Bayesian Ideas and Modern Bayesian Methods – An Introduction

ICPSR 2017 at York University

Georges Monette

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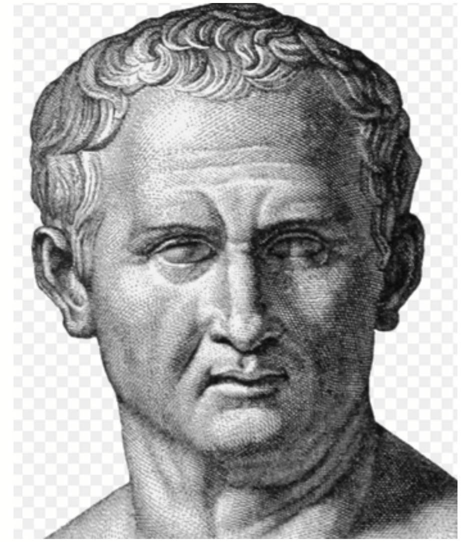
Cicero (106 BCE – 43 BCE)
gave two definitions for *probabile:*



- That which usually happens

- That which is commonly believed

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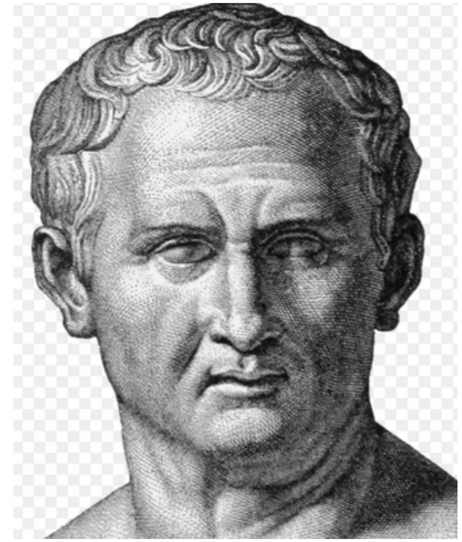

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  - frequentist objective interpretation

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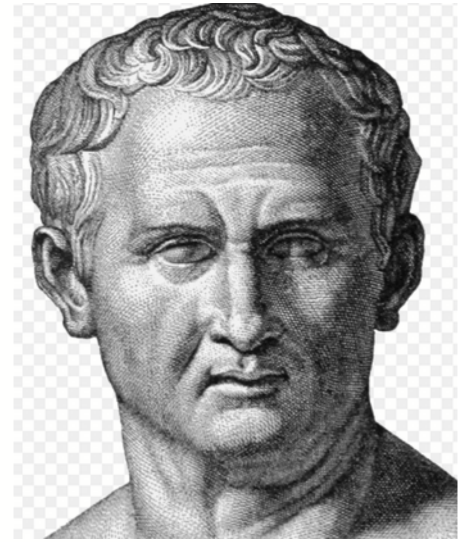

- That which usually happens
  - relative frequency
  - frequentist objective interpretation

- That which is commonly believed
  - degree of belief in a hypothesis
  - Bayesian subjective interpretation

What's the difference between Bayesians and Frequentists? An oversimplification:

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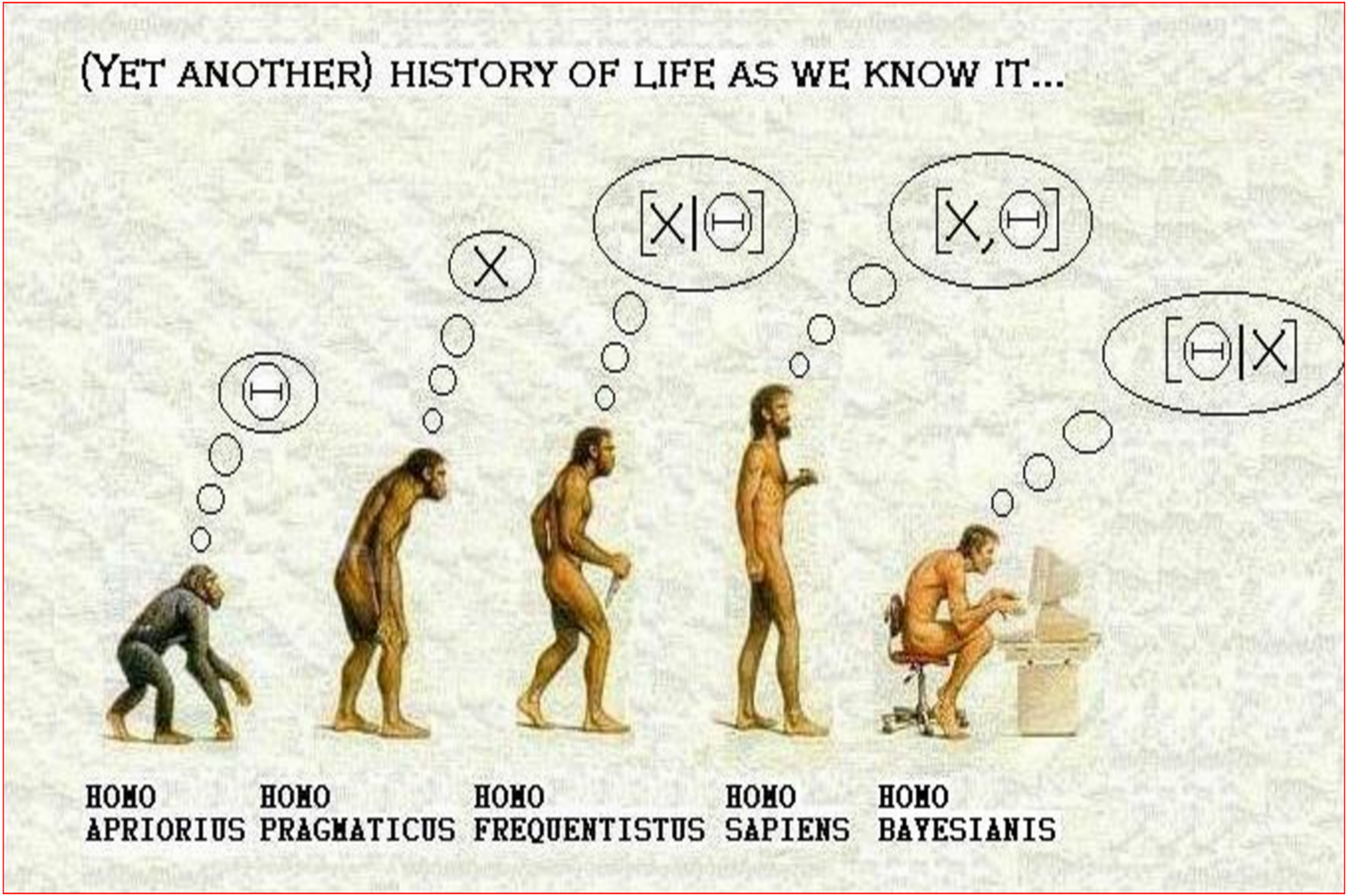
From ideology to utility

- Until recently the debate was mainly philosophical. F methods were much easier
- With improvements in MCMC, B methods have become more feasible and surpass F methods for many complex problems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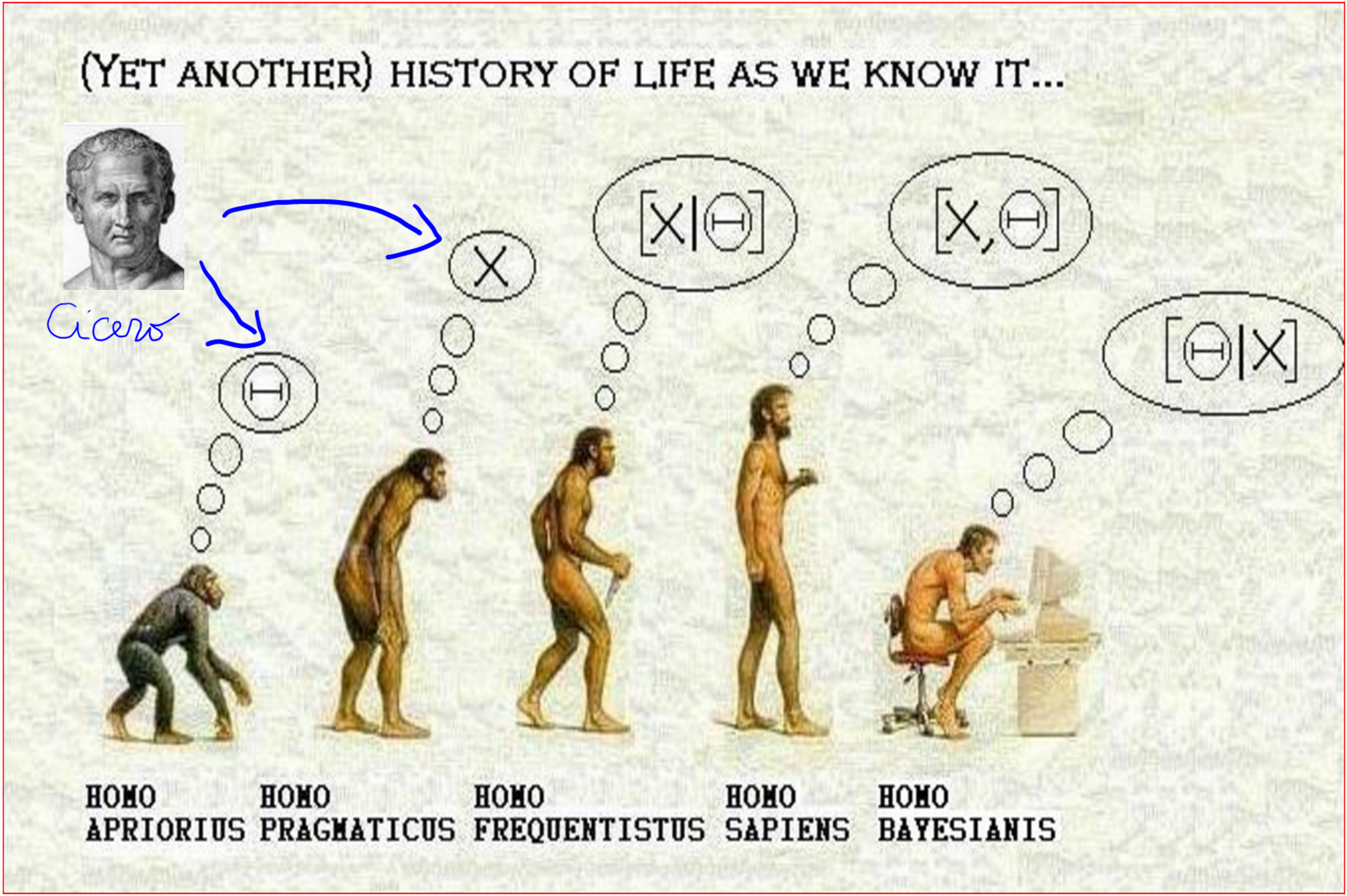
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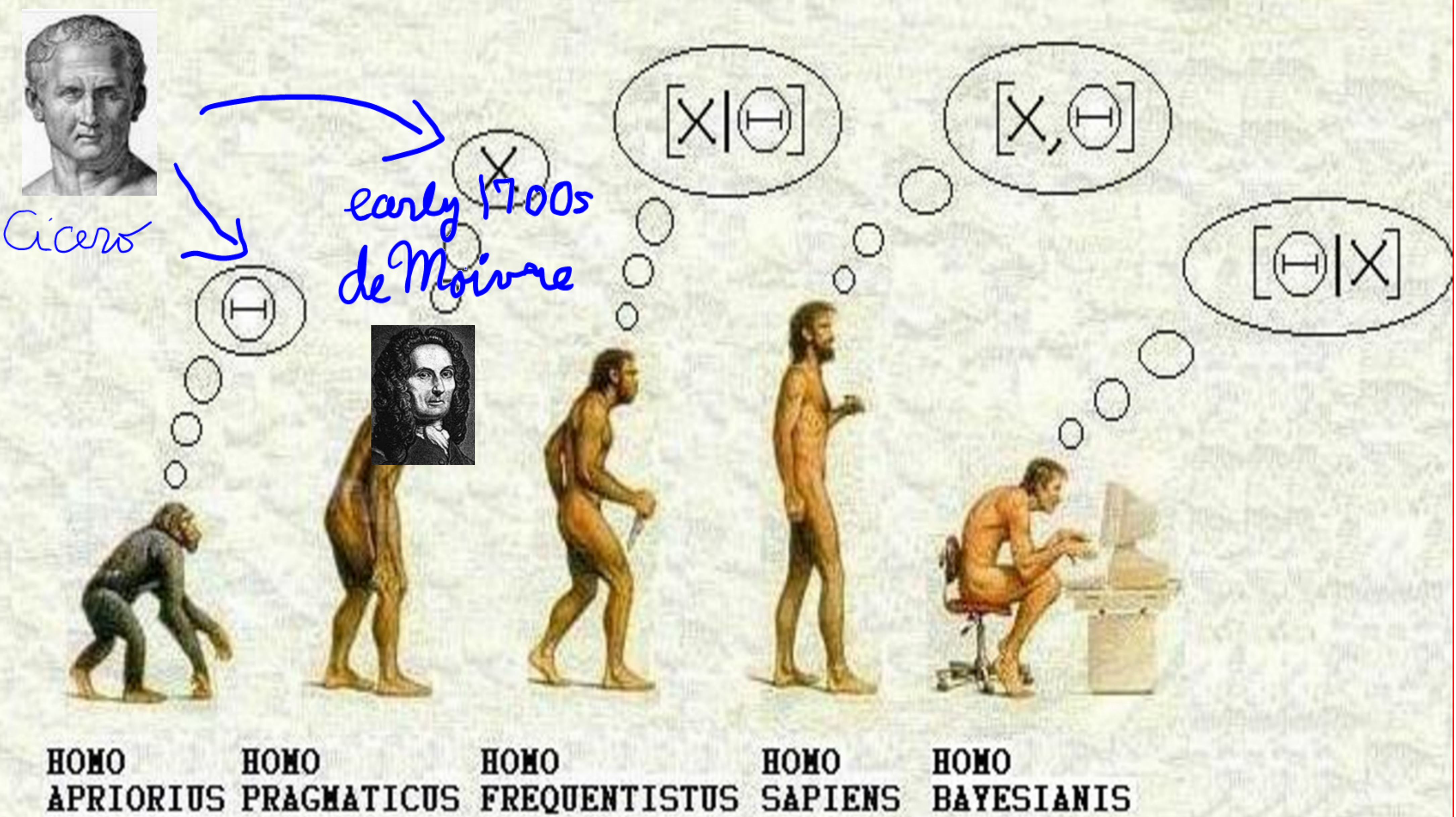

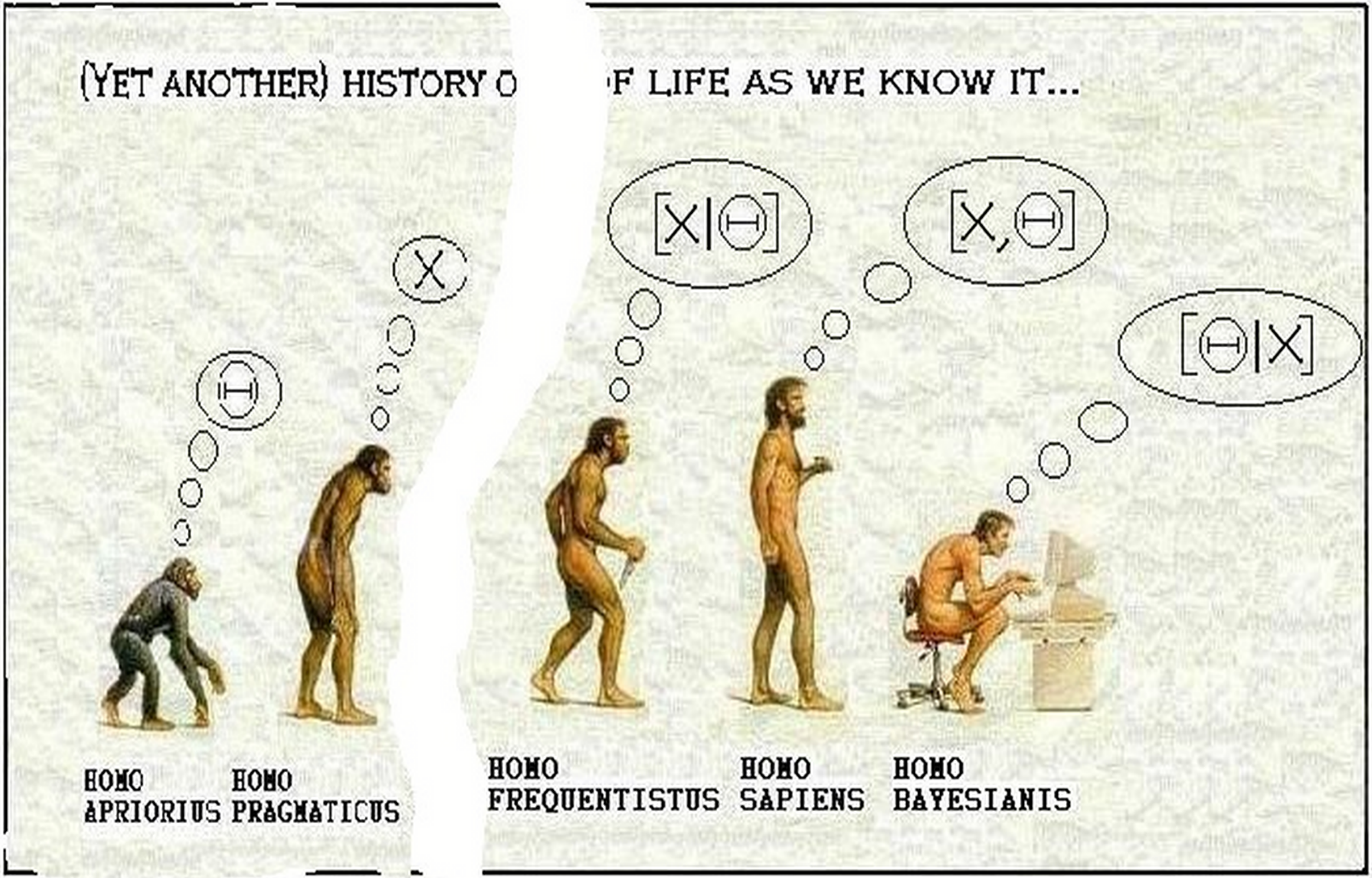

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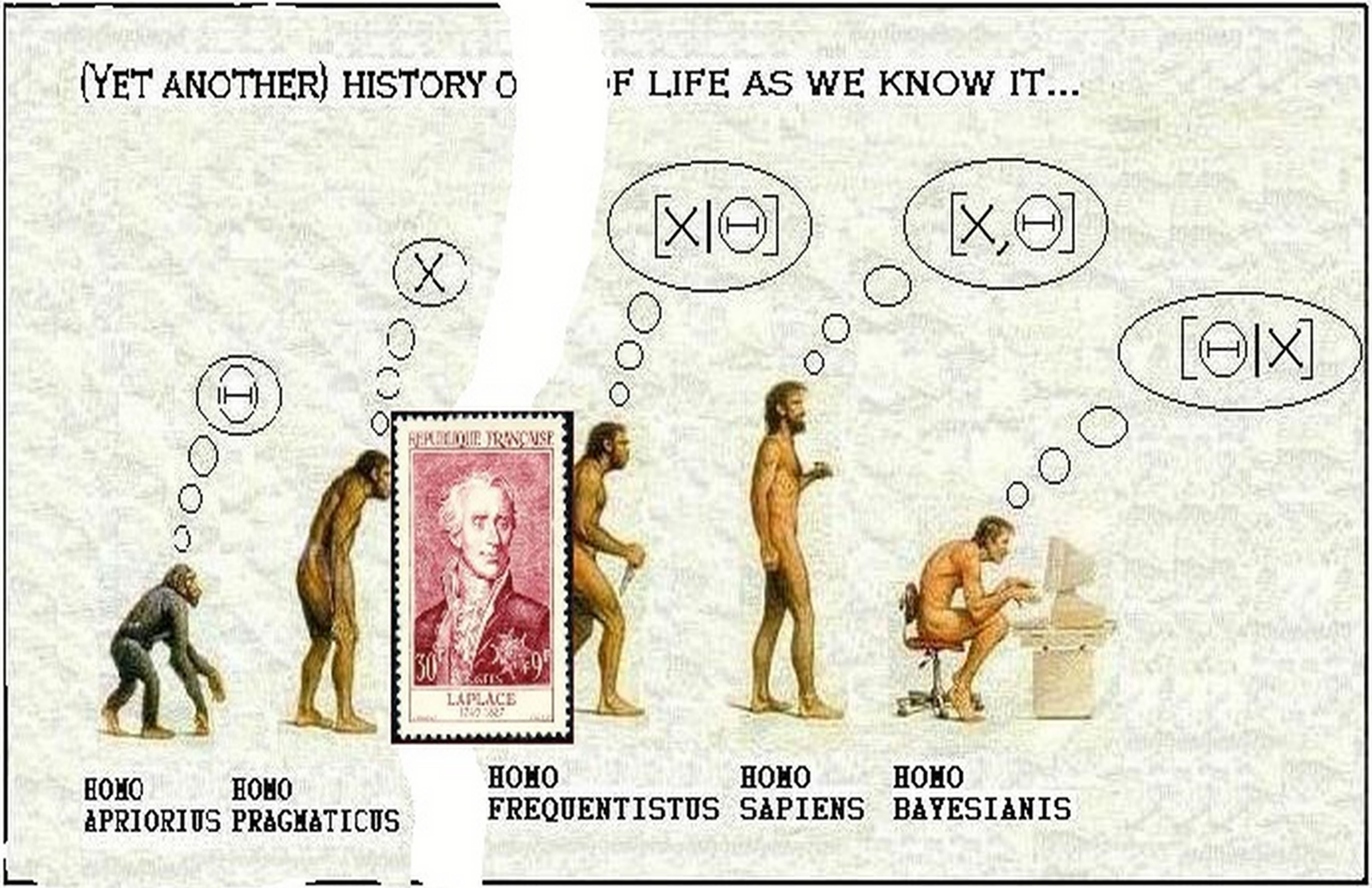

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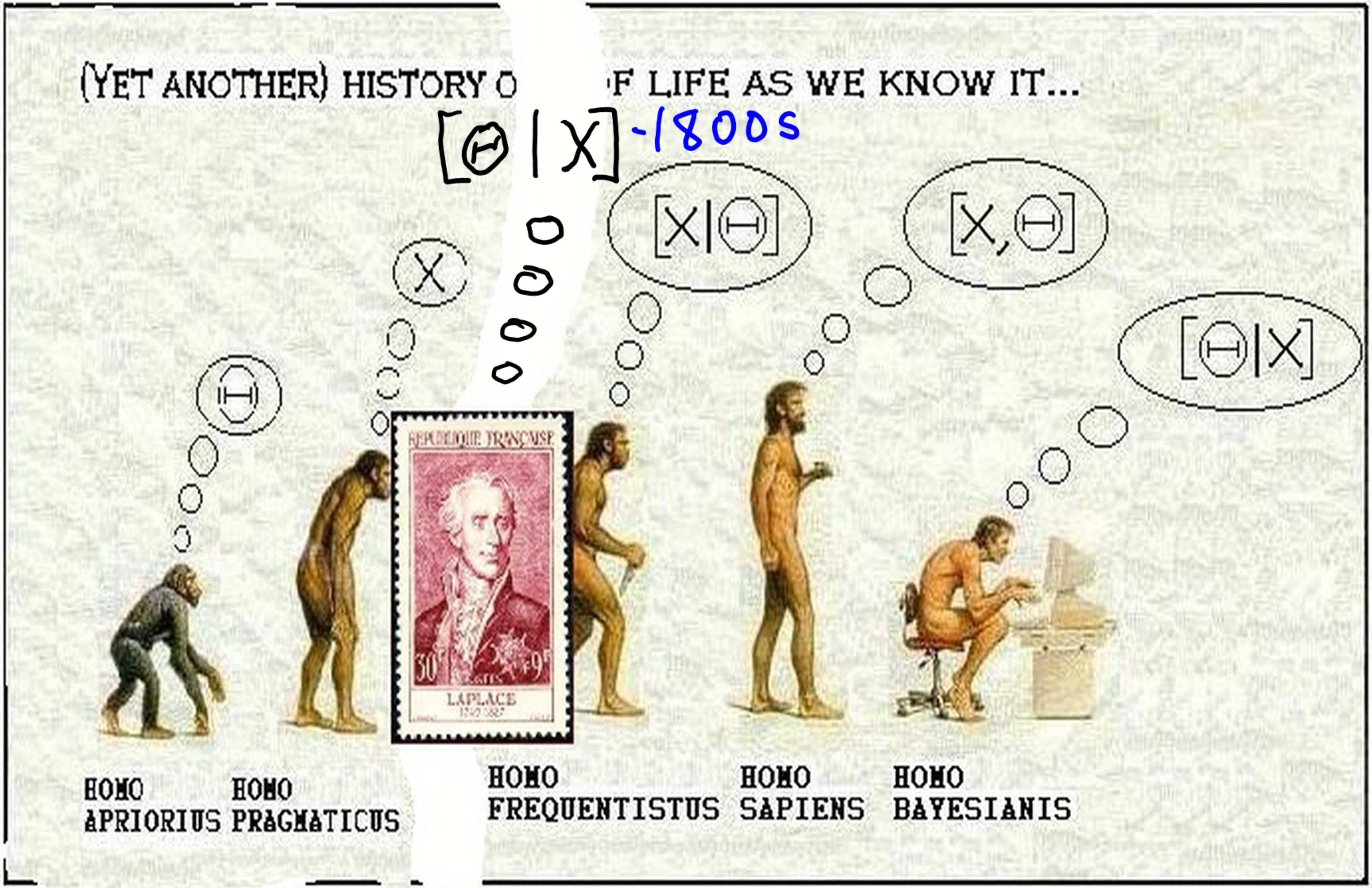

