underset{\sim}{0} + \text{tr} \sum_{i=1}^{n} \Sigma^{-1} (\bar{x} - \mu)(\bar{x} - \mu)^T$$

$$= n \operatorname{tr} \Sigma^{-1} \hat{\Sigma} + n \operatorname{tr} \Sigma^{-1} (\bar{x} - \mu)(\bar{x} - \mu)^T$$

$$= n \operatorname{tr} \Sigma^{-1} \left\{ \hat{\Sigma} + (\bar{x} - \mu)(\bar{x} - \mu)^T \right\}$$

$$f = \frac{1}{(2\pi)^{pn/2}} \frac{1}{|\Sigma|^{n/2}} \exp\left\{ -\frac{n}{2} \operatorname{tr} \Sigma^{-1} \left\{ \hat{\Sigma} + (\bar{x} - \mu)(\bar{x} - \mu)^T \right\} \right\}$$

$$\ln f = a - \frac{n}{2} \ln|\Sigma| - \frac{n}{2} \operatorname{tr} \Sigma^{-1} \left\{ \hat{\Sigma} + (\bar{x} - \mu)(\bar{x} - \mu)^T \right\}$$

Suppose we're missing half the x_2 - for the largest x_1's - we have the x_1's.

Using imaginary data

$$\ln f \doteq a - \frac{n}{2}\ln|\Sigma| - \frac{n}{2}\operatorname{tr}\Sigma^{-1}$$

Recall Sample of n from $N(\mu, \sigma)$

$$\ell(y_i \mid \mu, \sigma) = \log\left(\frac{1}{\sqrt{2\pi}\cdot\sigma} \frac{1}{} \exp\left\{ -\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2 \right\} \right.$$

$$\ell(\underset{\sim}{y}_i \mid \underset{\sim}{\mu}, \Sigma) = \log\left(\frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2}(\underset{\sim}{x}_i - \underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{x}_i - \underset{\sim}{\mu}) \right\} \right.$$

$\mathbb{R}^2$

$$= k - \frac{1}{2}\log|\Sigma| - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)$$

$$l(\underset{\sim}{y}_1 \dots \underset{\sim}{y}_n \mid \mu, \Sigma) =$$

$$= k' - \frac{n}{2} \log|\Sigma| - \frac{n}{2} \operatorname{tr}\left(\Sigma^{-1}\left(\hat{\Sigma} + (\bar{\underset{\sim}{y}} - \mu)(\bar{\underset{\sim}{y}} - \mu)^T\right)\right)$$

n_c = n complete n_m = n missing y_2

$$n = n_c + n_m$$

$$= b' - \frac{n_c}{2} \log|\Sigma| - \frac{n_c}{2} \operatorname{tr}(\Sigma^{-1}(\Sigma$$

$$E\left(l(\underset{\sim}{y}_1 \dots y_2 \mid \mu, \Sigma) \mid \underset{\sim}{x}_0, \mu_t, \Sigma_t\right)$$

Because l is an exp. family $\Rightarrow$ y is the canonical variable

ℓ is linear in y.

$$E(\ell(y; \mu, \Sigma) \mid z) = \ell(E(y \mid z))$$

$$E(\underset{\sim}{y} \mid \underset{\sim}{y}_0, \Sigma, \mu)$$

For an exponential family:

$$\ell(\underset{\sim}{y} \mid \underset{\sim}{\theta}) = \{\underset{\sim}{\eta}(\underset{\sim}{\theta}) \cdot \underset{\sim}{T}(\underset{\sim}{y}) - A(\underset{\sim}{\theta}) + B(\underset{\sim}{y})\}$$

$$E(\ell(y \mid \underset{\text{at } \hat{}}{\theta}) \mid y_0, \theta_t)$$

$$= E\left(\eta(\theta)\cdot T(y) - A(\theta) + b \mid y_0, \underset{\sim}{\theta}_t\right)$$

$$= \eta(\theta)\, E(T(y) \mid y_0, \theta_t) - A(\theta)$$

Need $T(y)$

$$f(y \mid \underset{\sim}{\mu}, \Sigma) = \frac{1}{(2\pi)^{p/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu)' \Sigma^{-1}(x-\mu)\right\}$$