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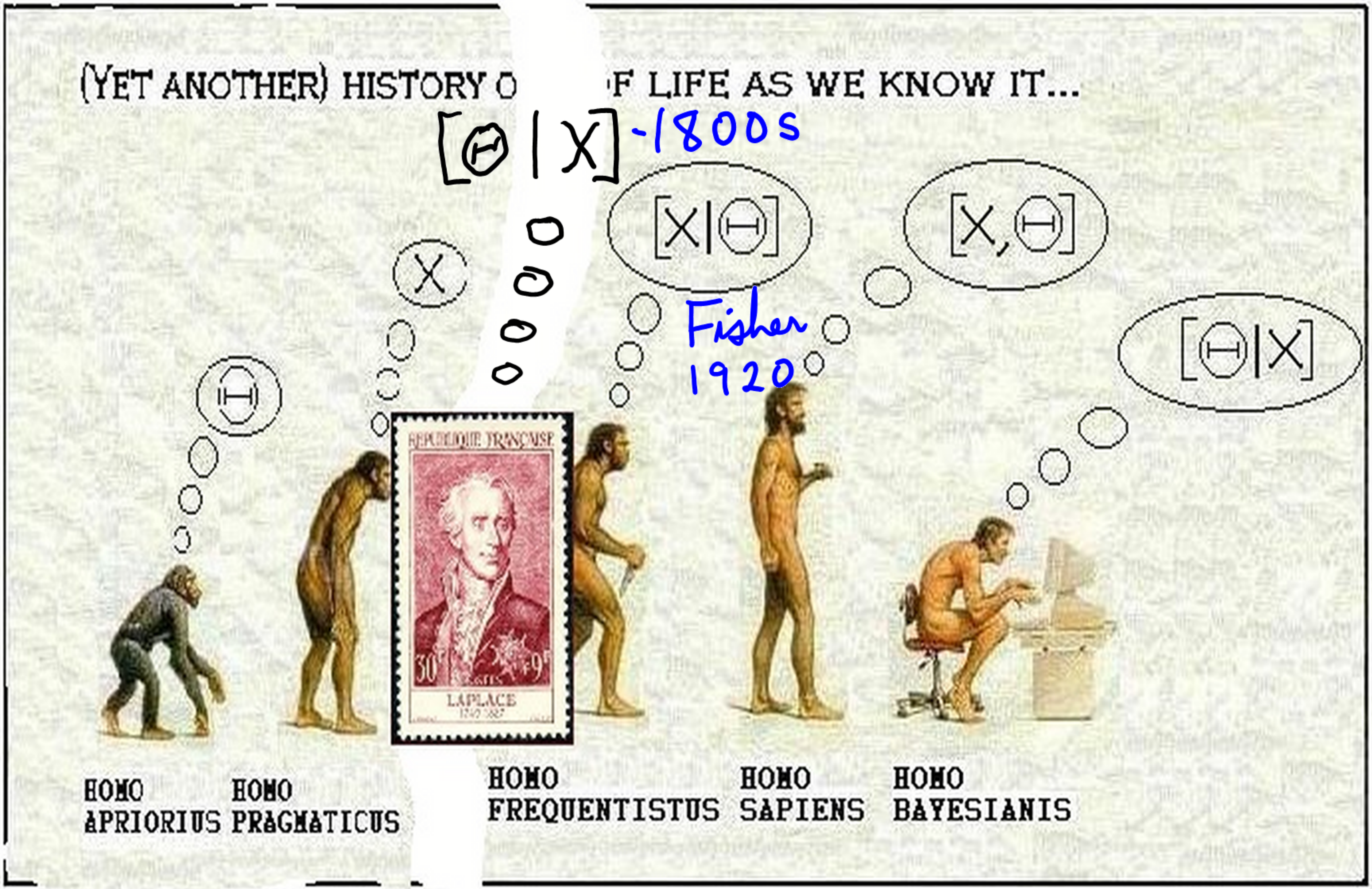
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R. A. Fisher's clever idea

The Lady Tasting Tea and the p-value: a frequentist basis for inference

In 1919, Dr. Muriel Bristol at Rothampsted Experimental Station claimed she could tell whether the milk was poured in first or the tea first.

- Imagine that she was offered 12 cups of tea in random order – 6 prepared milk first and 6 tea first
- She got 10 of the 12 right

What does this tell us about her ability to tell the difference?

Rationale behind the p-value

How can we quantify the evidence that she can tell the difference?

- Pretend that she can't tell the difference: 'null hypothesis' H_0
- The probability of getting 10 out 12 right is $p(y|H_0) = 0.038961$
- But the probability of any single outcome, even one consistent with H_0, might be very small and might say nothing against H_0
- Fisher's idea: use the *tail probability*, the probability of y as or more extreme than the observed value of y

p-value:

$$\begin{aligned} \Pr\left(y^+|H_0\right) &= p(y=10|H_0) + p(y=12|H_0) \\ &= 0.038961 + 0.001082 \\ &= 0.040043 \end{aligned}$$

Proof by contradiction/implausibility

Contradiction	**Implausibility**
A implies not B	A implies B is improbable
B true	B is observed
Therefore A is false	Therefore A is unlikely

Courtroom analogy: presumption of innocence

H_0: Innocence

Consider probability of data (evidence) | innocence

If evidence inconsistent with innocence, then reject innocence and find guilt

Sally Clark

- Young lawyer, gives birth to first son in September 1996
- son dies, apparently of SIDS, at 10 weeks
- second son born a year later
- dies, apparently of SIDS, at 8 weeks
- only evidence of trauma consistent with resuscitation attempts
- charged with two counts of murder

Sir Roy Meadow

- distinguished pediatrician
- as expert witness testifies:
  - probability of one SIDS death: $\frac{1}{8{,}500}$
  - probability of two: $\left(\frac{1}{8{,}500}\right)^2 = \frac{1}{72{,}250{,}000}$
  - 'if she's innocent, the chances of this happening are 1 in 72 million'
- jury convicts Sally Clark of murder in November 1999
- first appeal lost in October 2000
- second appeal succeeds and Sally Clark is released in January 2003
- she dies in 2007 at the age of 42

H_0: Sally is innocent

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Y : 2 children die for no apparent cause

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$$\text{p-value} = \Pr(Y^+ \mid H_0)$$

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Meadow's calculation

$$\approx \frac{1}{8{,}500} \times \frac{1}{8{,}500} = \frac{1}{72{,}250{,}000}$$

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Criticism :

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$$\approx \underbrace{\frac{1}{8{,}500} \times \frac{1}{8{,}500}}_{\text{assumes independence}} = \frac{1}{72{,}250{,}000}$$

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2) $\frac{1}{8,500}$ too small

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Correct p-value is larger - maybe $\frac{1}{10{,}000}$!

So anyways:

$$p < 0.0001$$

Therefore guilty beyond a reasonable doubt.

BUT:

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Do we really want $P(y^+ \mid H_0)$?

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- Must be close!?
- Is $P(Y^+ | H_0)$ a good proxy for $P(H_0 | Y)$?

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$$P(H_0|Y) = \frac{P(H_0, Y)}{P(Y)} = \frac{P(Y|H_0)P(H_0)}{P(Y)}$$

Bayesian tree:

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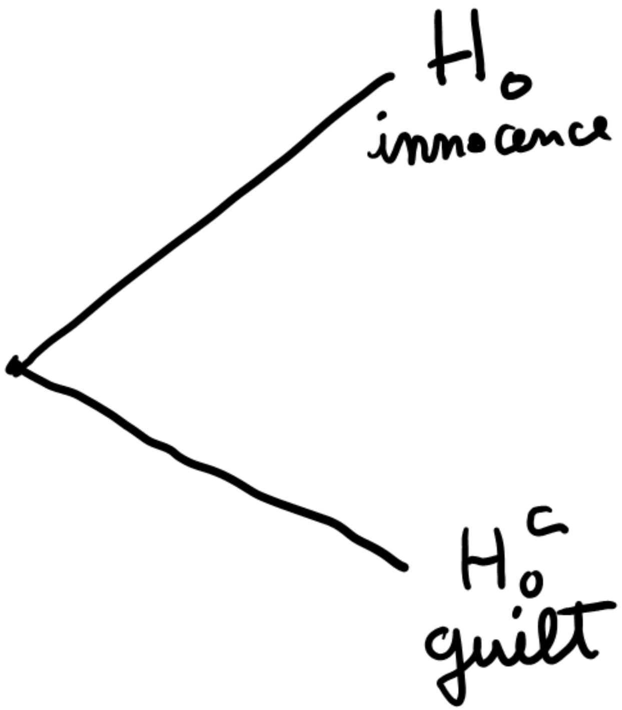

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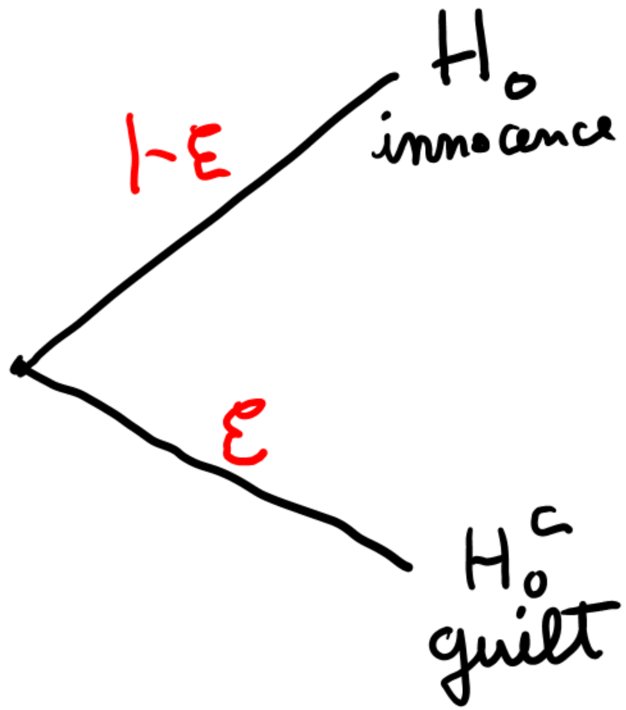

ε = very small number

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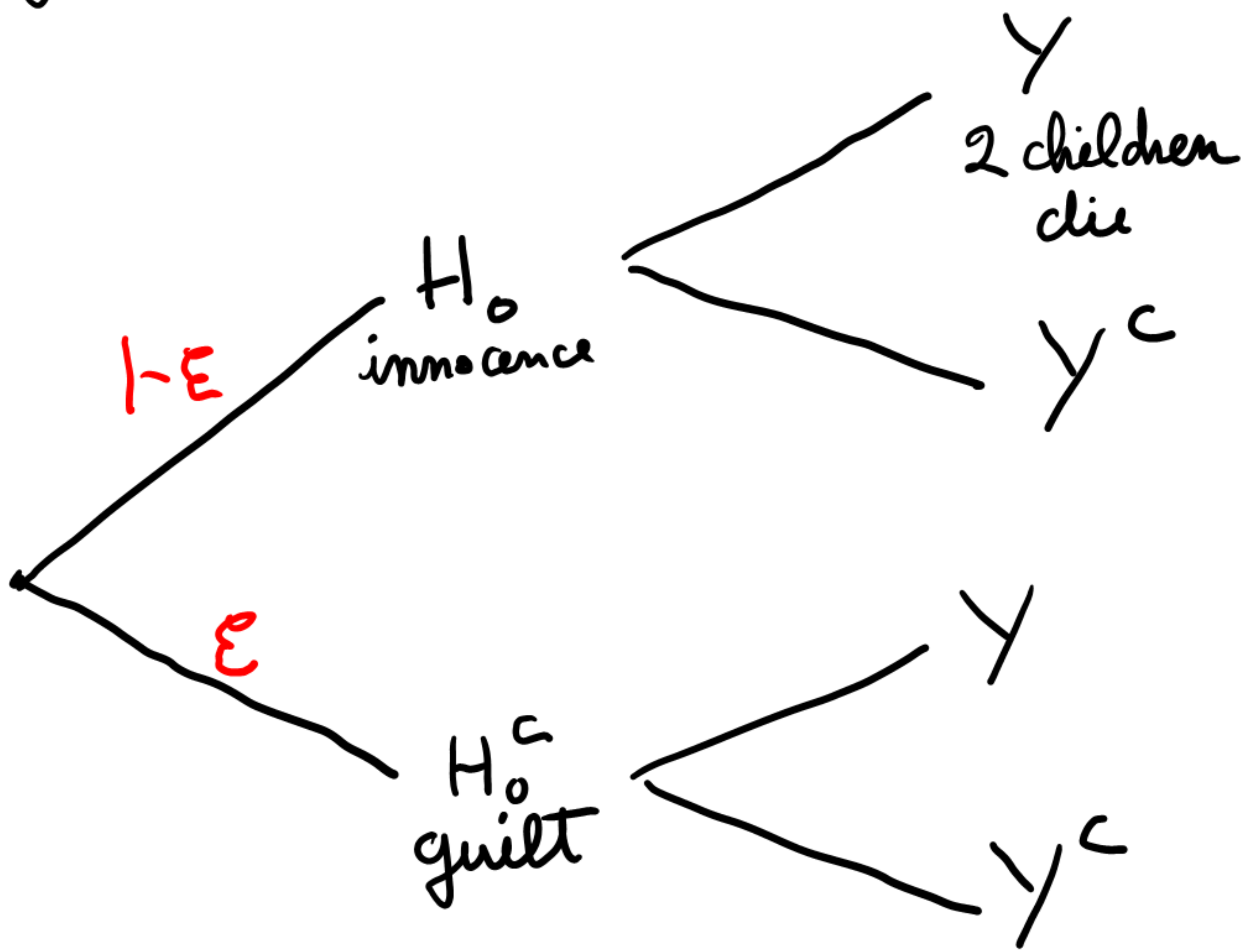

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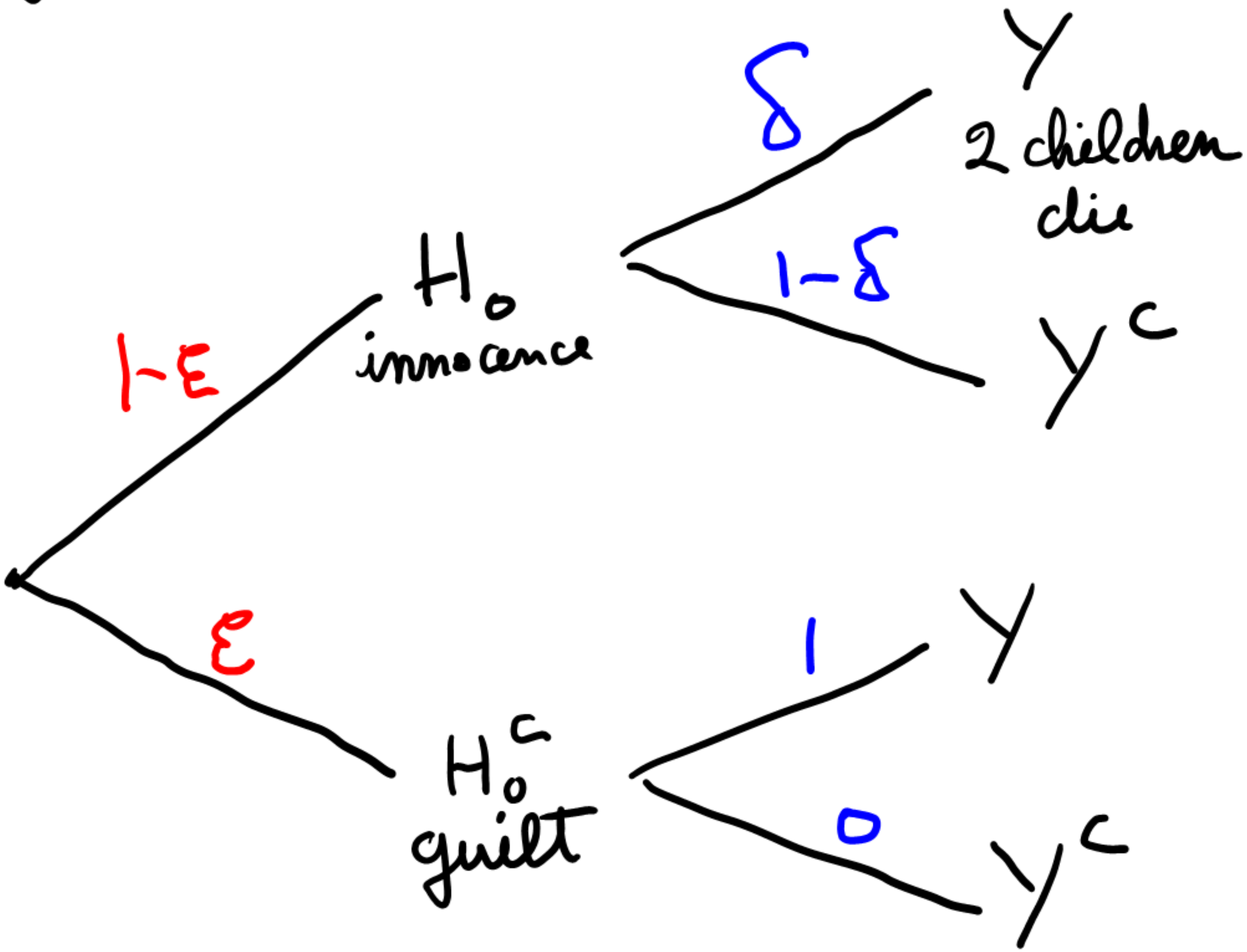

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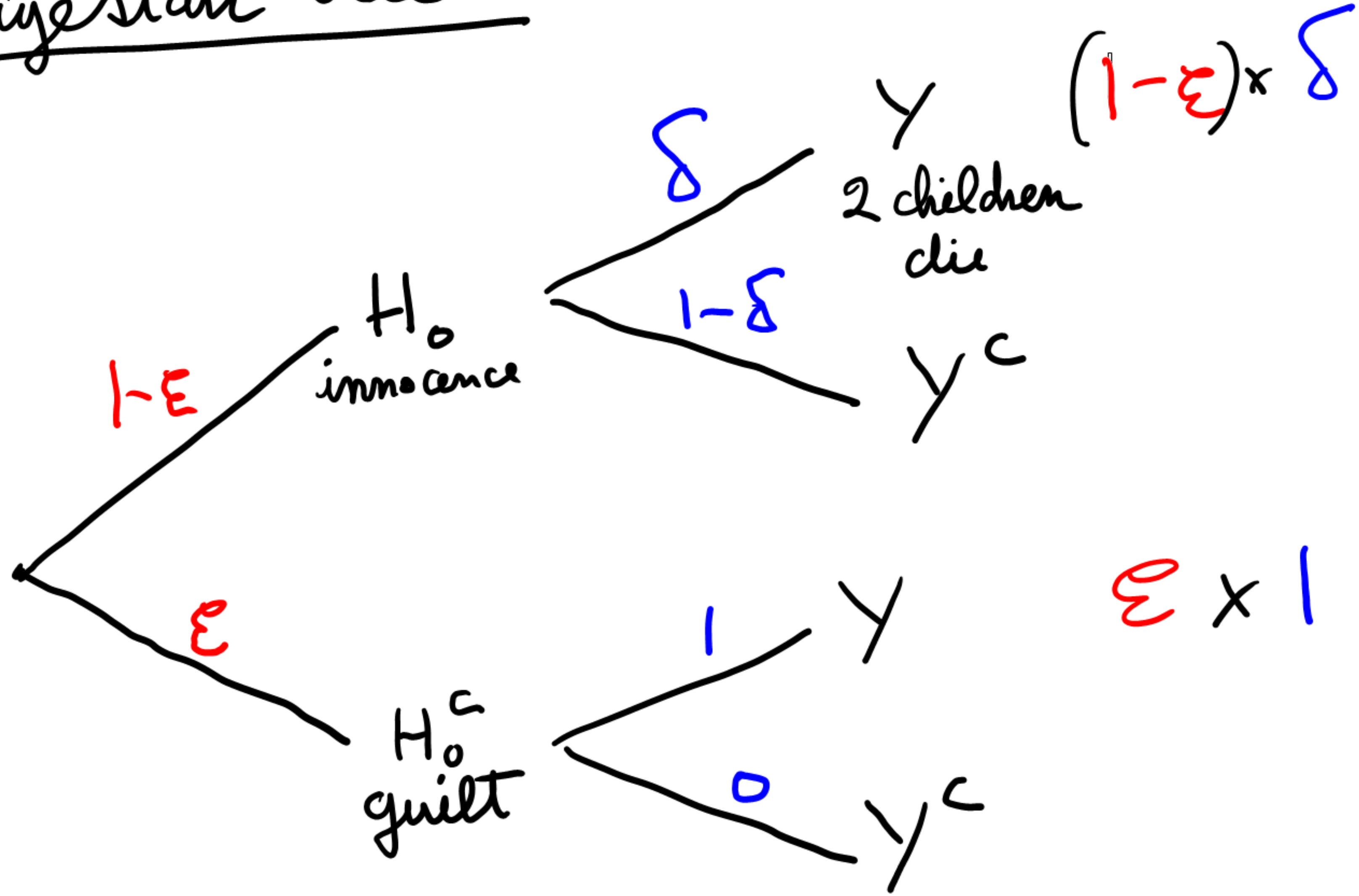