$$\ell f = k - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}(y-\underset{\sim}{\mu})' \Sigma^{-1}(y-\underset{\sim}{\mu})$$

$$= k - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}\operatorname{tr}\Sigma^{-1}(yy' - 2\mu y' + \mu\mu')$$

$$= k - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}\mathrm{tr}\left(\underbrace{\Sigma^{-1}yy'}_{\Sigma,\mu} - \underbrace{2\Sigma^{-1}\mu y'}_{\Sigma^{-1}\mu,y} + \Sigma^{-1}\mu\mu'\right)$$

$$= k - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}\mathrm{tr}\,\Sigma^{-1}\mu\mu' - \frac{1}{2}\mathrm{tr}\,\Sigma^{-1}\left(\underset{\sim}{y}\underset{\sim}{y}^{T} - 2\mu y'\right)$$

When y_2 is missing we need

$$E(y_2 \mid y_1, \underset{\sim}{\theta}),\quad E(y_1 y_2 \mid y_1, \underset{\sim}{\theta}),\quad E(y_2^2 \mid y_1, \underset{\sim}{\theta})$$

$$E(y_2 \mid y_1, \theta) = (y_1 - \mu_1)\frac{\sigma_{12}}{\sigma_{11}} + \mu_2$$

$$E(y_1 y_2 \mid y_1, \theta) = y_1 E(y_2 \mid y_1, \theta)$$

$$= y_1 (y_1 - \mu_1) \frac{\sigma_{12}}{\sigma_{11}} + \mu_2$$

$$E(y_2^2 \mid y_1, \theta)$$

$$= E(y_2 \mid y_1, \theta)^2 + Var(y_2 \mid y_1, \theta)$$

$$= \left\{ (y_1 - \mu_1)^2 \frac{\sigma_{12}}{\sigma_{11}} + \mu_2 \right)^2 + \sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}$$

Try it as a function of Y

where $Y = \begin{bmatrix} y_{11} & y_{12} \\ \\ y_{n1} & y_{n2} \end{bmatrix}$ 1st obs

nth obs

Let $\underset{\sim}{y}_i = [y_{i1}\ y_{i2}]$ $\mu = (\mu_1\ \mu_2)$

log lik for one observation

$$f(y \mid \mu, \Sigma) = \frac{1}{(2\pi)^{p/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(y-\mu)\Sigma^{-1}(y-\mu)'\right\}$$

$$\ln f = k - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}\operatorname{tr}\Sigma^{-1}(y-\mu)^T(y-\mu)$$

For a sample of n in Y

$$= k - \frac{n}{2}\ln|\Sigma| - \frac{n}{2}\operatorname{tr}\Sigma^{-1}\underbrace{\sum_{i=1}^{n}(y_i-\mu)^T(y_i-\mu)}$$

$$\left(\sum_{i=1}^{n}\left(y_i y_i^T - y_i\mu^T - \mu y_i^T + \mu\mu^T\right)\right)$$