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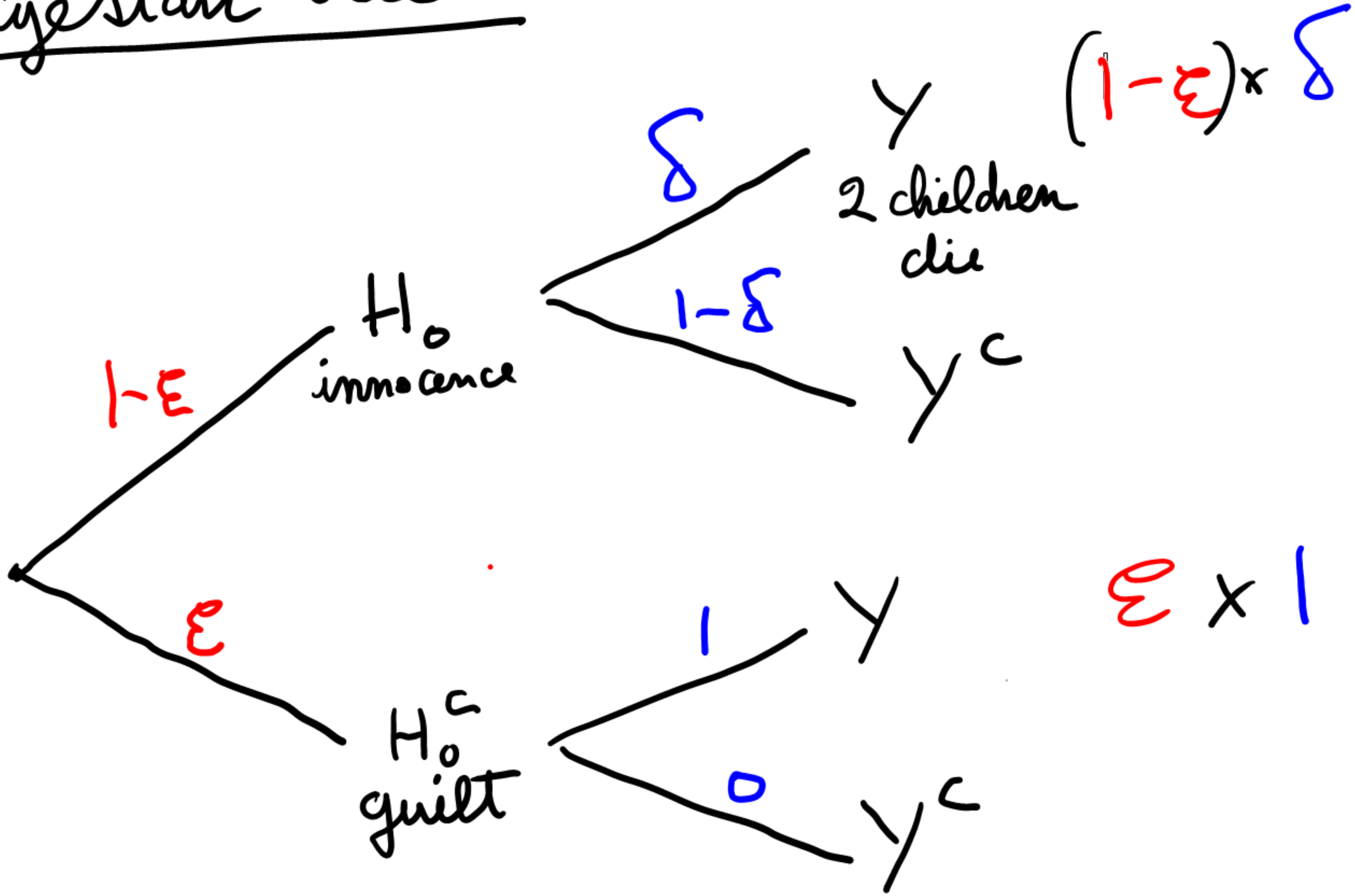

$$P(H_0 \mid Y) = \frac{(1-\varepsilon)\delta}{(1-\varepsilon)\delta + \varepsilon} \approx \frac{\delta}{\delta + \varepsilon}$$

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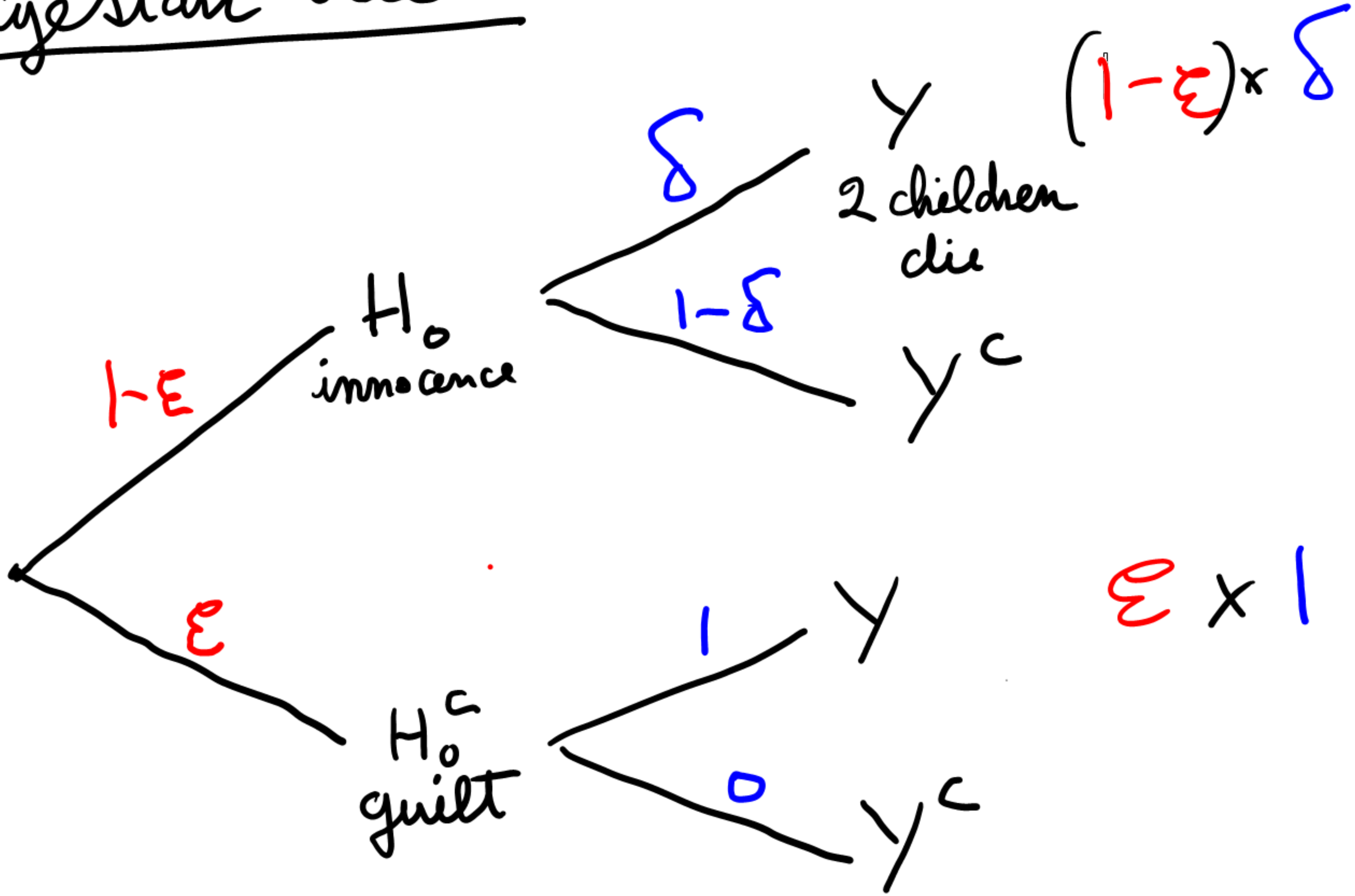

$$P(H_0|Y) = \frac{(1-\varepsilon)\delta}{(1-\varepsilon)\delta+\varepsilon} \approx \frac{\delta}{\delta+\varepsilon} \approx \frac{1}{2}?$$

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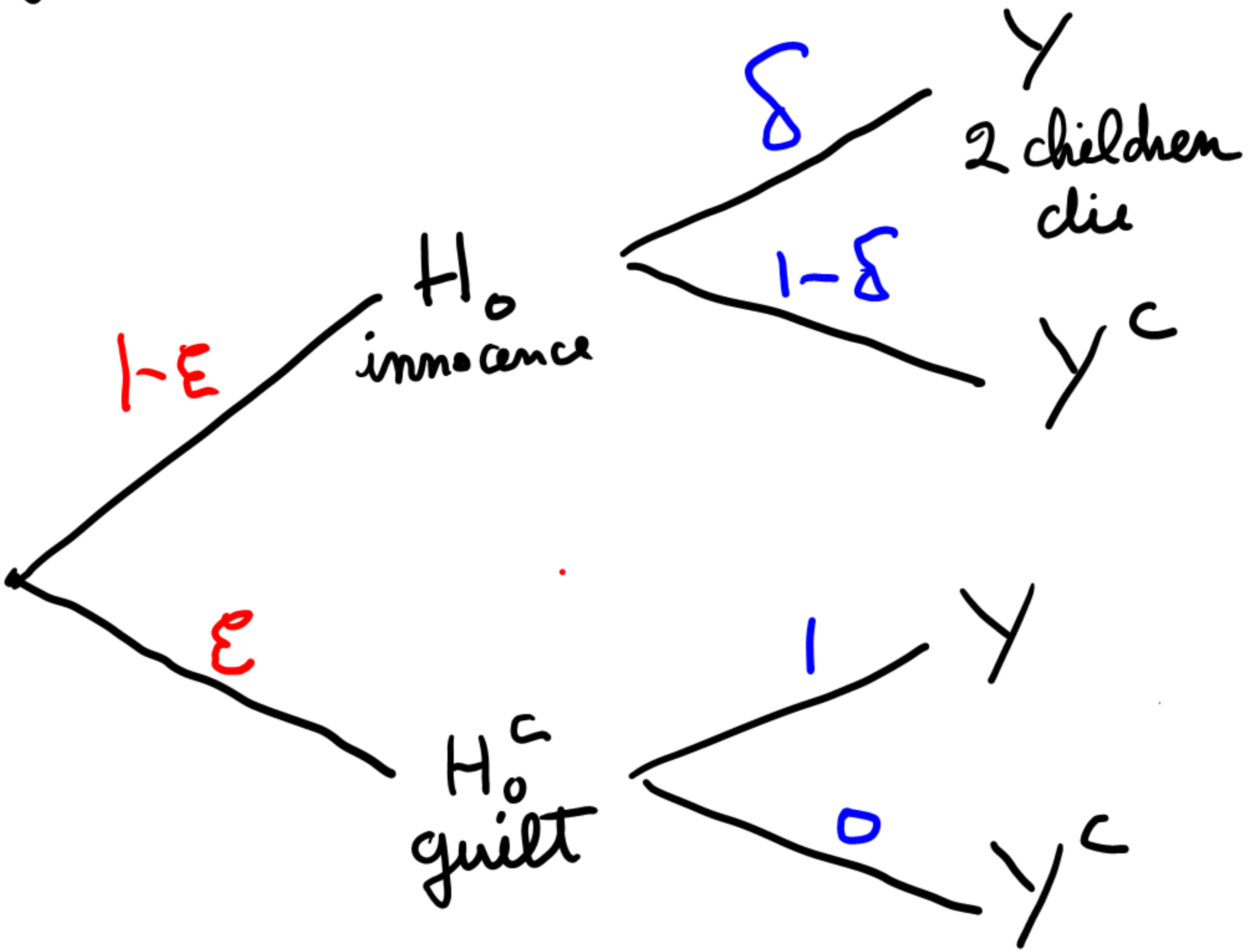

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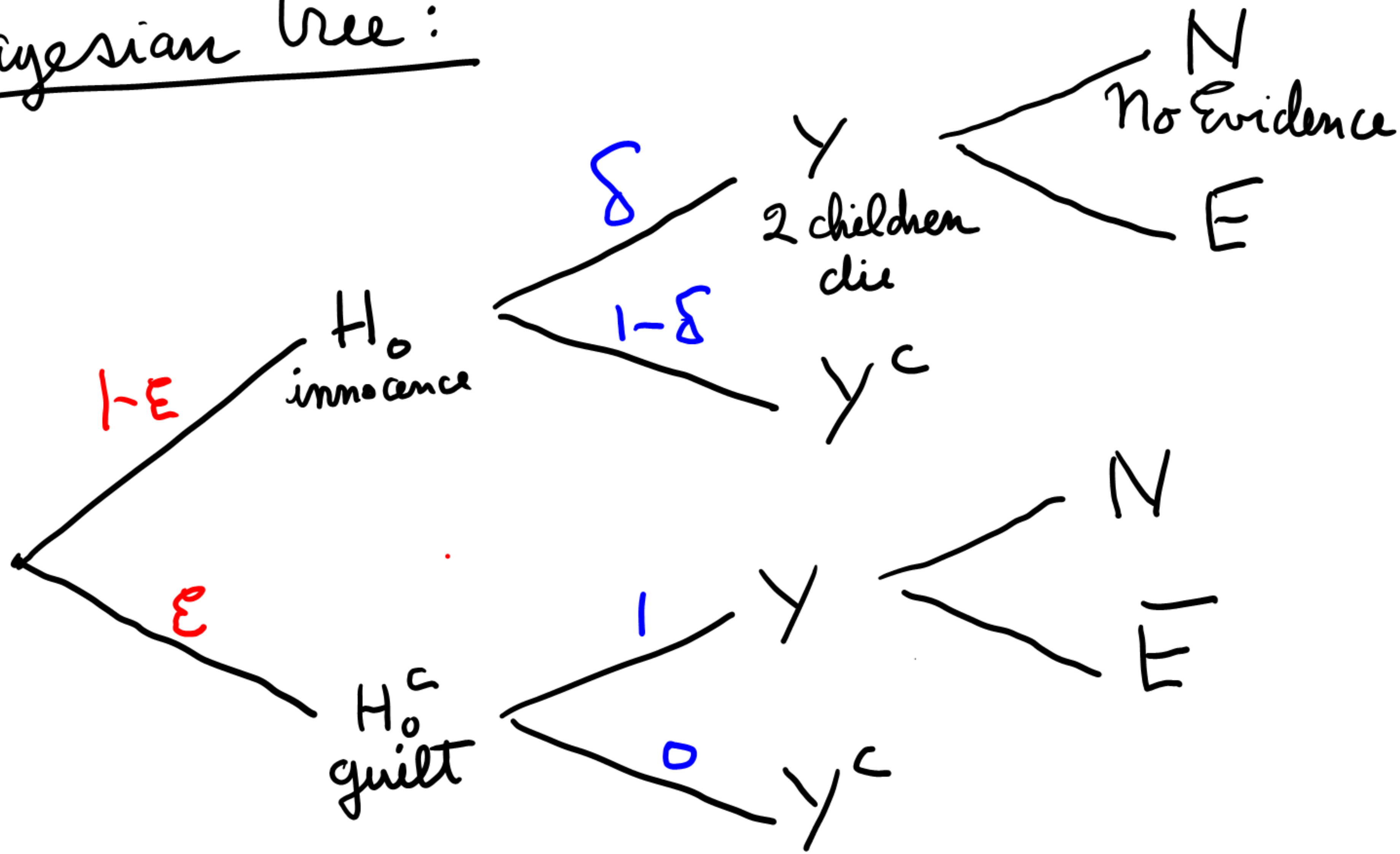

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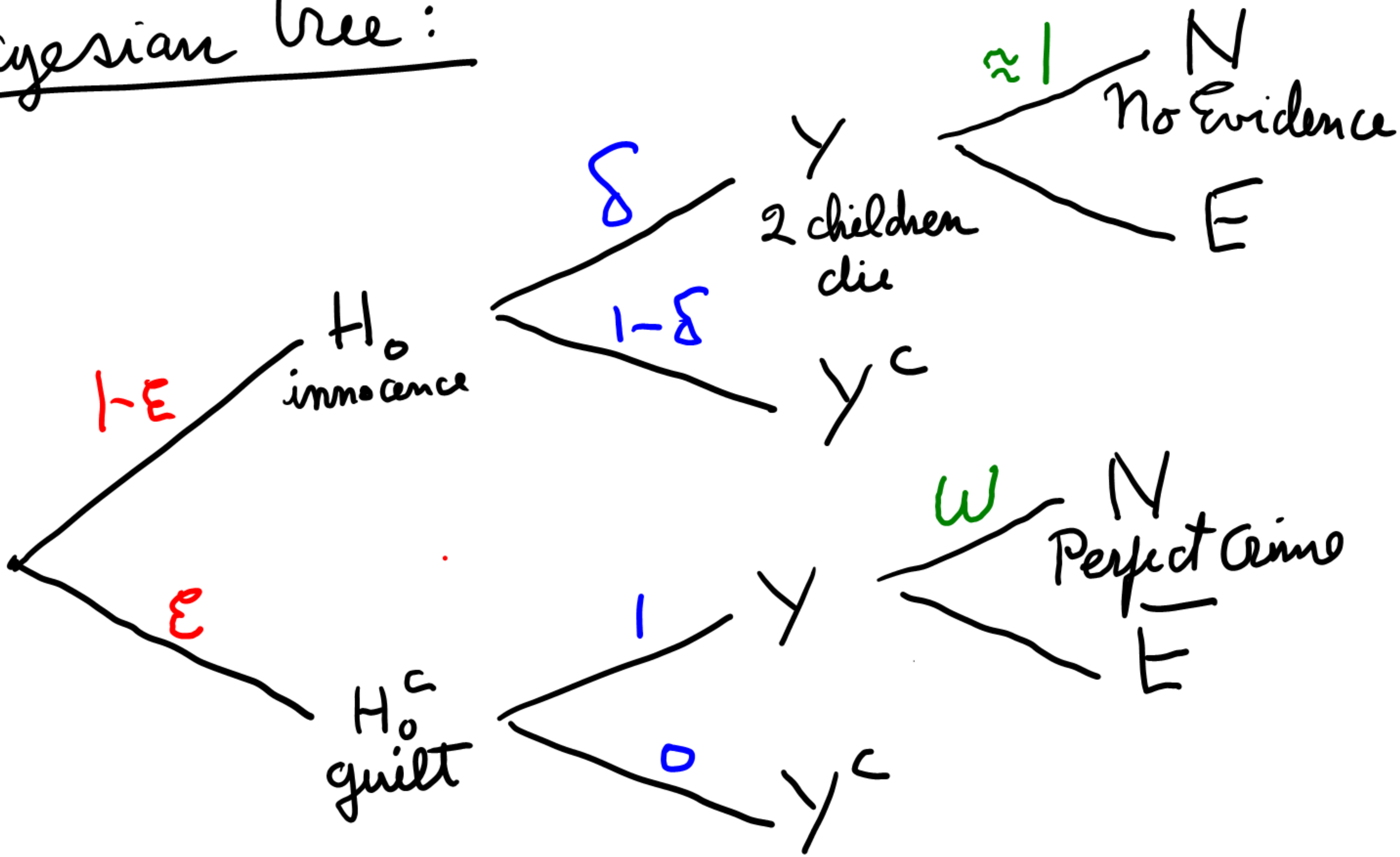

ω = another small number

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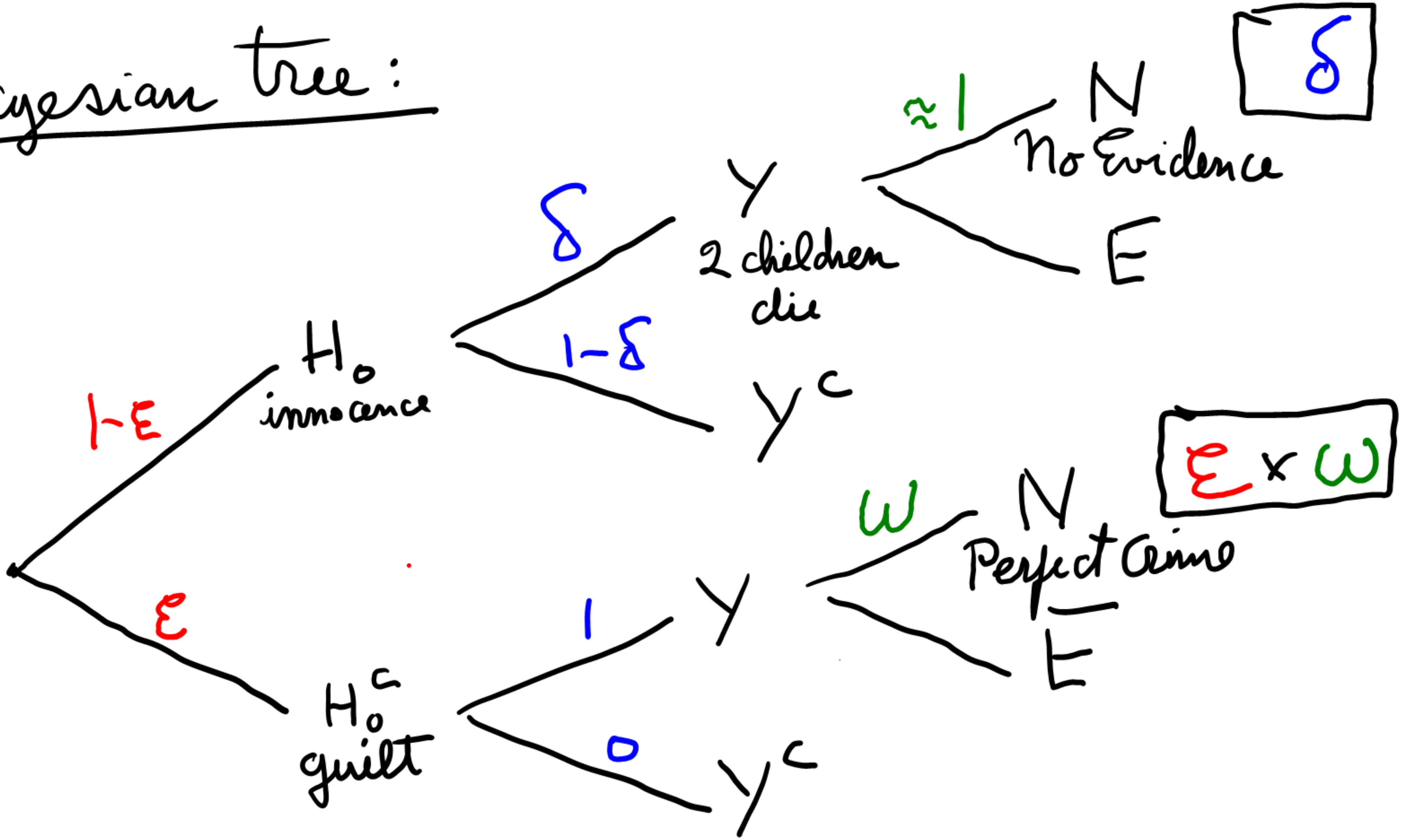

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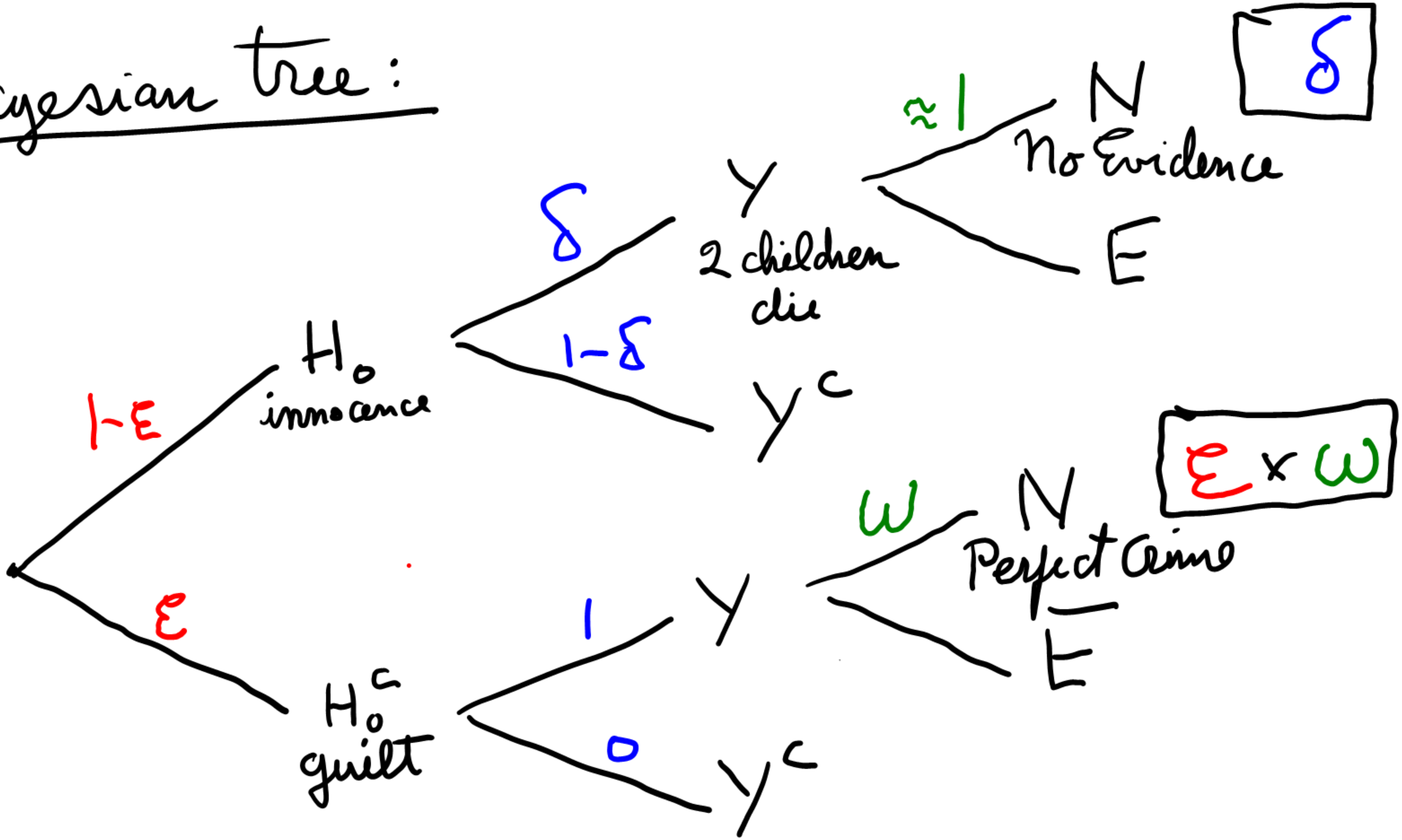

$$\Pr(H_0 \mid Y, N) \approx \frac{\delta}{\delta + \varepsilon \times \omega}$$

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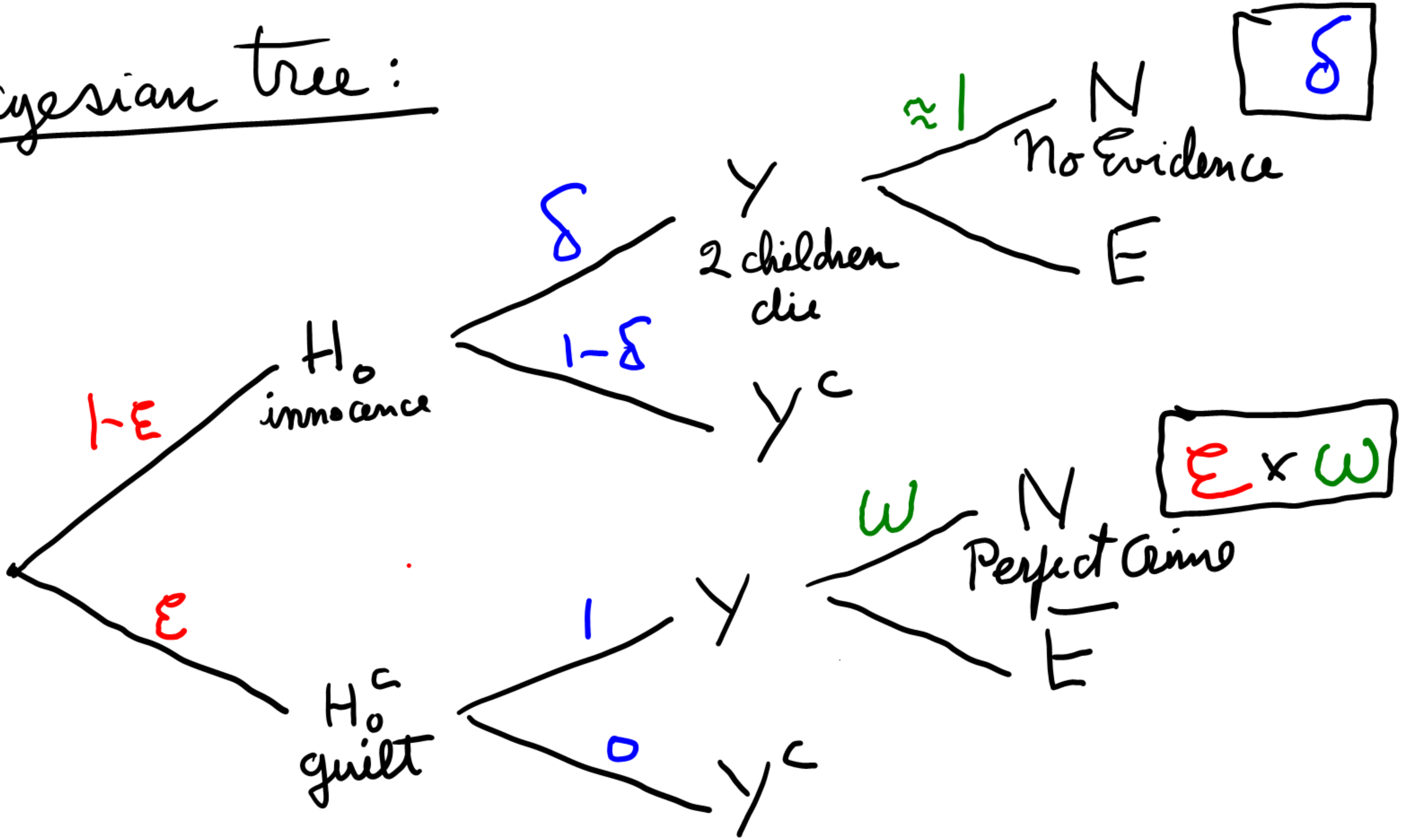

$$\Pr(H_0 \mid Y, N) \approx \frac{\delta}{\delta + \varepsilon \times \omega} = \text{close to } 1$$

Sally Clark is

innocent

beyond a reasonable doubt.

GUILTY

GUILTY

INNOCENTE

Proof beyond a reasonable doubt?

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A very small value of the probability of innocence ‘given’ the evidence?

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$$\text{Probability(Innocence} \mid \text{Evidence)?}$$

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$$\text{Probability(Innocence} \mid \text{Evidence)?}$$

What did Roy Meadow learn from stats?

How to calculate:

$$\text{Probability(Evidence}^{+} \mid \text{Innocence)}$$

the *p-value*, the probability of obtaining evidence *as or more contradictory* assuming innocence.

The fundamental neurosis of statistics

- We really want $p(\theta|y)$ but we'd have to accept $p(\theta)$
- So we give the world $p(y^+|\theta))$
  - Most people quietly think it's a proxy for $p(\theta|y)$
  - if not, what in the world could it be?
- Gigerenzer:
  - the confusion created by this unresolved conflict among statisticians, which is both suppressed and inherent in statistics textbooks, leads to a systemic neurosis in science for which the ritual of NHST is a form of conflict resolution – like compulsive hand washing – which makes it resistant to logical arguments
- One is most strongly committed to the beliefs one does not understand

$$P(Y^+ \mid H_0)$$

was a poor proxy for

$$P(H_0 \mid Y)$$

If p-values are so bad, why do we still use them?

- For many common problems they are consistent with Bayesian answers
    - see Fiducial and Structural Inference

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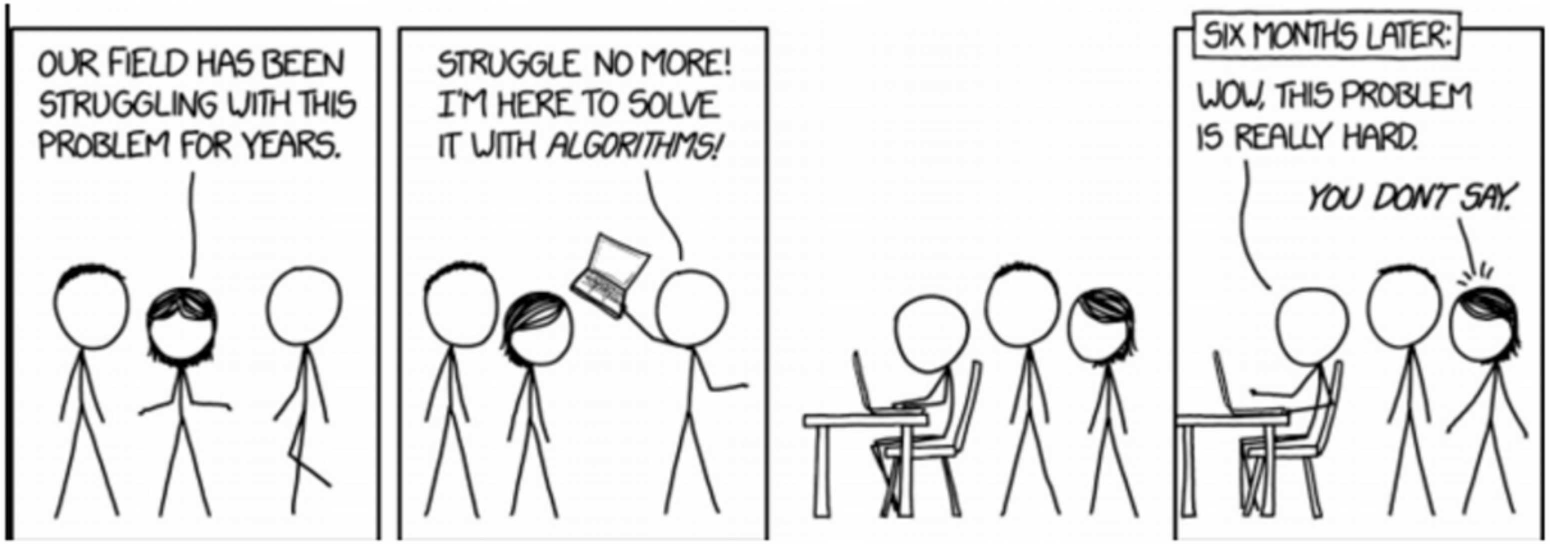
  - This is changing thanks to MCMC
- You don’t need to justify a choice of priors
- Fisher finally cautioned to use p-value only if there is little other information on H_0

If you feel puzzled, you are not alone: (Reid, 2017)



From a 1996 interview:

Nancy Reid: Why is conditional inference so hard?

Sir David Cox: I expect we're all missing something but I don't know what it is.

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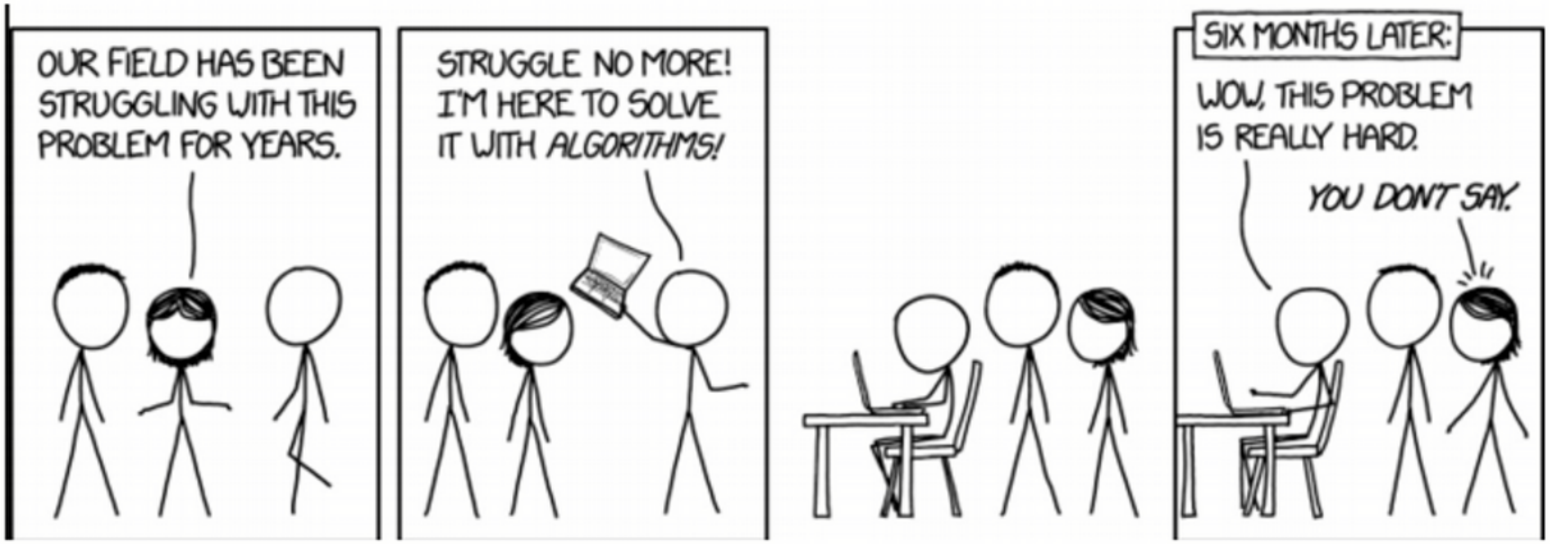

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– cited in 2017

Philosophical ~~Basic~~ problem

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Given a model $P(X|\theta)$

~~Basic~~ Philosophical problem

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To get $P(\theta|X)$

you need to be willing

to specify $P(\theta)$

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Given a model $P(X|\theta)$

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Then $P(X,\theta) = P(X|\theta)P(\theta)$

and $P(\theta|X) = \dfrac{P(X,\theta)}{P(X)}$

Philosophical ~~Basic~~ problem

Given a model $P(X|\theta)$ Model

To get $P(\theta|X)$

you need to be willing to specify $P(\theta)$ Prior

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Posterior

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Posterior

- You need a prior to get a posterior.
- Can we justify a particular prior?

Frequentists only use $P(X \mid \theta)$ and don't need $P(\theta)$

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Your methods are subjective. You have no objective justification for your prior

Frequentists only use $P(X \mid \theta)$ and don't need $P(\theta)$

Your methods are subjective. You have no objective justification for your prior

Your methods may be "objective" but they answer the wrong question $P(Y^+ \mid \theta)$ instead of $P(\theta \mid Y)$

Practical problem:

$$p(X, \theta) = p(X \mid \theta)\, p(\theta)$$

Practical problem:

$$p(X,\theta) = p(X|\theta)\,p(\theta)$$

$$p(\theta|X) = \frac{p(X,\theta)}{p(X)}$$

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$$\int p(x,\theta)\,d\theta$$

$$\theta \in \mathbb{R}^{huge}$$

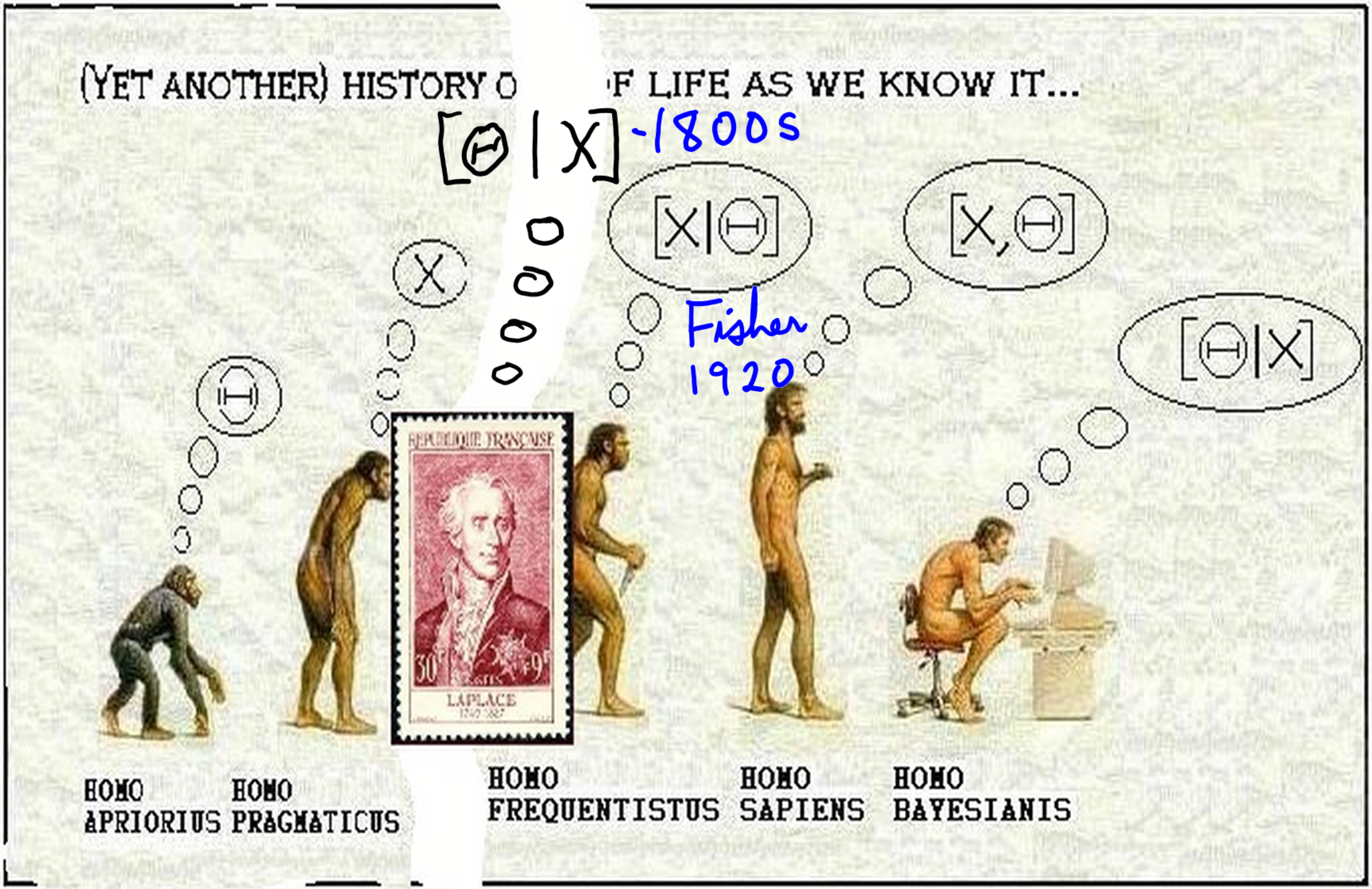
Practical problem:

$$p(x, \theta) = p(x \mid \theta)\, p(\theta)$$

$$p(\theta \mid x) = \frac{p(x, \theta)}{p(x)}$$

$$\int p(x, \theta)\, d\theta$$

If θ has high dimension this becomes easily impossible.



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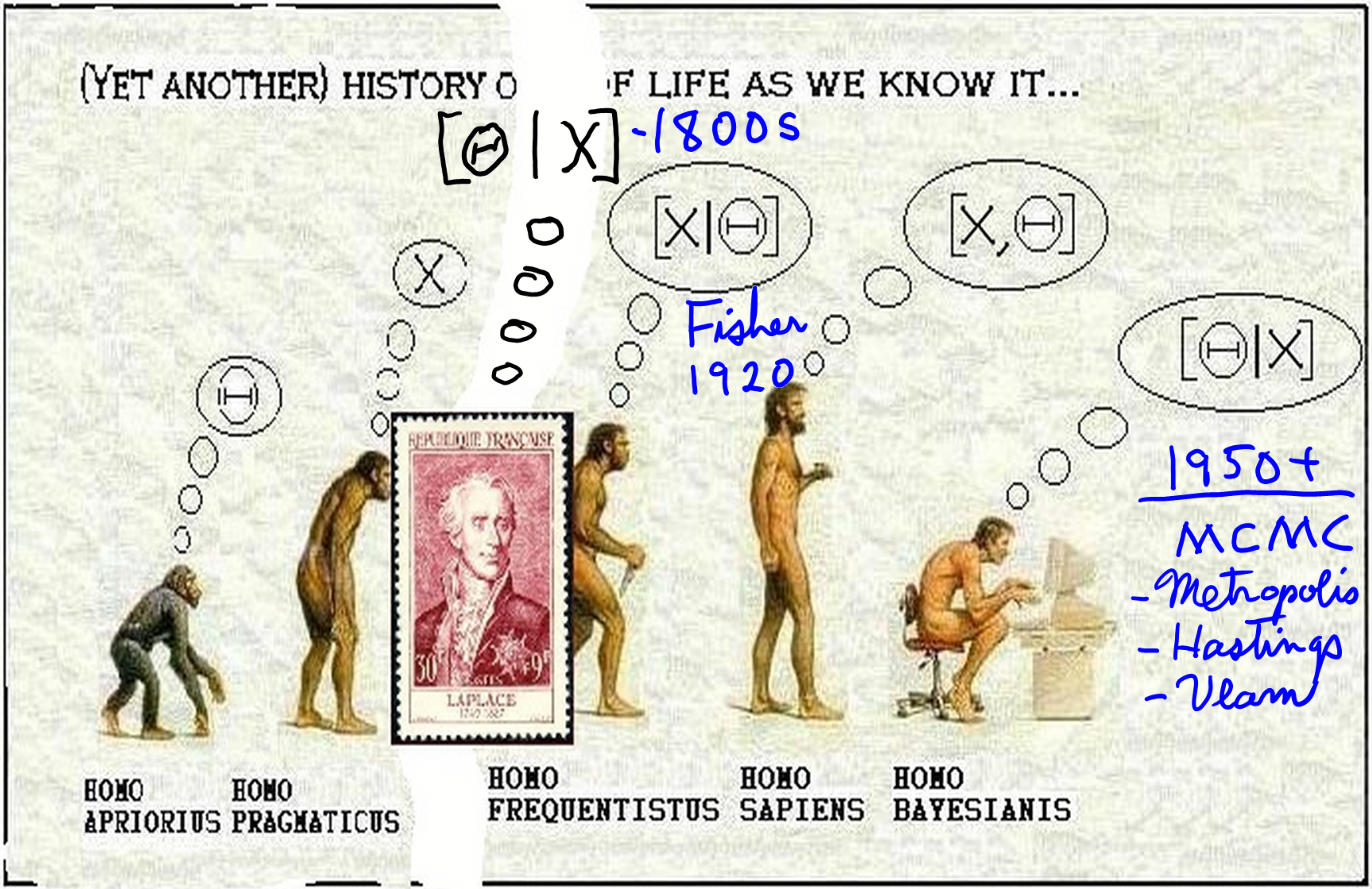
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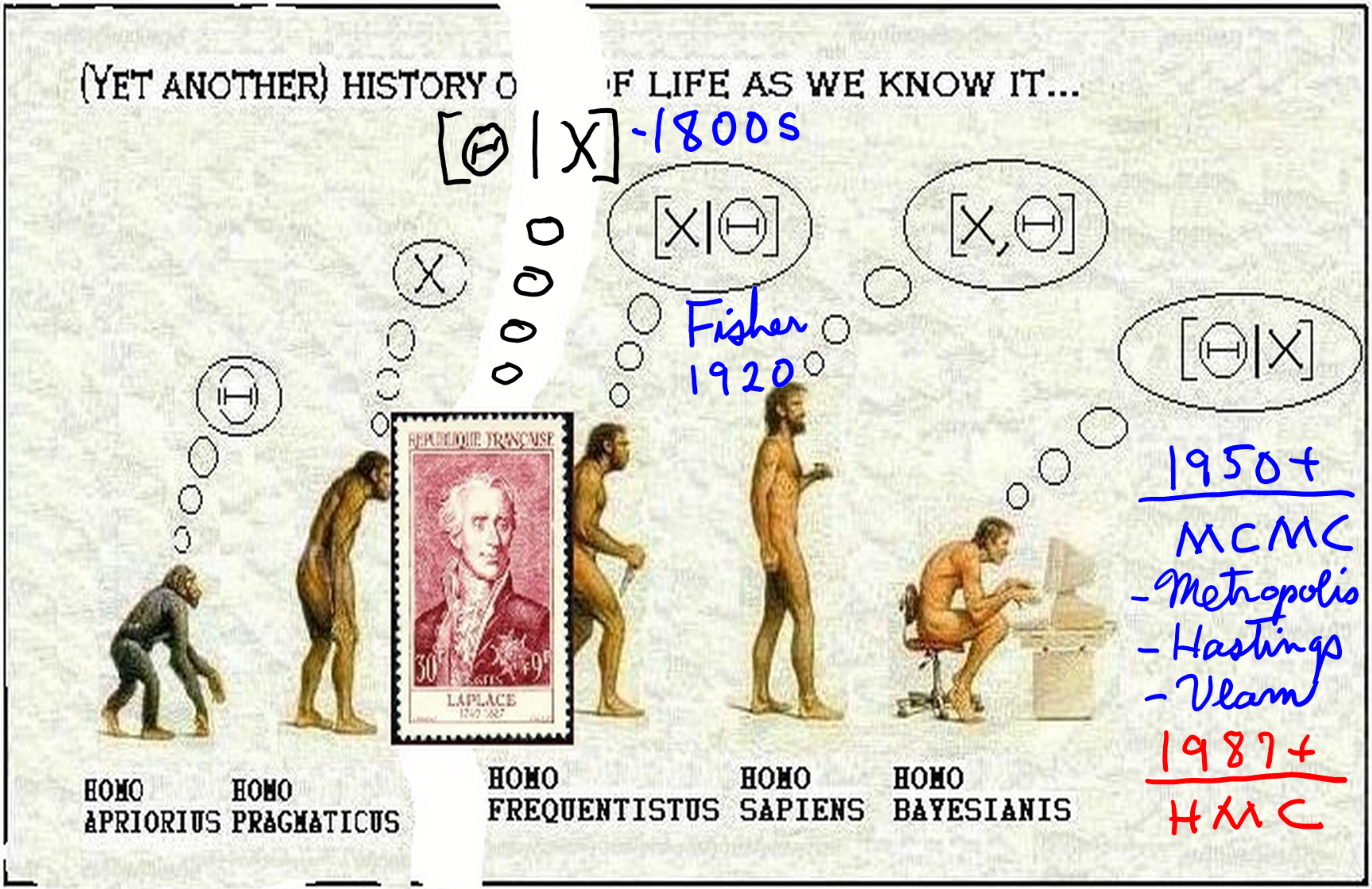
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Practical problem:

$$p(X, \theta) = p(X|\theta)\, p(\theta)$$

$$p(\theta|X) = \frac{p(X, \theta)}{p(X)}$$

MCMC (mid 20th C.) comes to the rescue: It's possible to sample from $p(\theta|X)$ knowing only $p(X, \theta)$

Posteriors without priors?