$$= Y^TY - Y^T\mathbf{1}\mu^T - \mu\mathbf{1}^TY + \mu\mu^T$$

$$= Y_0^T Y_0 + Y_m^T Y_m - Y_0^T 1 \mu^T - Y_m^T 1 \mu^T$$

$$- \mu 1^T Y_0 - \mu 1^T Y_m + \mu \mu^T$$

Now $E(\mu 1^T Y_m)$

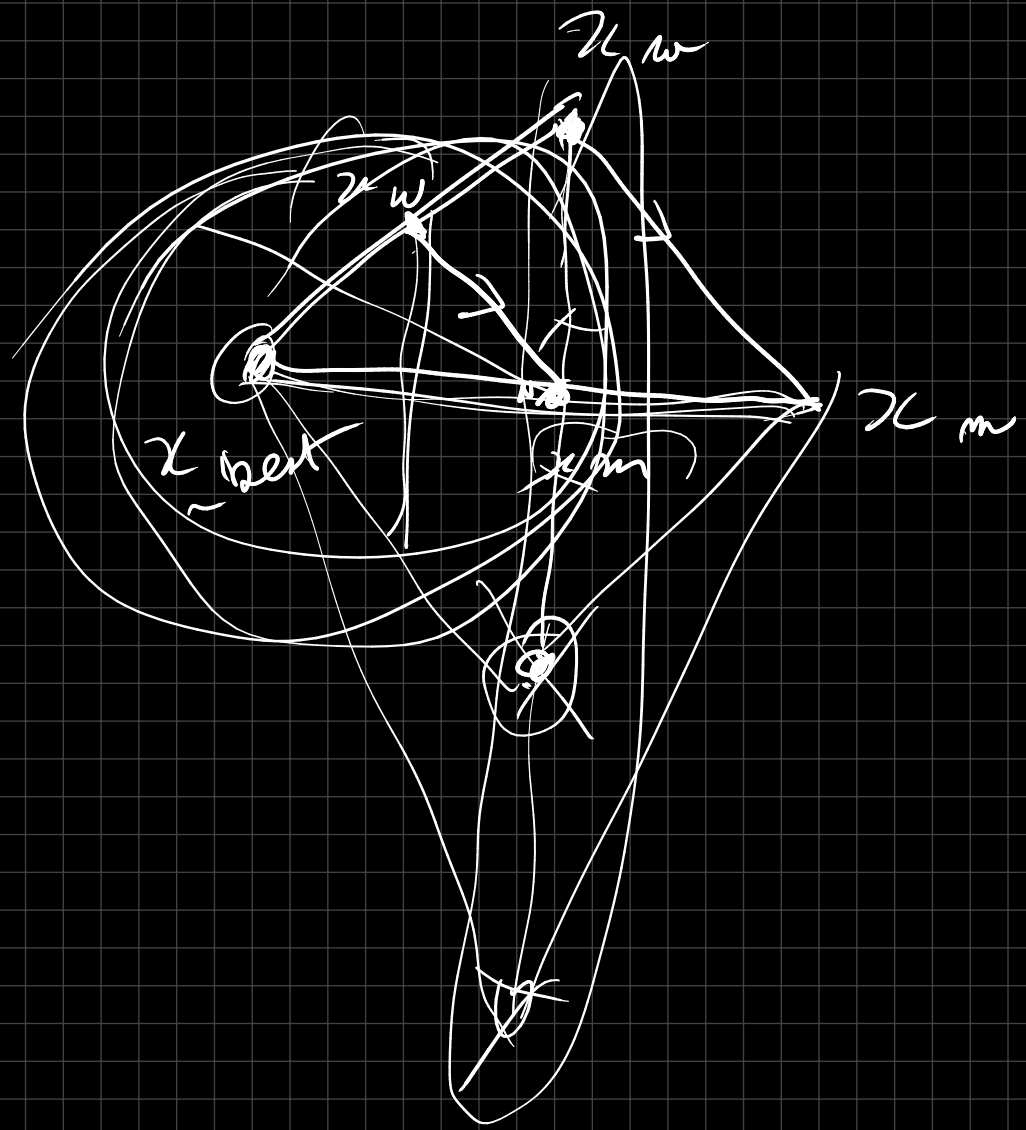
need $E(Y_2 | \mu_1, \Sigma) = \mu_2 + \frac{\sigma_{12}}{\sigma_{11}}(y_1 - \mu_1)$

$$E\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} | \mu_1, \Sigma) = \begin{pmatrix} y_1 \\ \mu_2 + \frac{\sigma_{12}}{\sigma_{11}}(y_1 - \mu_1) \end{pmatrix}$$

and summing $\begin{pmatrix} \sum y_{i1} \\ n_m \mu_2 + \frac{\sigma_{12}}{\sigma_{11}}\left(\sum y_{i1} - n_m \mu_1\right) \end{pmatrix}$

$$E(m \mid Z, \mu_t, \Sigma_t)$$

$$y_m^T y_m = \sum \begin{pmatrix} y_{1i}^2 & y_{1i} y_{2i} \\ & y_{2i}^2 \end{pmatrix}$$


x_w
x_w
x_{ext}
x_m
x_m