Fisher - Fiducial inference

Fraser - Structural inference

Objective Bayesian inference

Baking the Bayesian omelette without breaking the Bayesian egg.

Emerging practice:

Using weakly informative priors.

Markov Chain Monte Carlo

Use $\quad P(\theta, X) = P(X|\theta)\, P(\theta)$

Markov Chain Monte Carlo

Use $$P(\theta, X) = P(X|\theta)\, P(\theta)$$

joint $\qquad$ model × prior

Samples from $P(\theta|X)$ using only $P(\theta, X)$

i.e. no need to find elusive $P(X)$

Markov Chain Monte Carlo

Use $$P(\theta, X) = P(X|\theta)\,P(\theta)$$

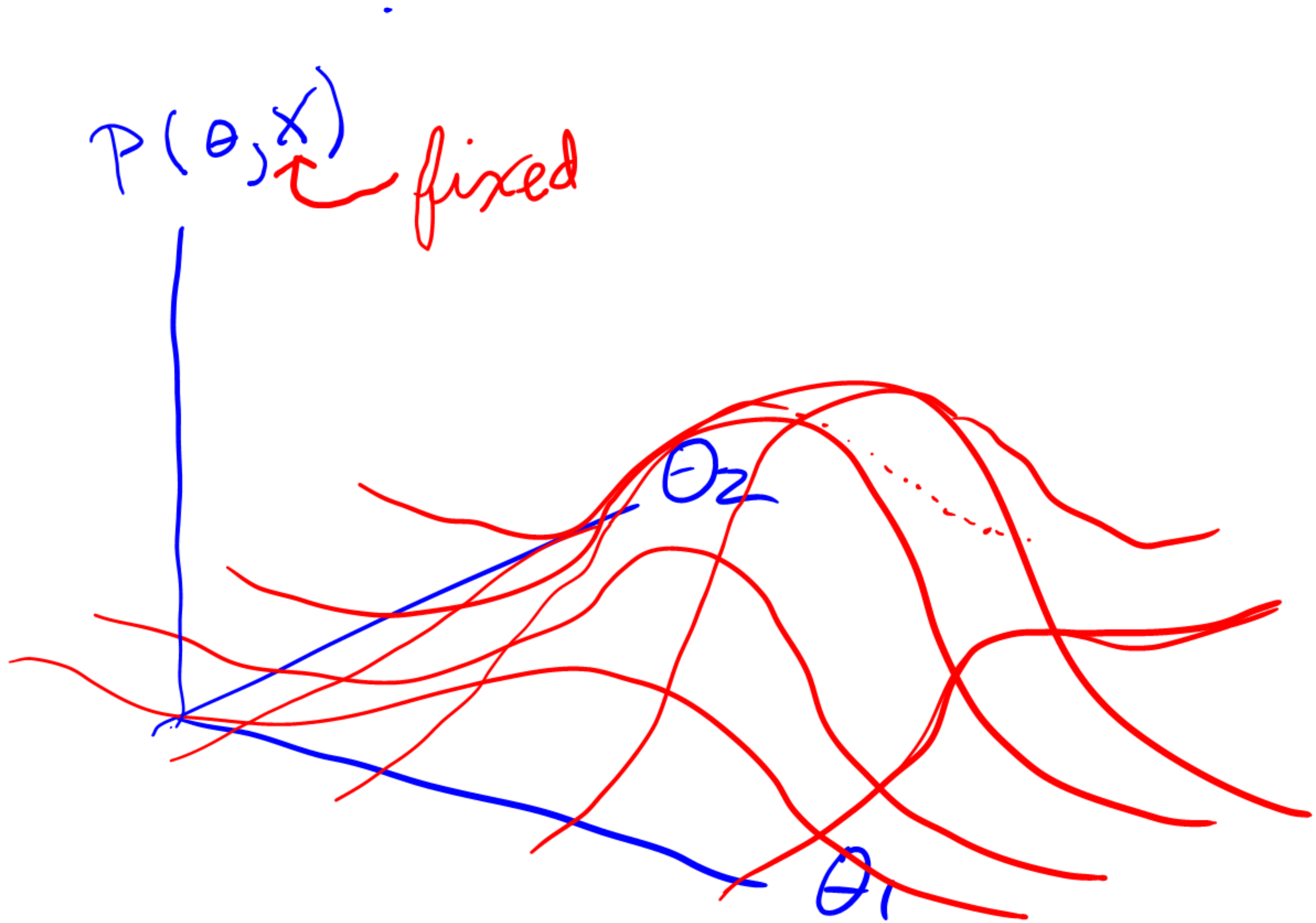
joint   model × prior

Samples from $P(\theta|X)$ using only $P(\theta, X)$

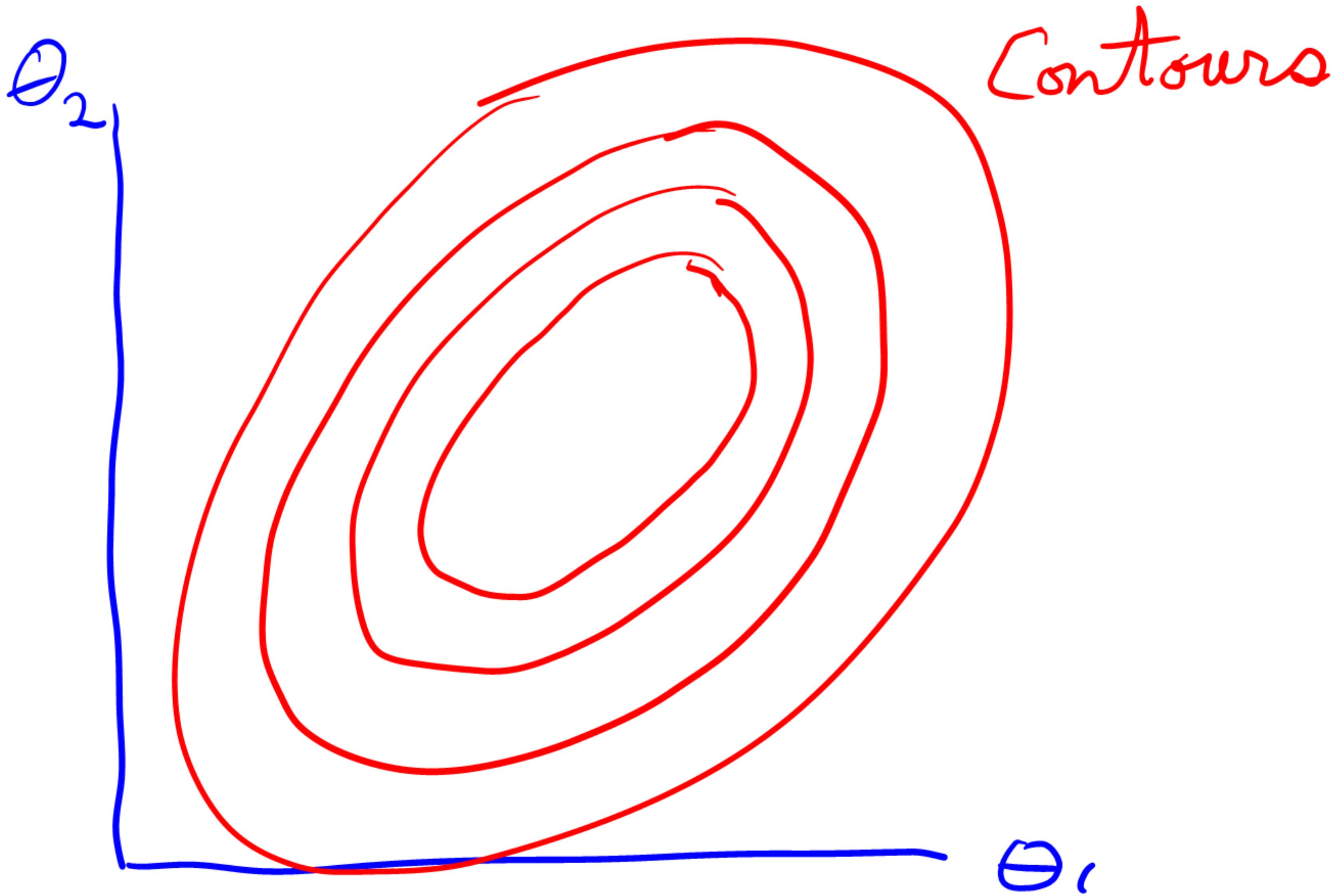
i.e. no need to find elusive $P(X)$

With X fixed, think of $P(\theta, X)$

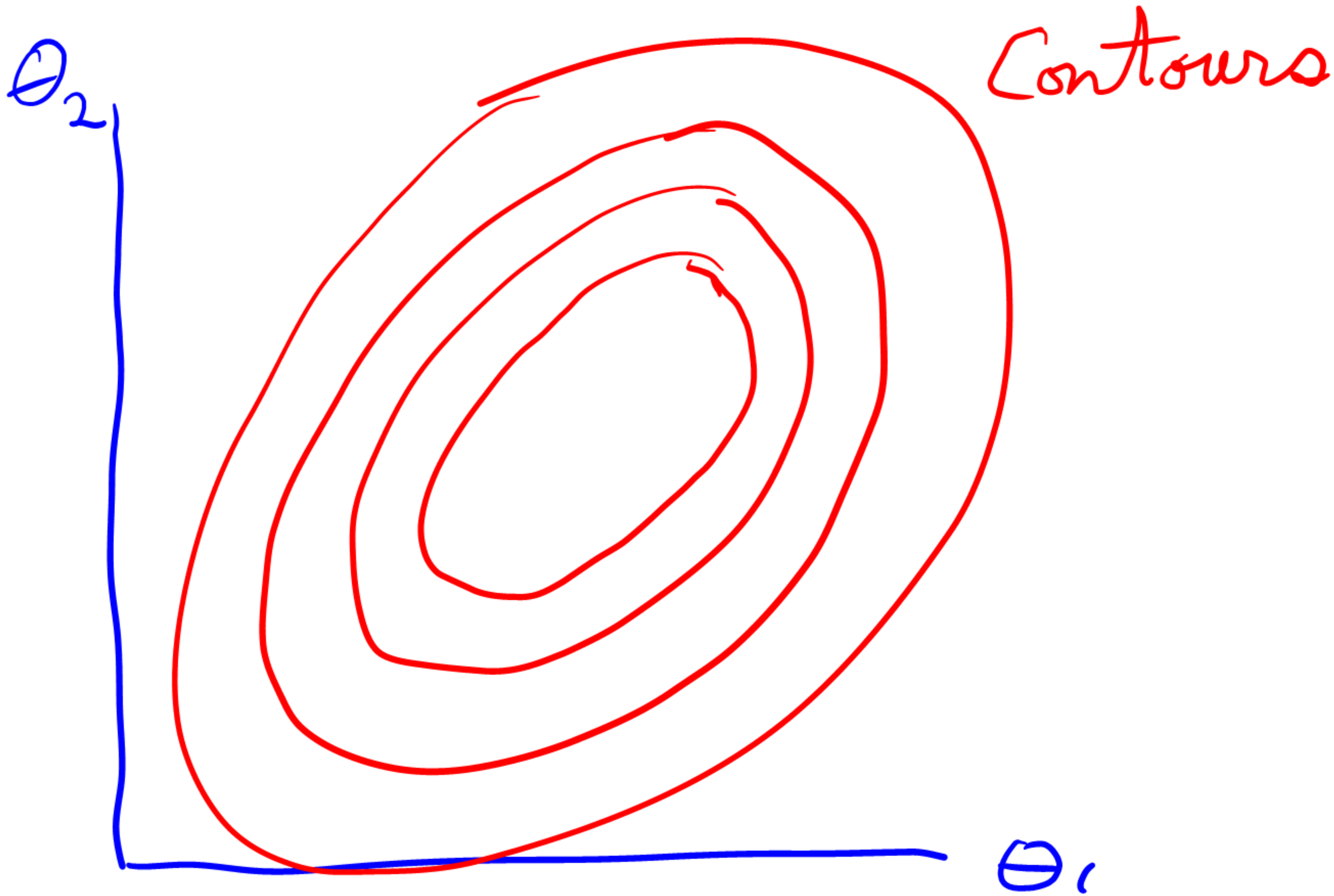
as defining a mountain over θ space



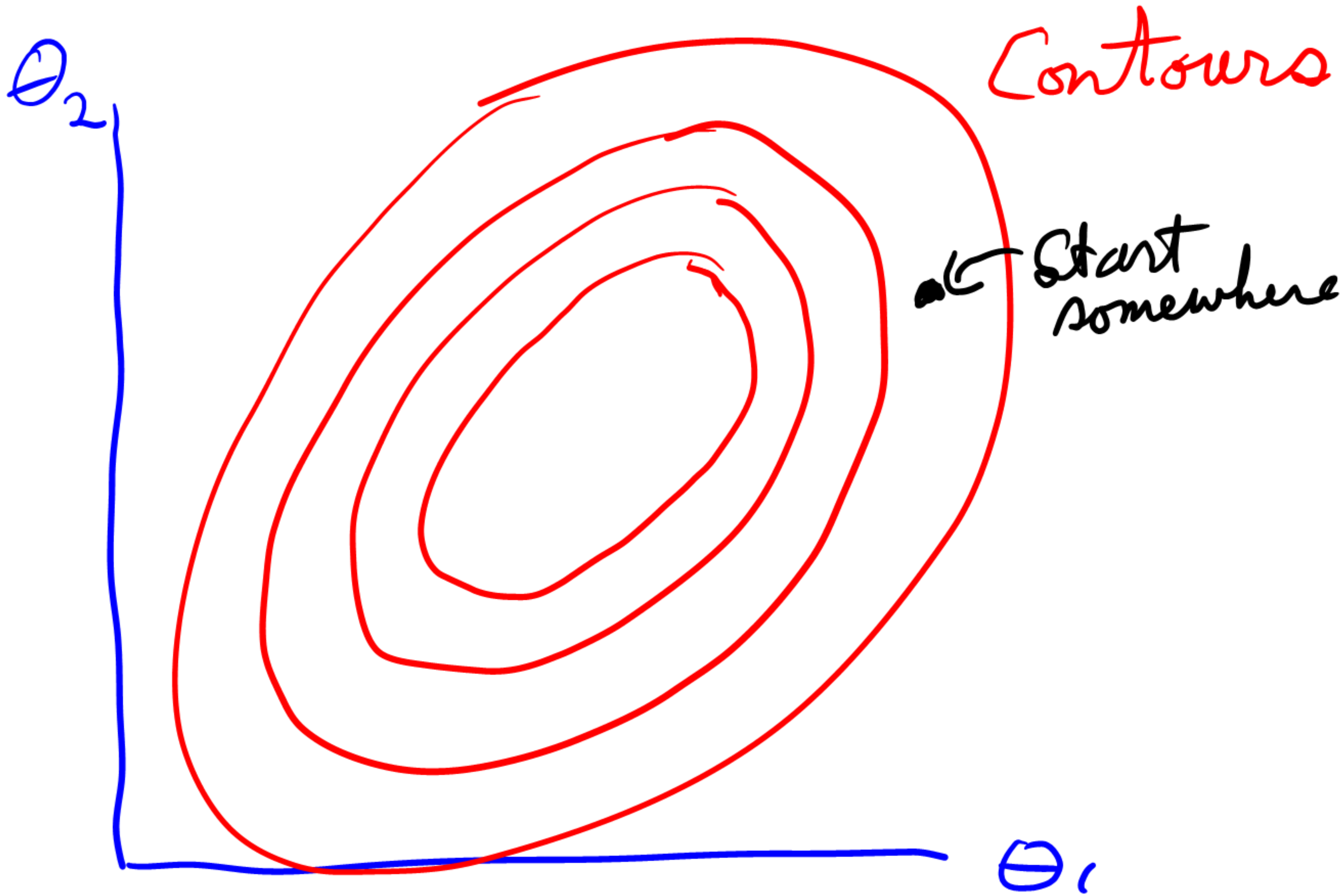
P(θ, x)
fixed
θ2
θ1



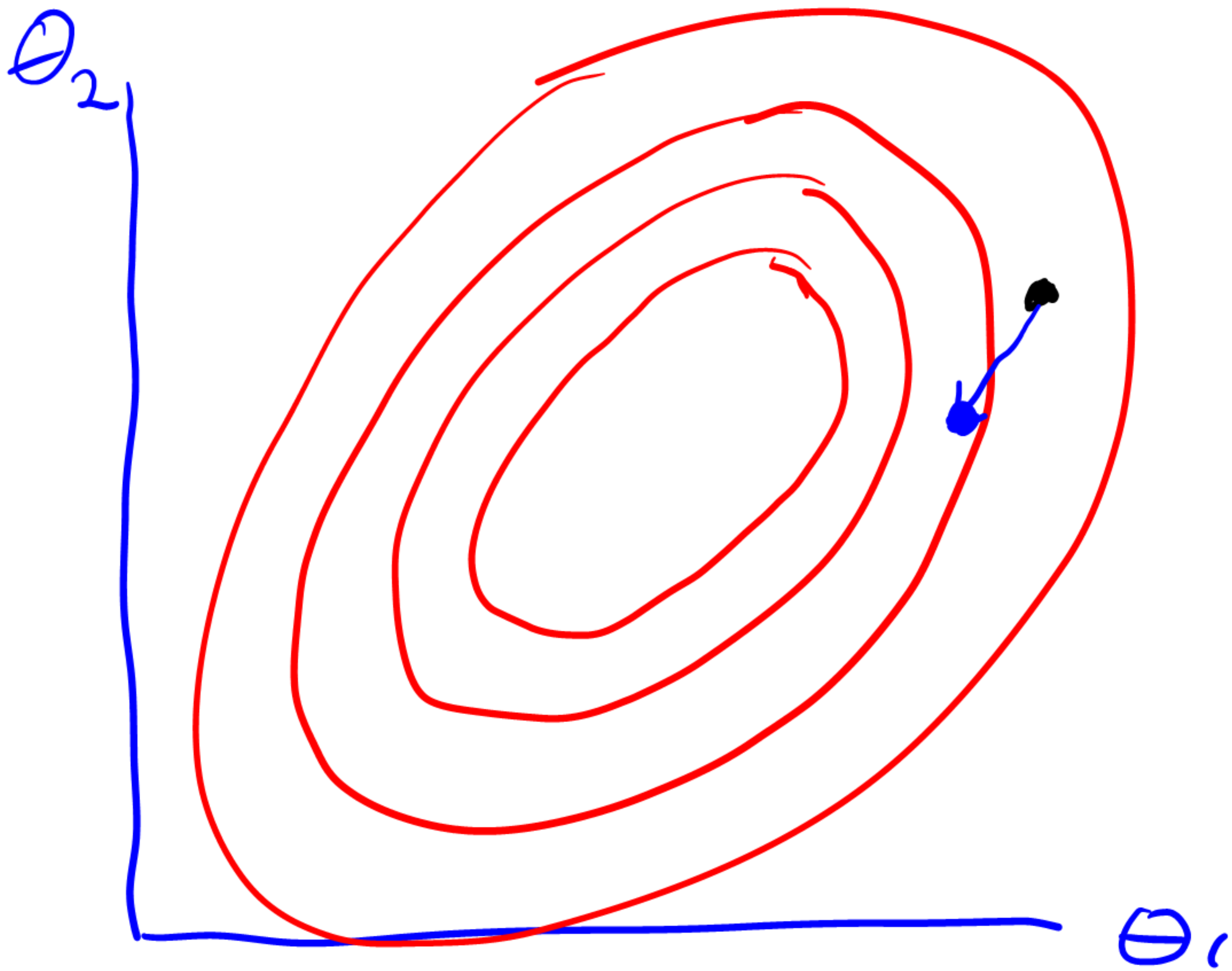
θ_2
Contours
θ_1



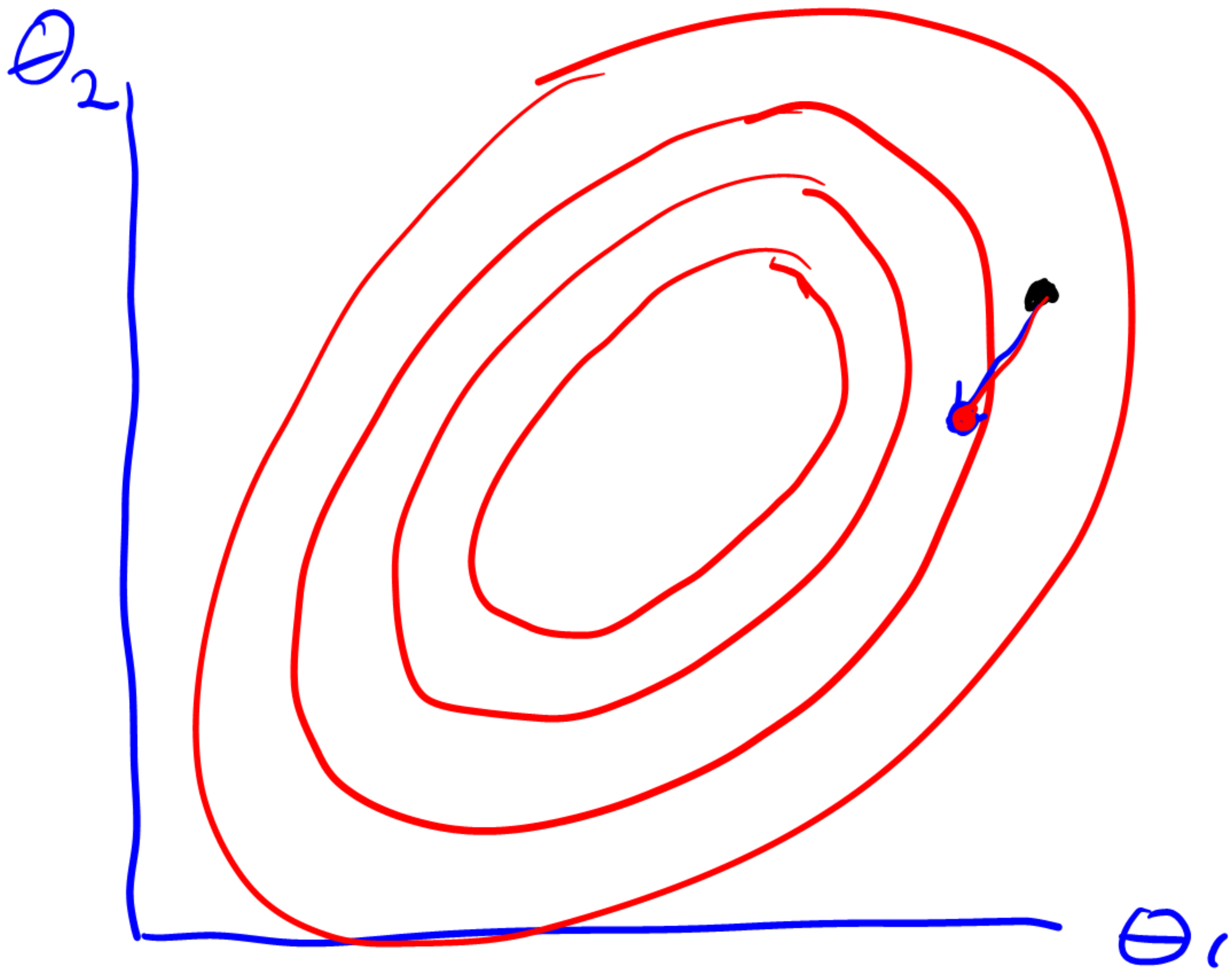
Metropolis-Hastings algorithm

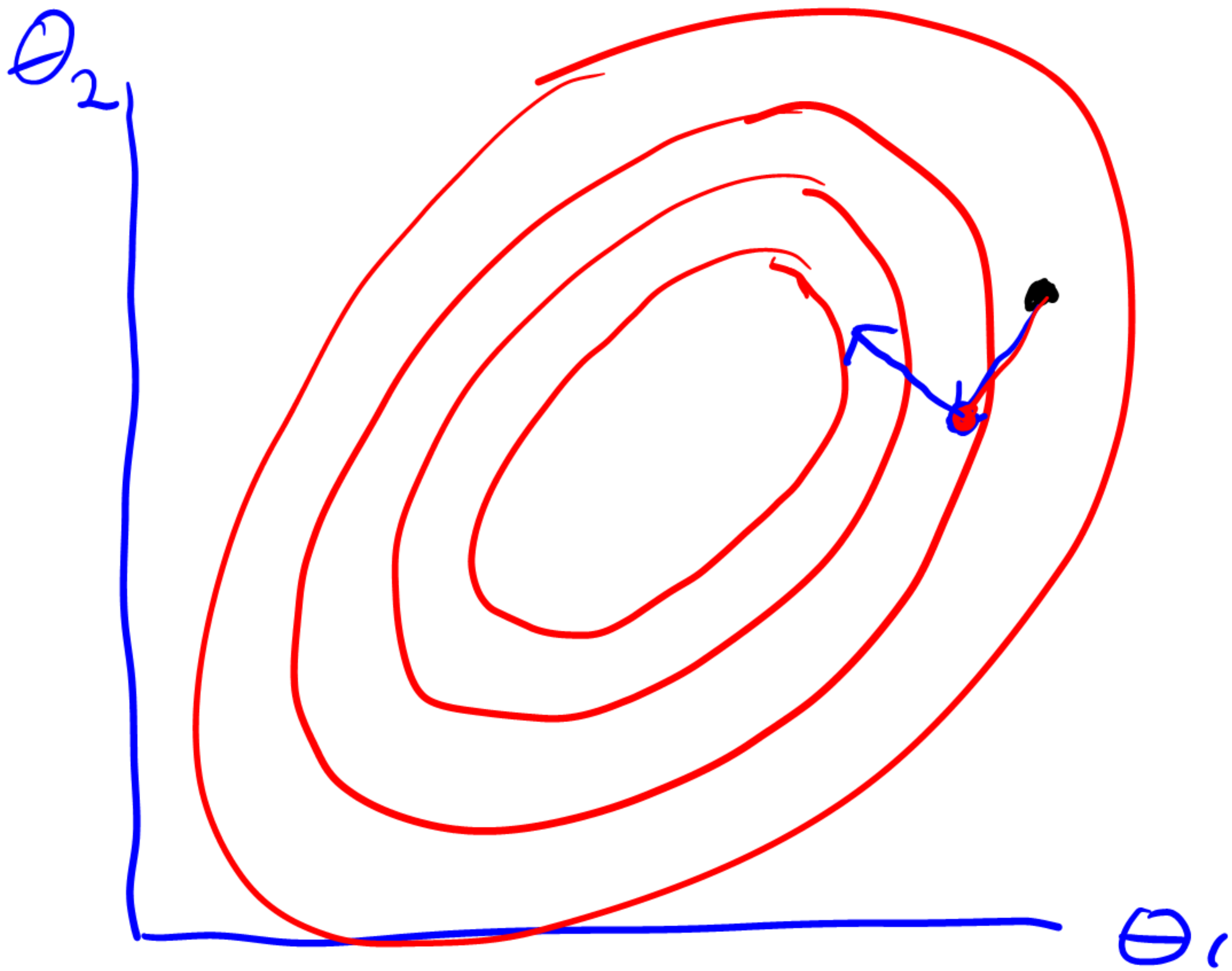


Metropolis-Hastings algorithm

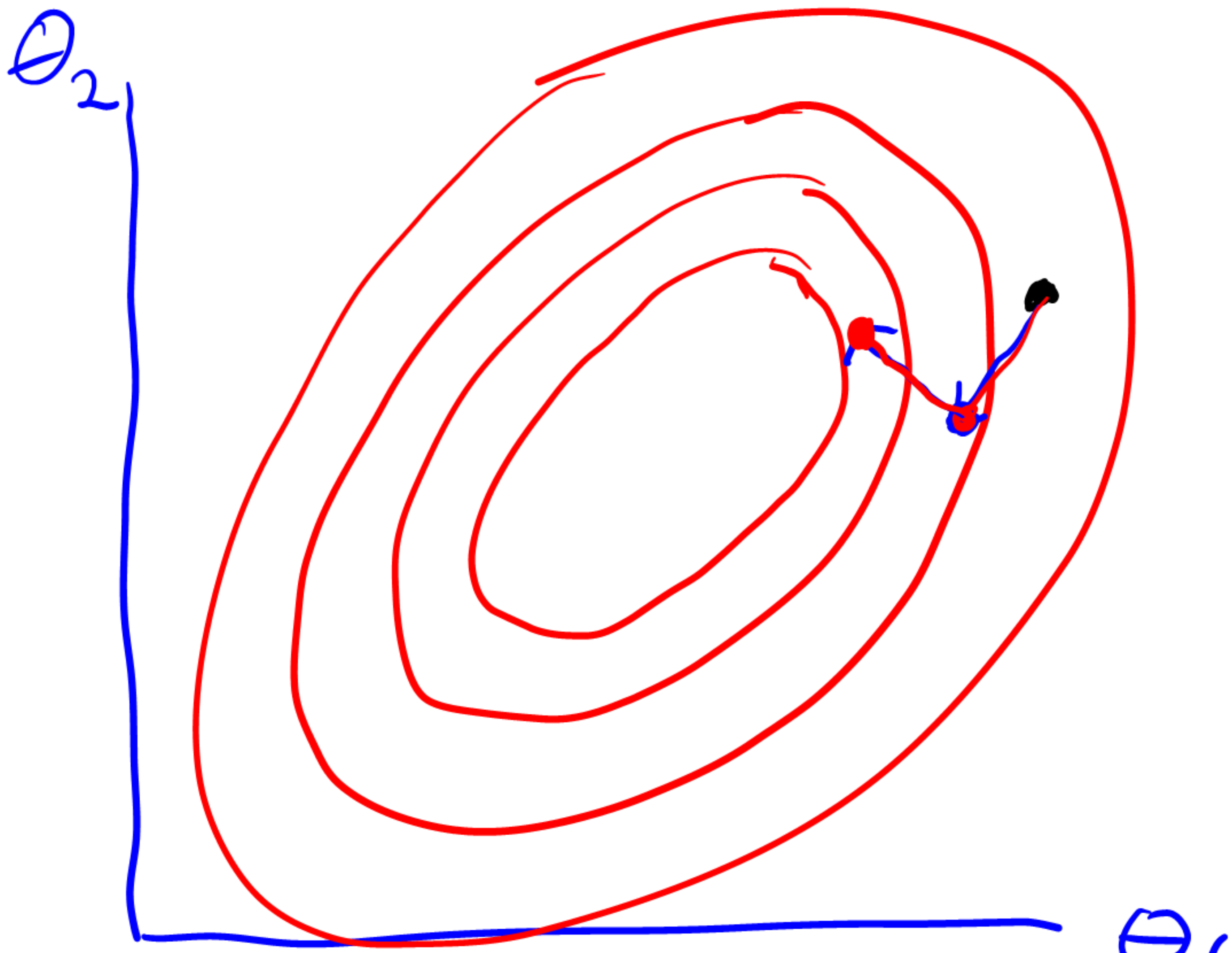


θ2
θ1

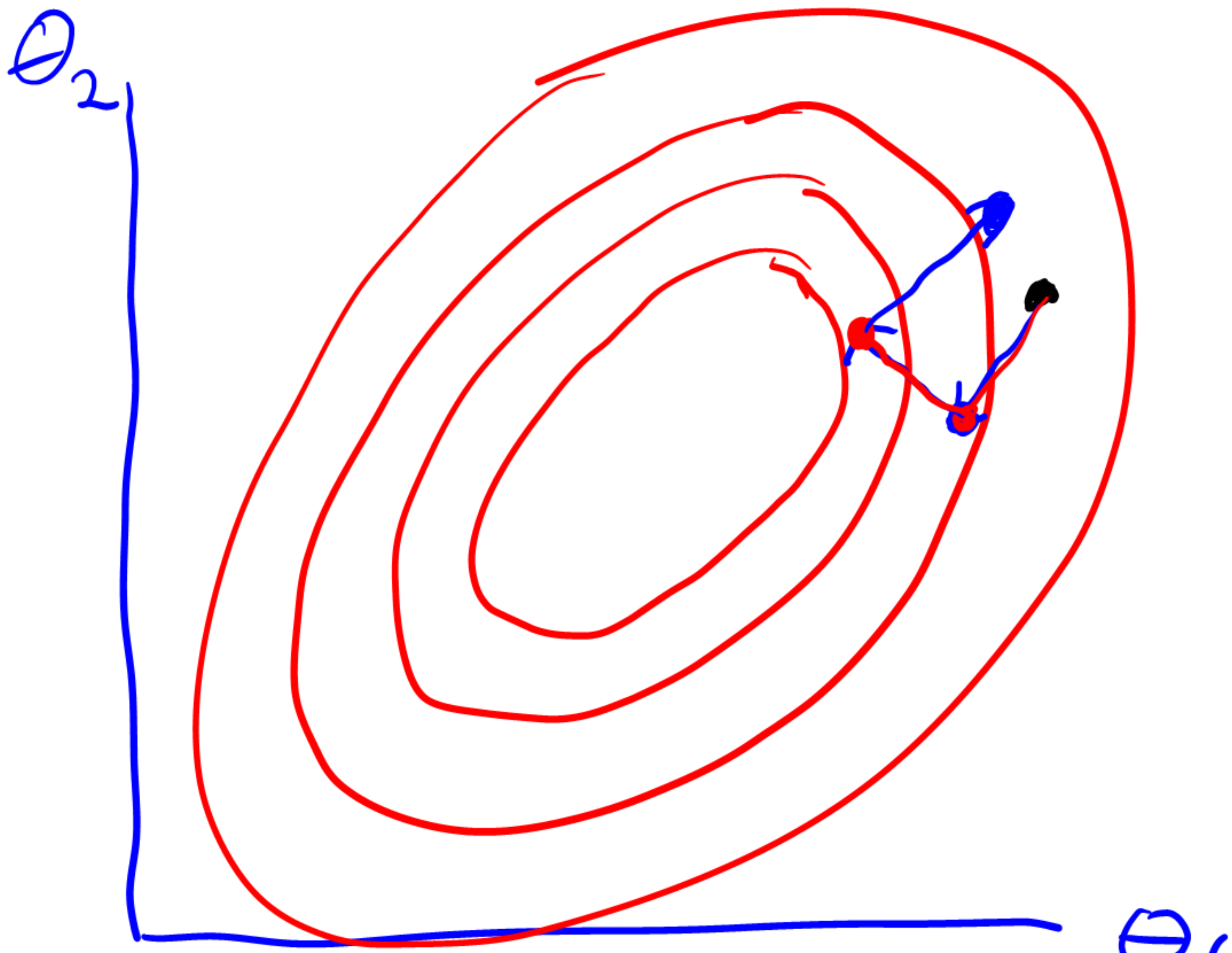

θ_2
θ_1



θ_2
θ_1



θ_2
θ_1



θ_2
θ_1

If you've walked downhill, you need to toss a biased coin with:

$$Pr(Heads) = \frac{p(\theta_{new}|Y)}{p(\theta_{last}|Y)}$$

Usually, it's very hard to compute the **numerator** and the **denominator** of this **ratio**.

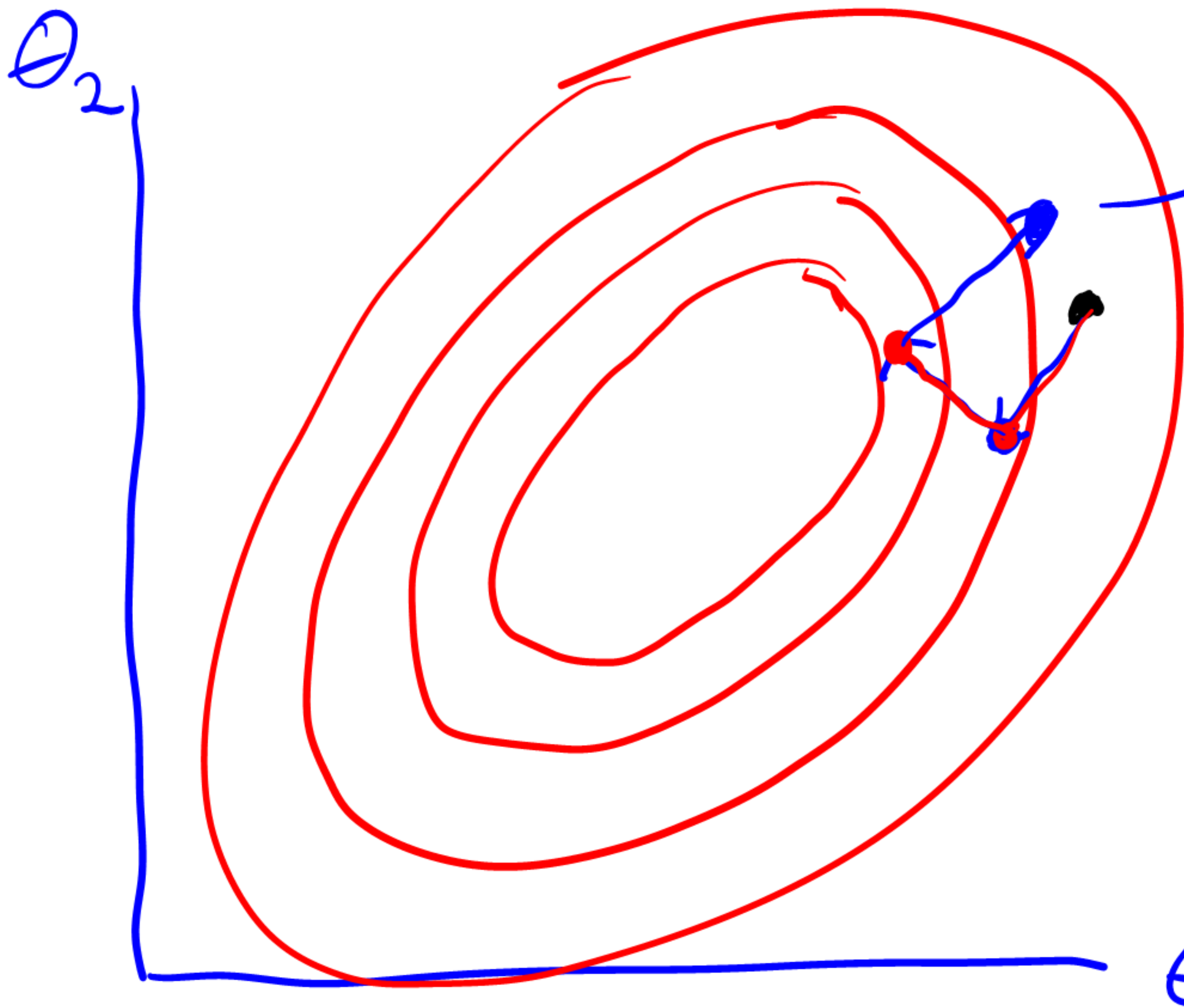
However, the **ratio** itself is, for many models, a relative cinch:

$$Pr(Heads) = \frac{p(\theta_{new}|Y)}{p(\theta_{last}|Y)} = \frac{p(\theta_{new}|Y)/p(Y)}{p(\theta_{last}|Y)/p(Y)} = \frac{p(Y,\theta_{new})}{p(Y,\theta_{last})} \times \frac{p(\theta_{new})}{p(\theta_{last})}$$

Which is just the **likelihood ratio** times the **prior ratio**.

The more you've gone down, the lower the probability of a Head.

- If you get a Head, plant a stake at your new position.
- If you get a Tail, step back to the last position and plant a second stake there.

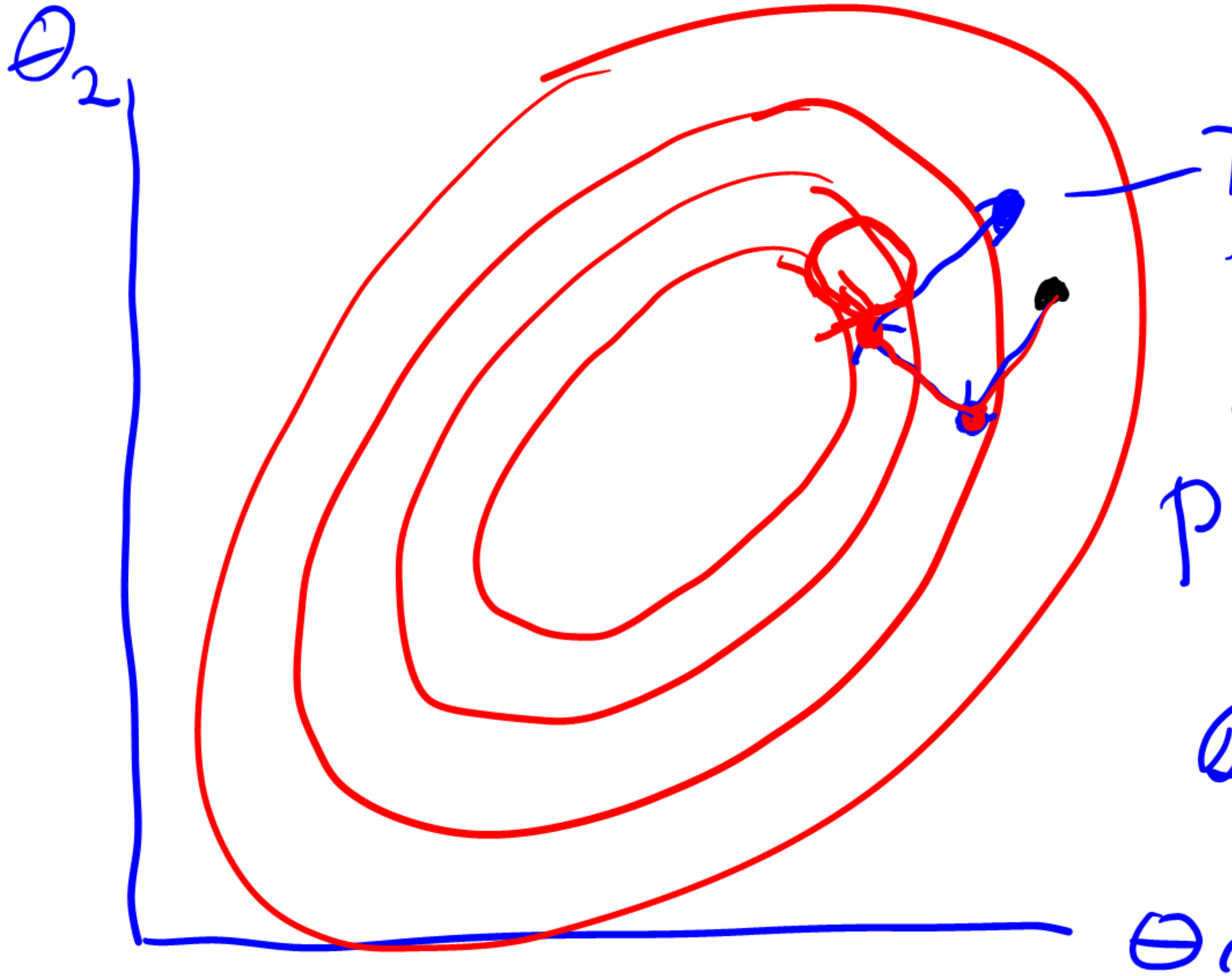


toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move

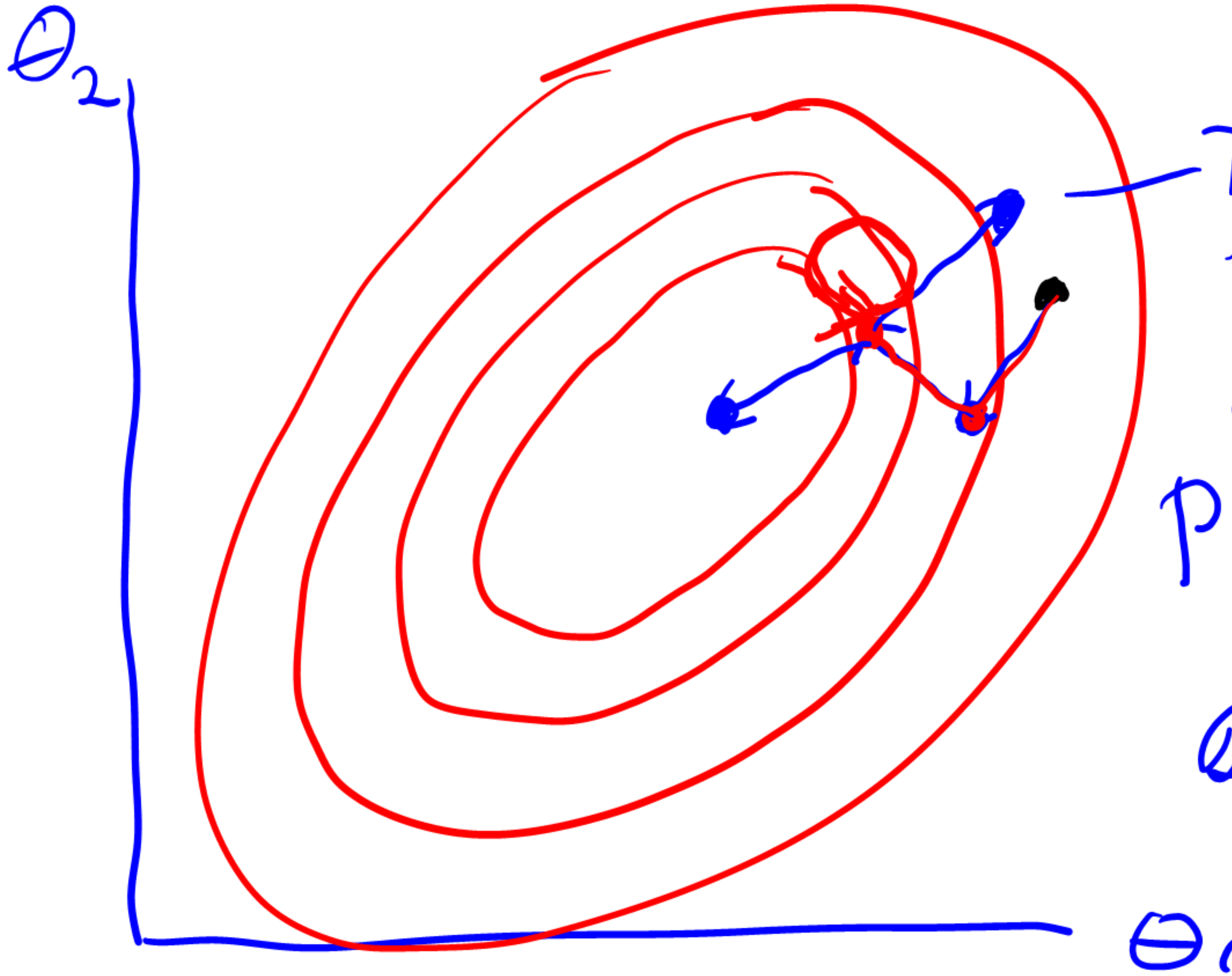
if Tails stay



toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

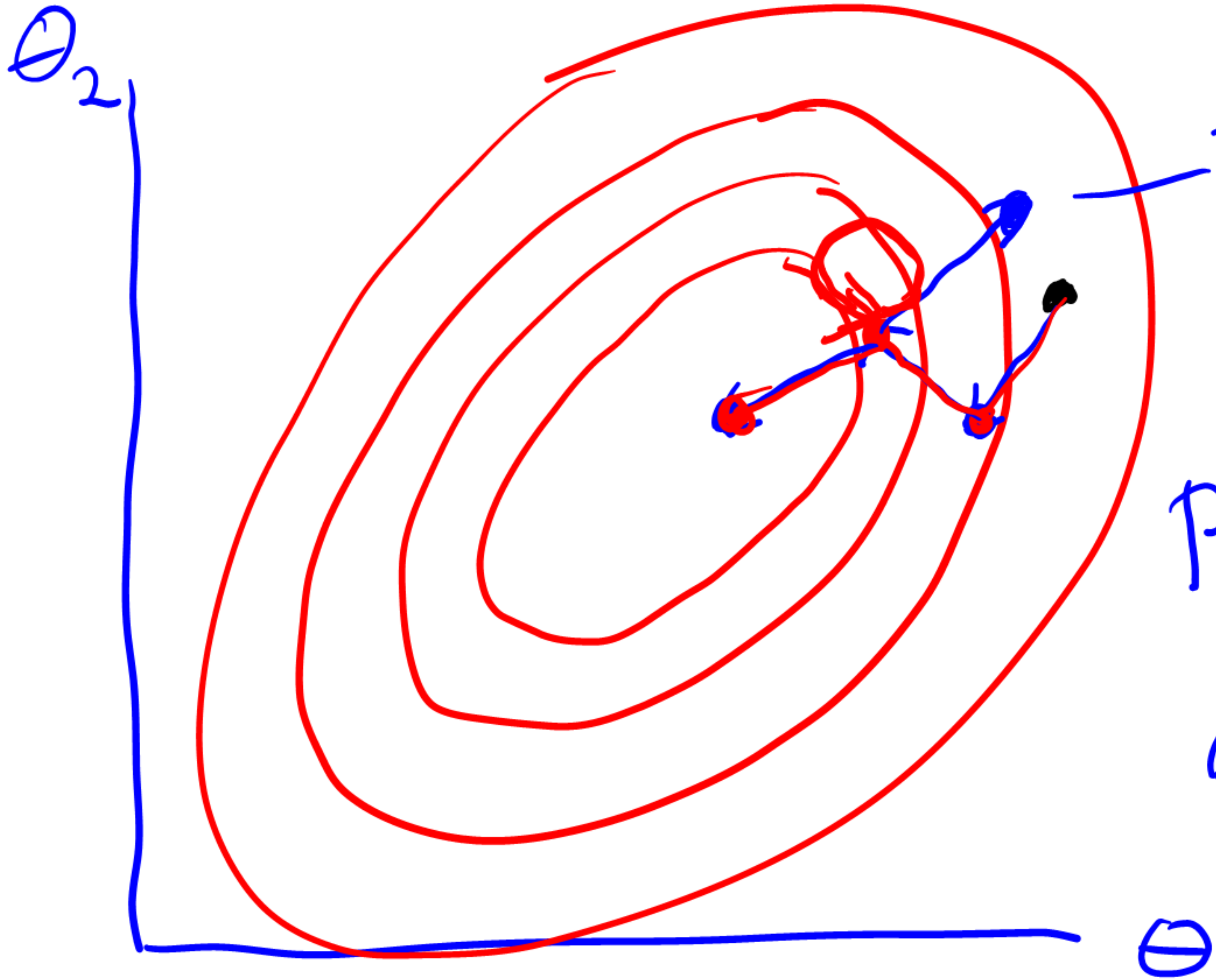
If Heads move
if Tails stay



toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

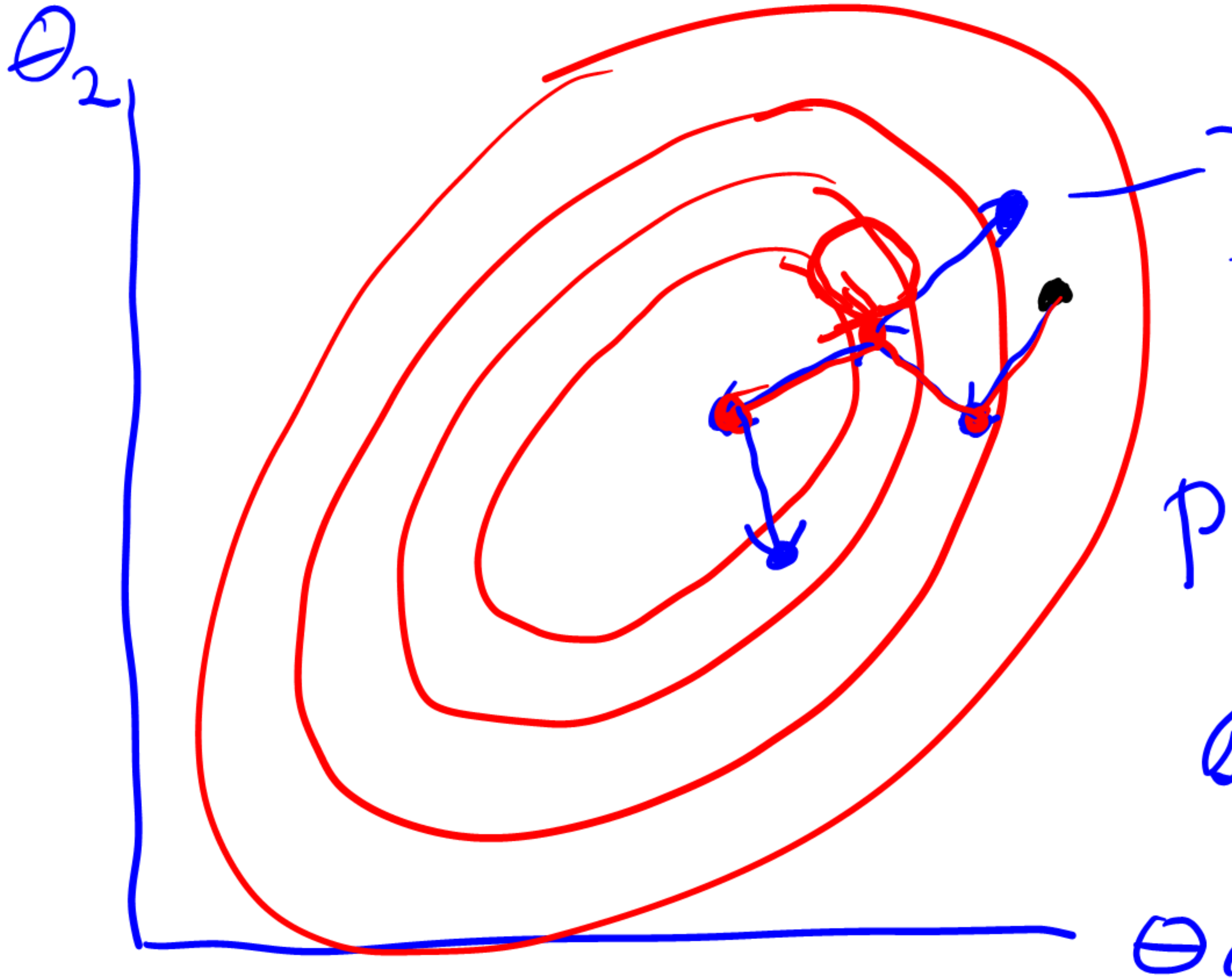
If Heads move
If Tails stay



toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

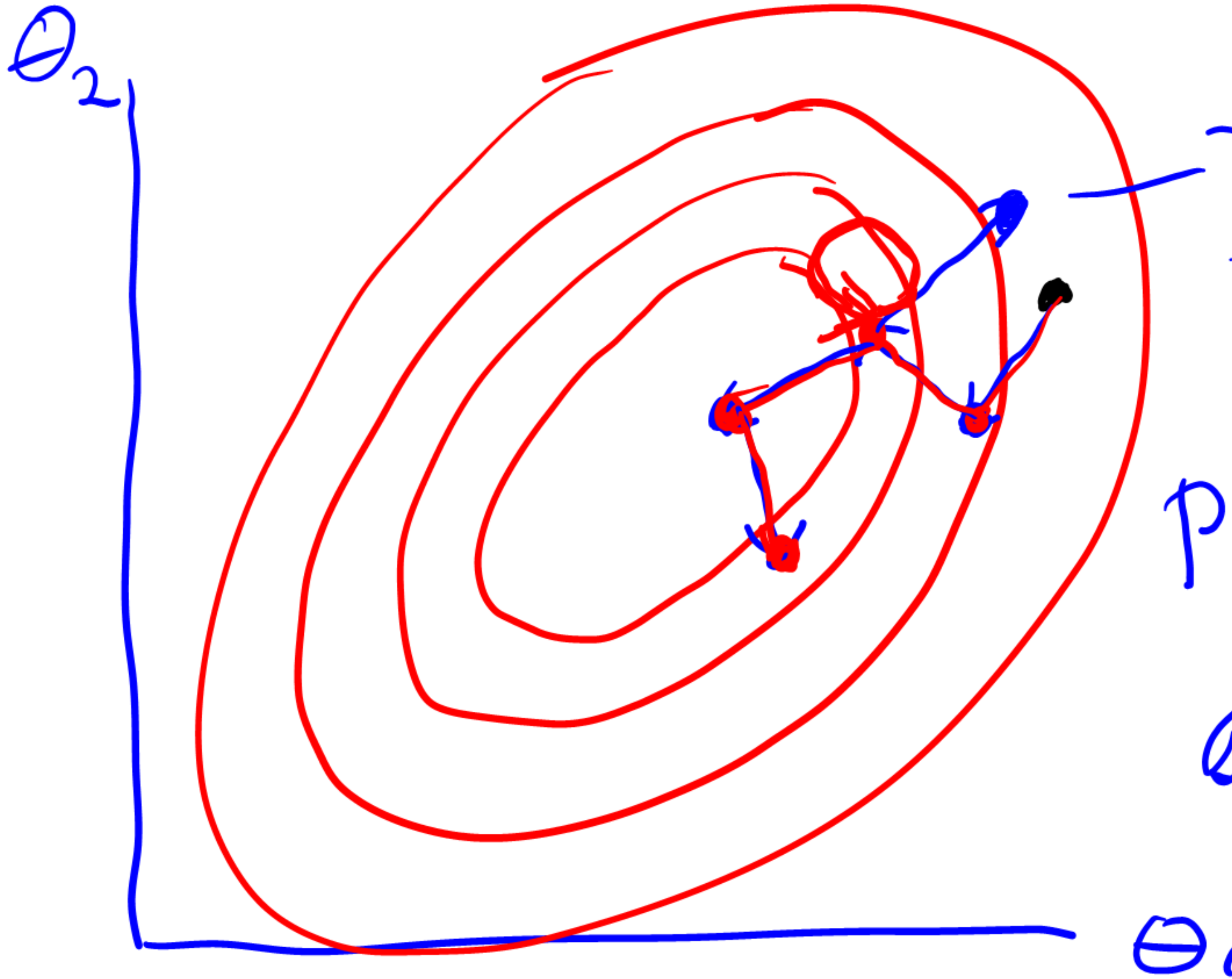
If Heads move
if Tails stay



toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

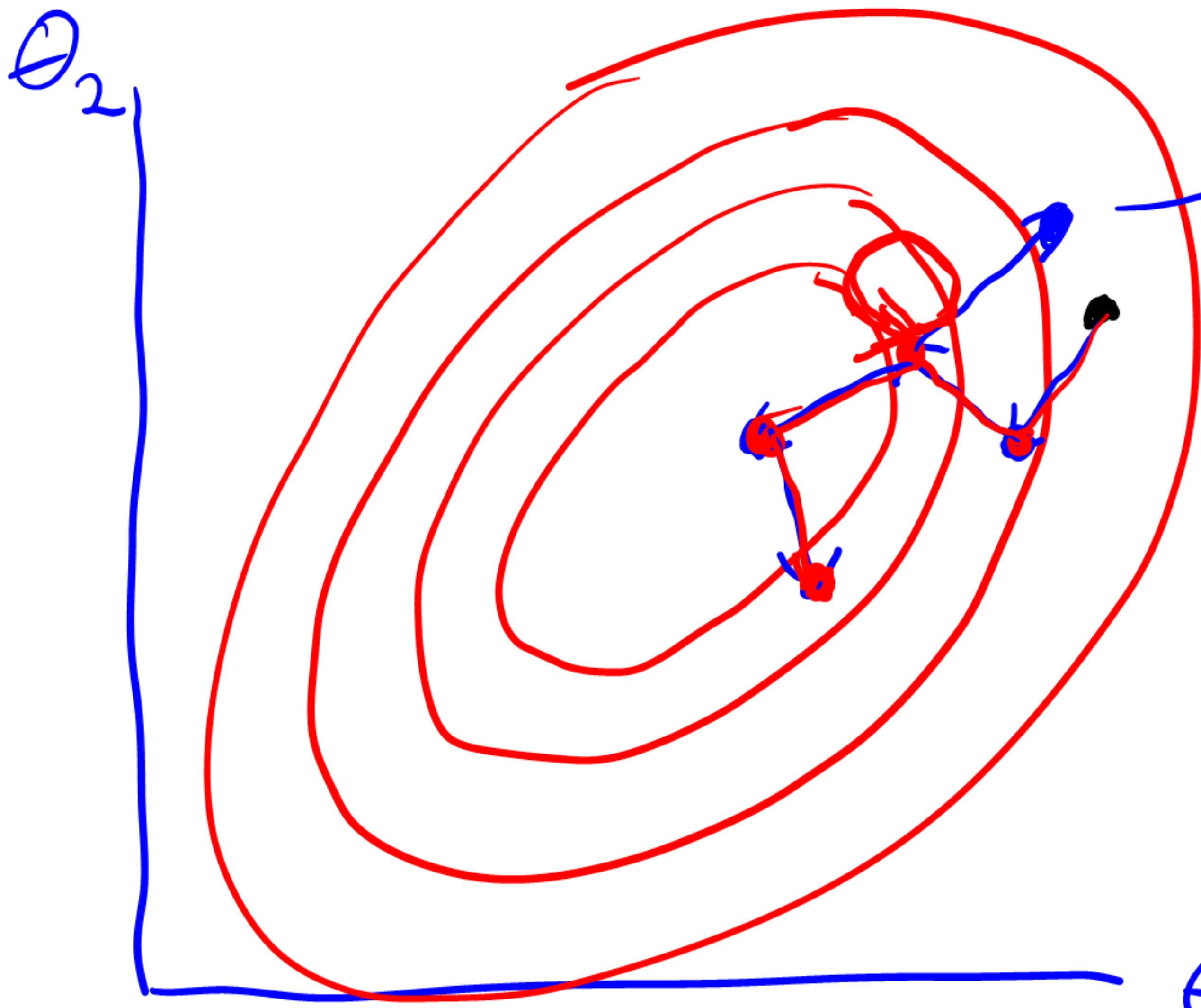
If Heads move
if Tails stay



toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move
if Tails stay

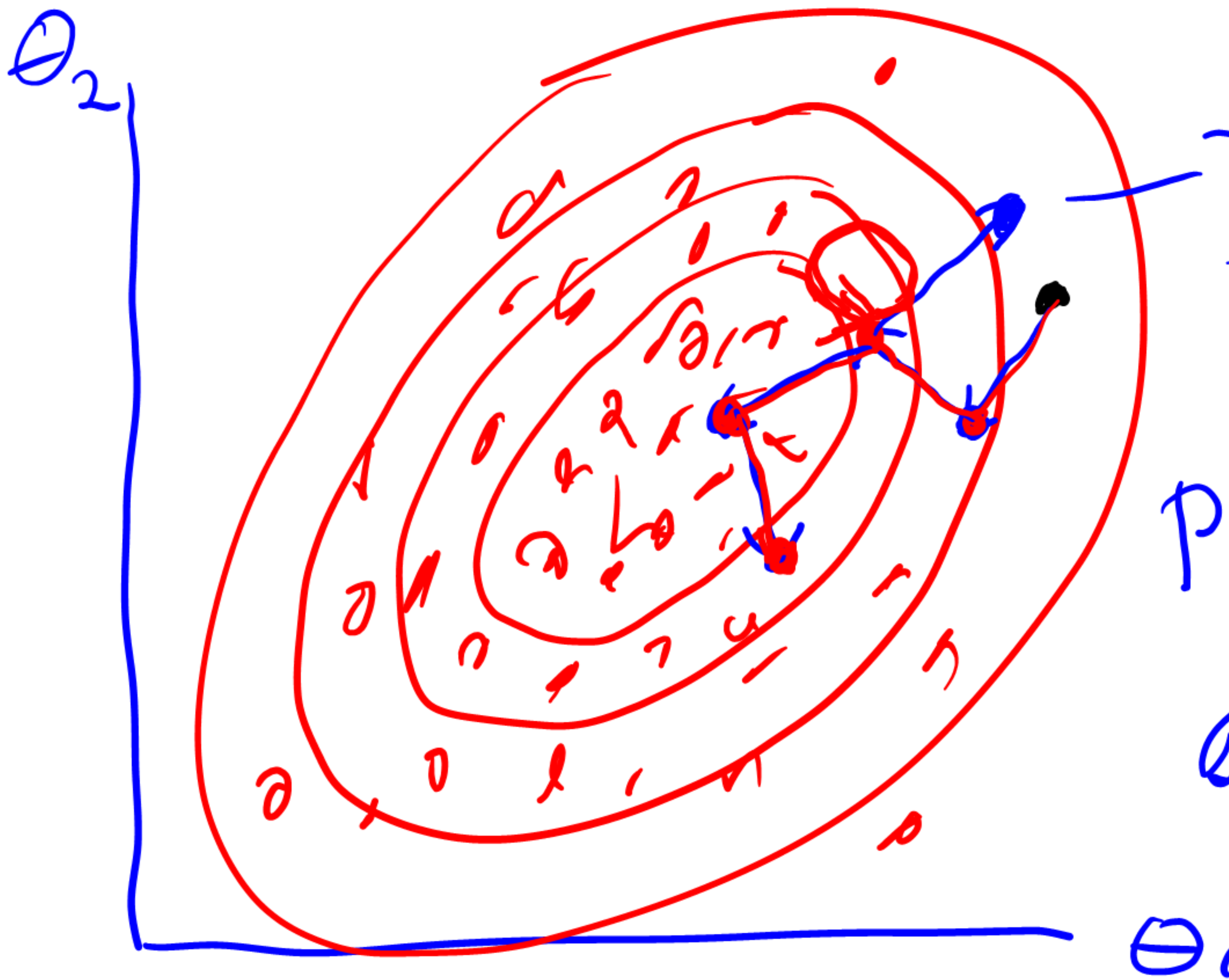


toss a biased coin with

$P(H) = \frac{P(\cdot)}{P(\cdot)}$

If Heads move
If Tails stay

Keep doing this for a very long time

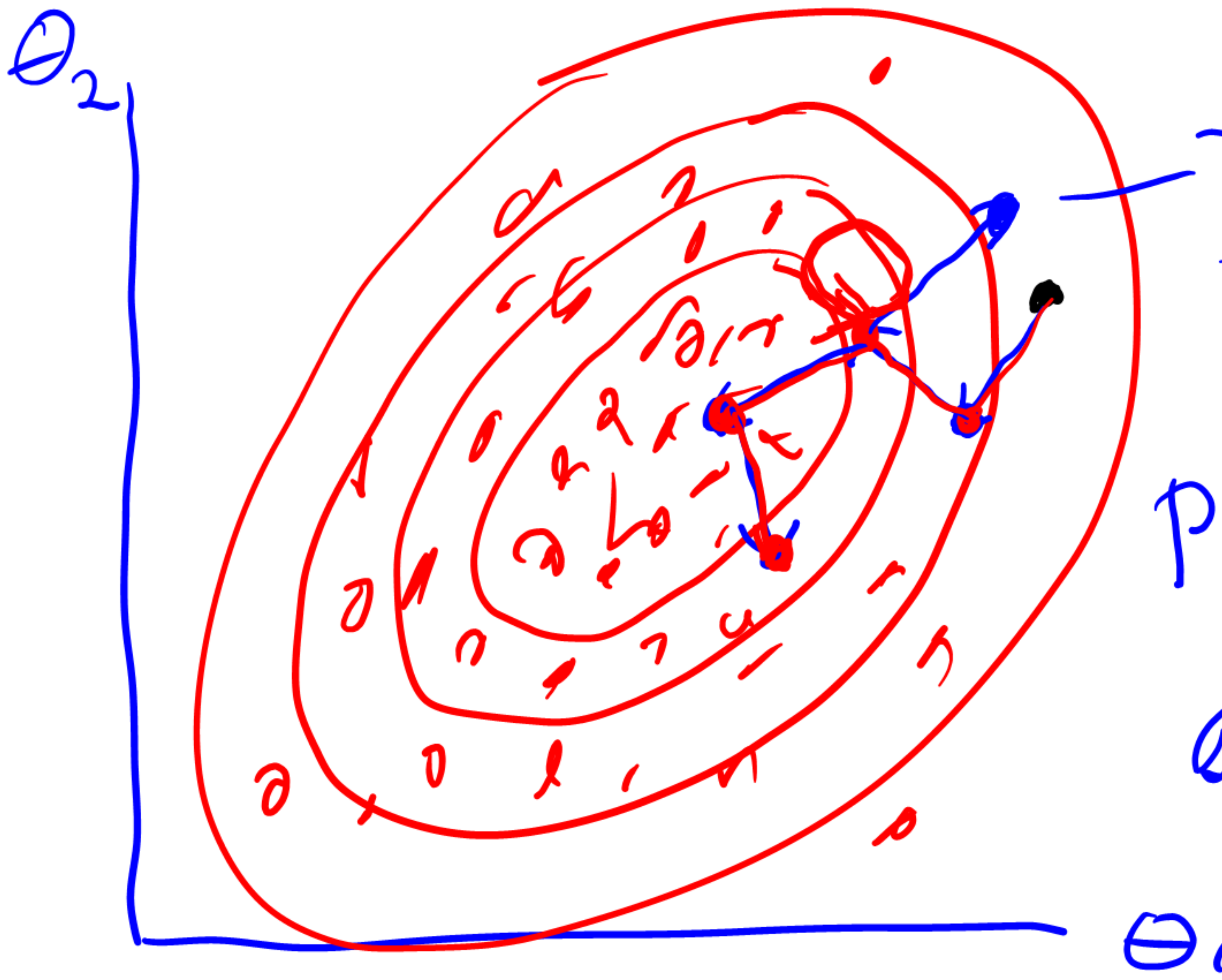


toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move
if Tails stay

Keep doing this for a very long time



toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move
if Tails stay

- Generates a sample from $P(\theta | X)$

That's the M-H algorithm

- points can be highly correlated
- very slow to cover distribution
  - especially with large # of parameters, e.g. multilevel models

- can get stuck in corners of distribution

$O(p^2)$ $O(p^2)$ $O(p^{5/4})$

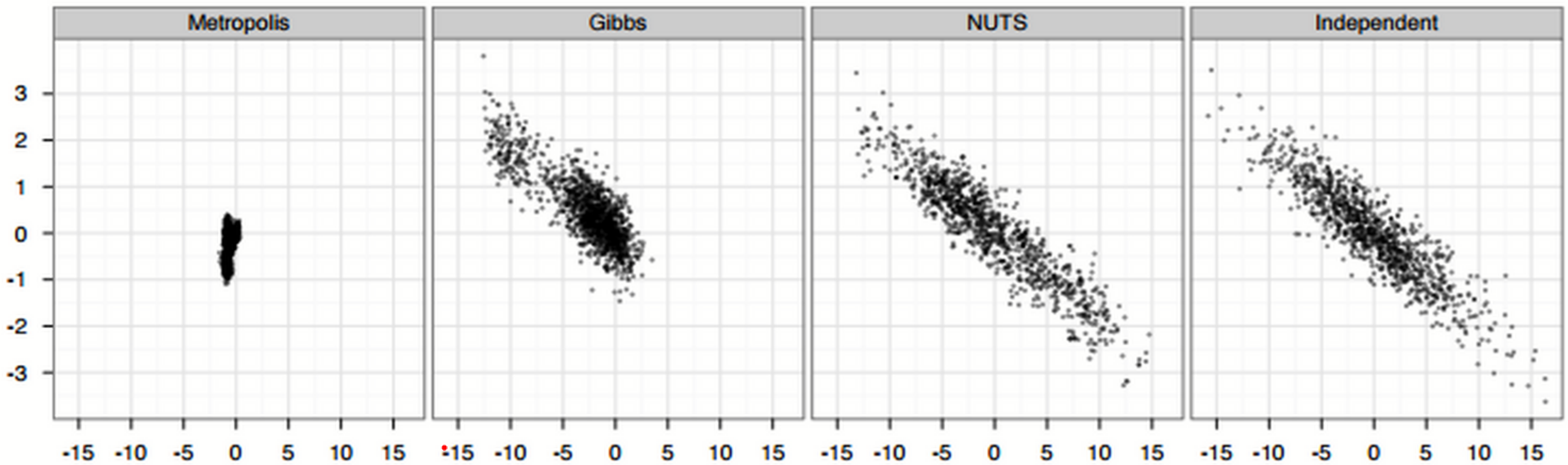


Figure 7: Samples generated by random-walk Metropolis, Gibbs sampling, and NUTS. The plots compare 1,000 independent draws from a highly correlated 250-dimensional distribution (right) with 1,000,000 samples (thinned to 1,000 samples for display) generated by random-walk Metropolis (left), 1,000,000 samples (thinned to 1,000 samples for display) generated by Gibbs sampling (second from left), and 1,000 samples generated by NUTS (second from right). Only the first two dimensions are shown here.

from Hoffman & Gelman (2014)

$O(p^2)$ $O(p^2)$ $O(p^{5/4})$

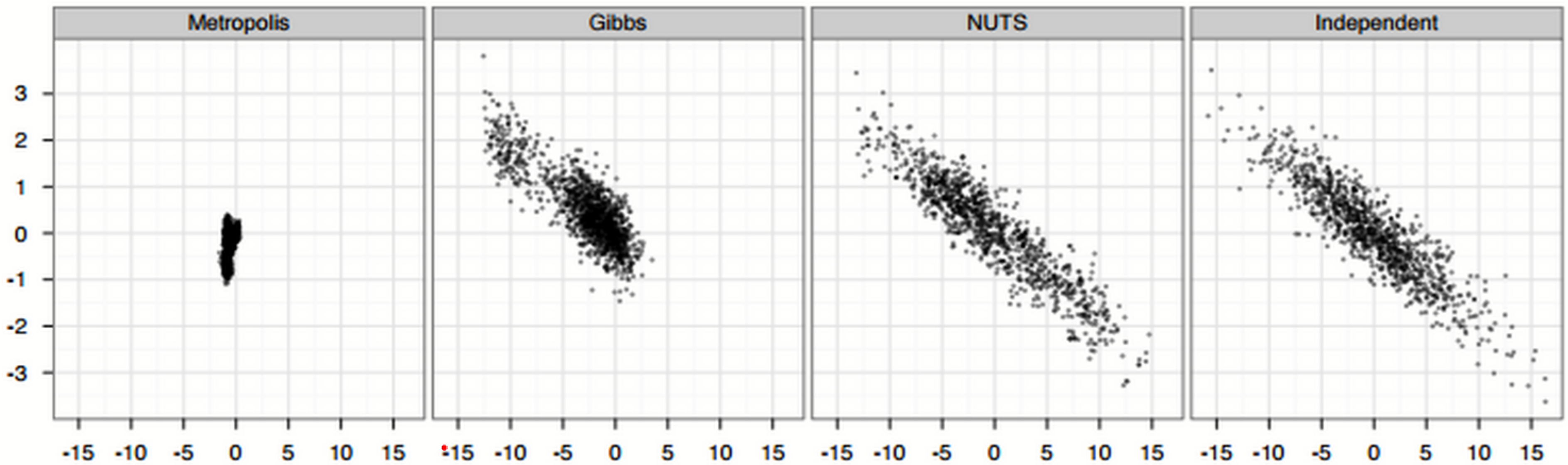
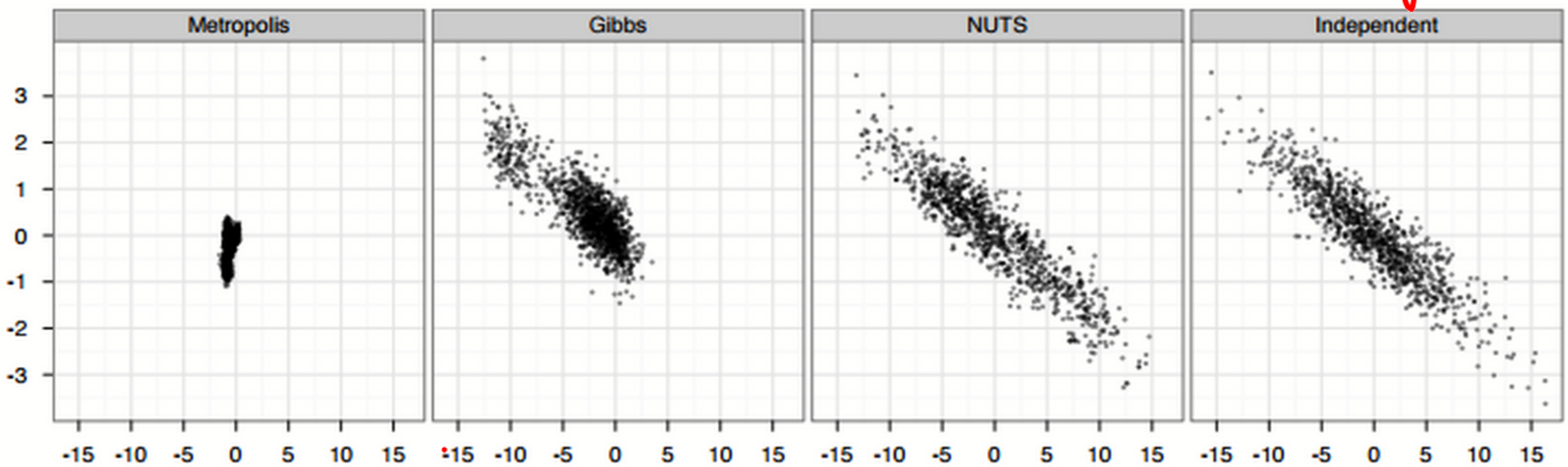


Figure 7: Samples generated by random-walk Metropolis, Gibbs sampling, and NUTS. The plots compare 1,000 independent draws from a highly correlated 250-dimensional distribution (right) with 1,000,000 samples (thinned to 1,000 samples for display) generated by random-walk Metropolis (left), 1,000,000 samples (thinned to 1,000 samples for display) generated by Gibbs sampling (second from left), and 1,000 samples generated by NUTS (second from right). Only the first two dimensions are shown here.

from Hoffman & Gelman (2014)

$O(p^2)$ $O(p^2)$ $O(p^{5/4})$ Target



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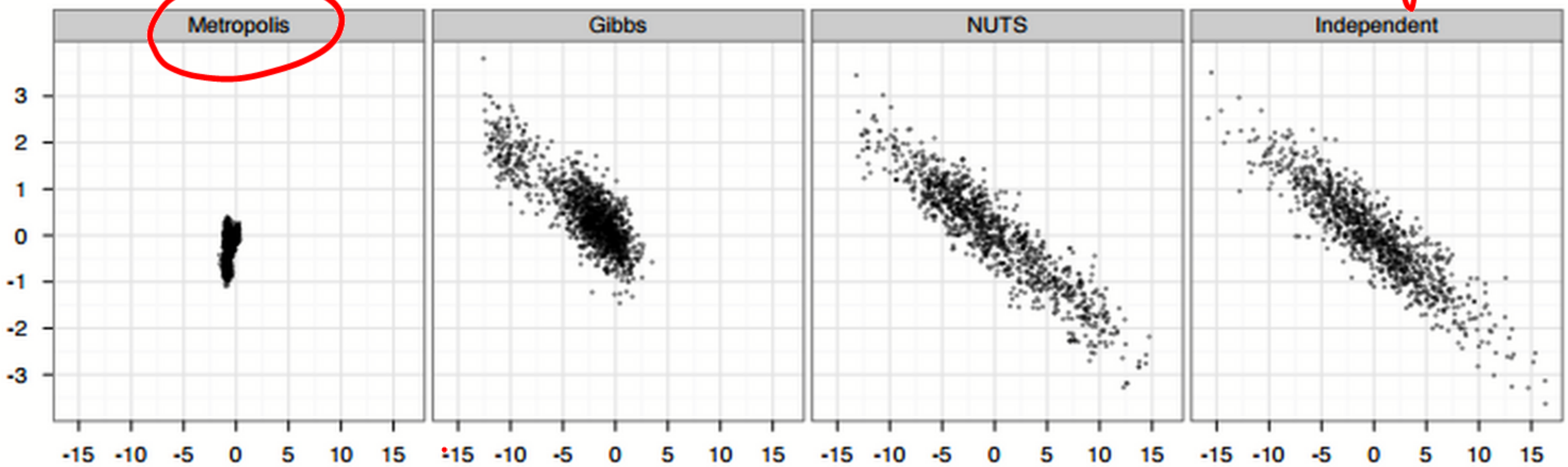

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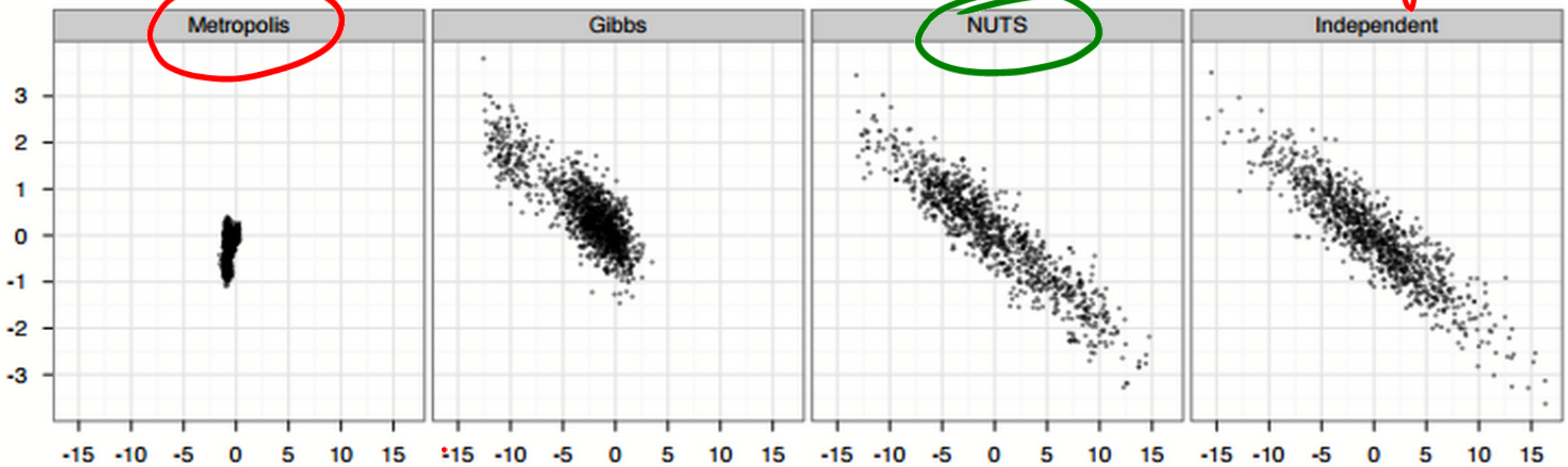

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$\Leftarrow$ HMC

from Hoffman & Gelman (2014)

Hamiltonian Monte Carlo

Hamiltonian Monte Carlo

fixed

- Turn the mountain  $\underset{\sim}{\theta}$ $\quad p(X, \underset{\sim}{\theta})$

into a bowl  $\underset{\sim}{\theta}$ by using $-\log p(X, \underset{\sim}{\theta})$

Hamiltonian Monte Carlo

- Turn the mountain  $p(x, \underset{\sim}{\theta})$

$\underset{\sim}{\theta}$ fixed

into a bowl  by using $-\log p(x, \underset{\sim}{\theta})$

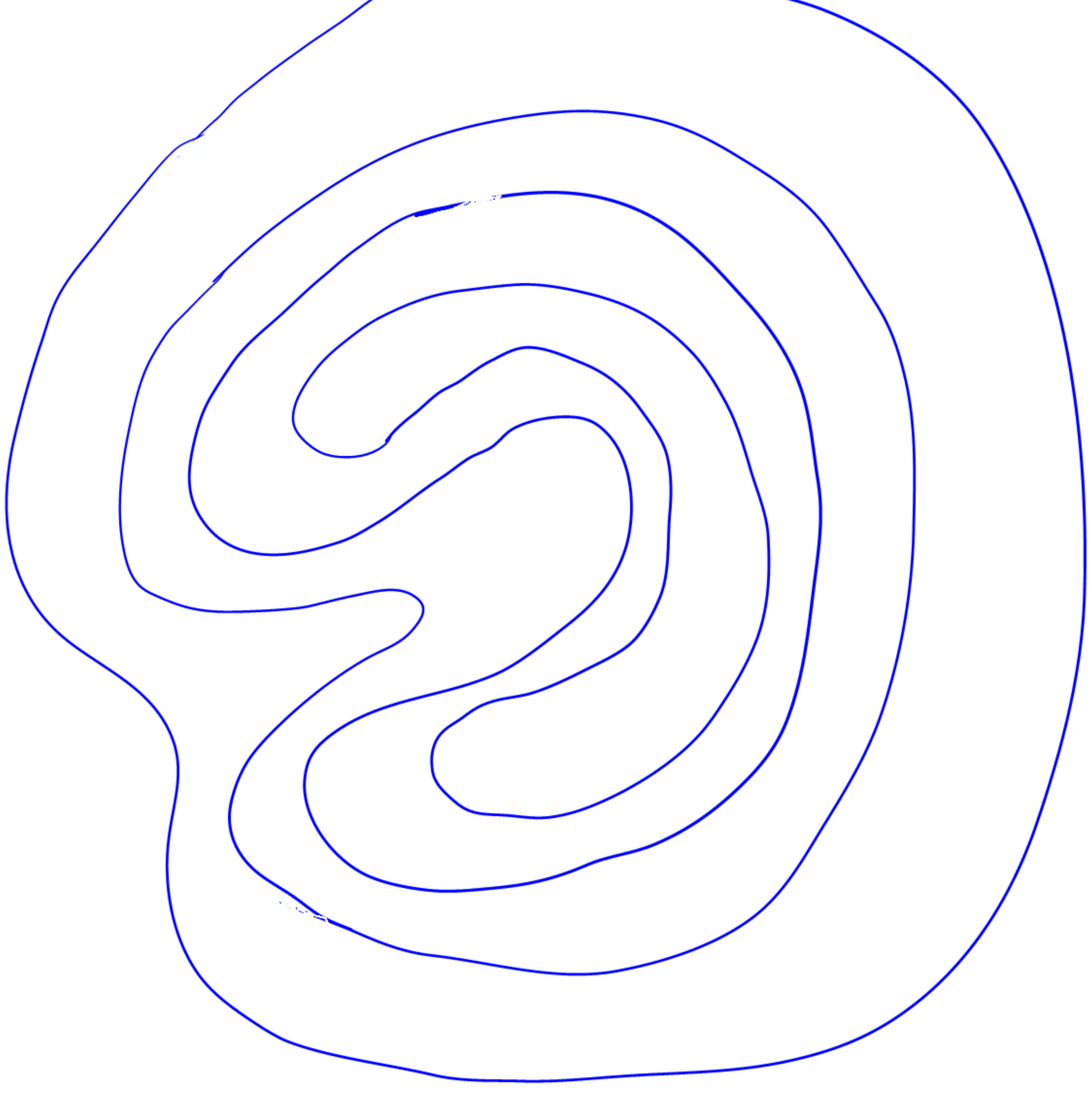
$\underset{\sim}{\theta}$

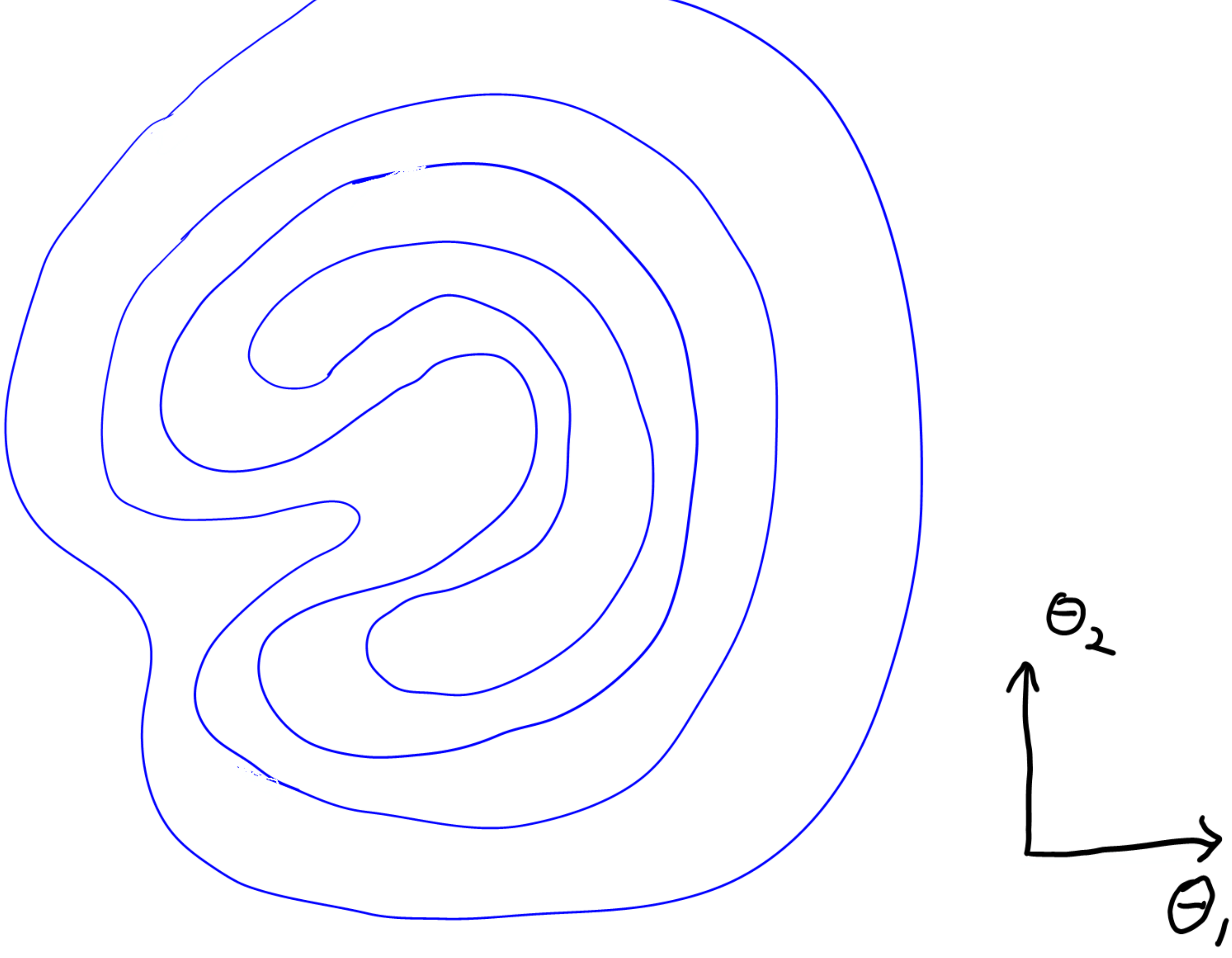
- Instead of taking random steps, go for a ride on a frictionless skateboard with swivel wheels — starting with a random push.


Red Bull
GENERATION

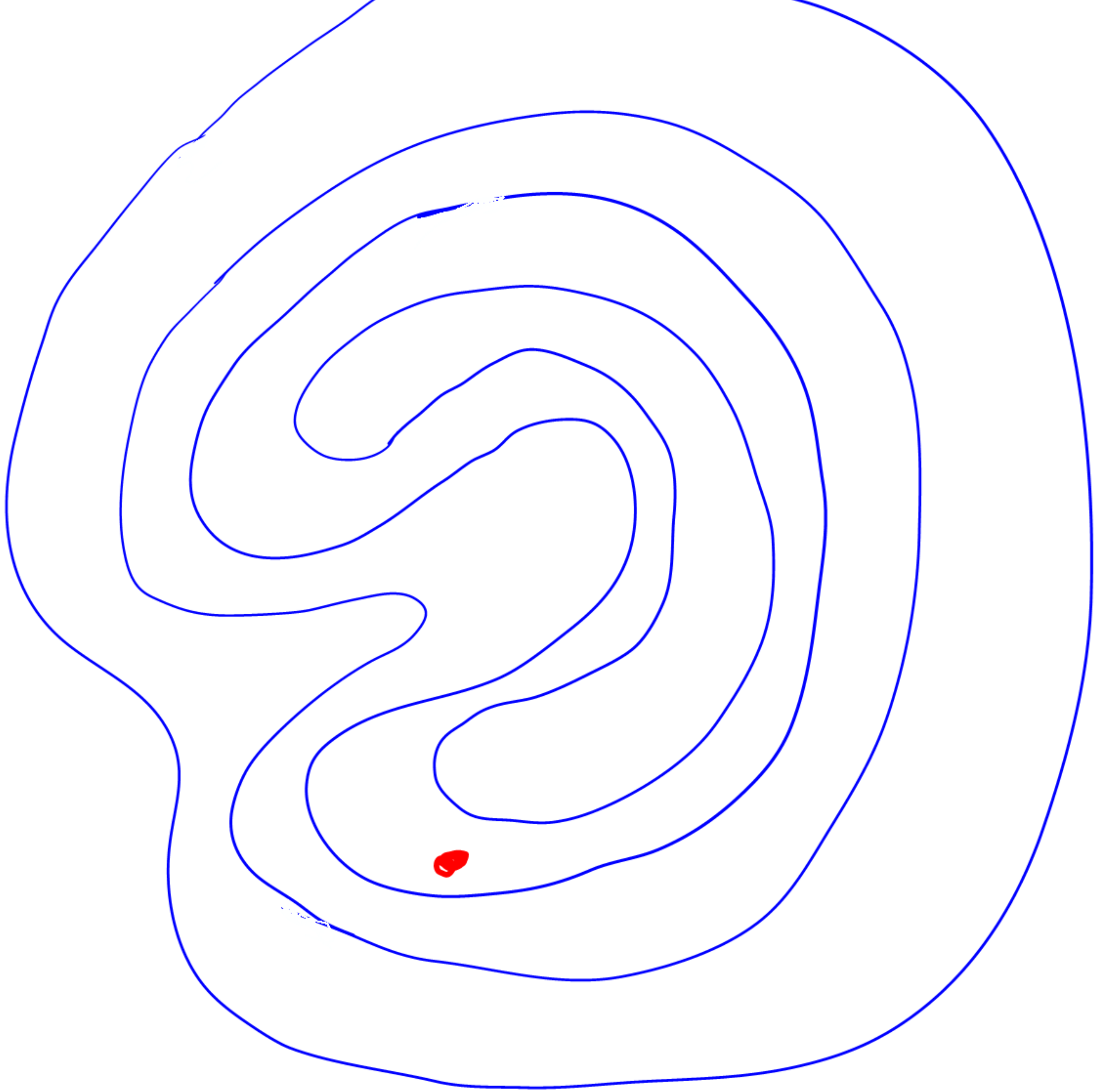
$-\log p(x,\theta)$

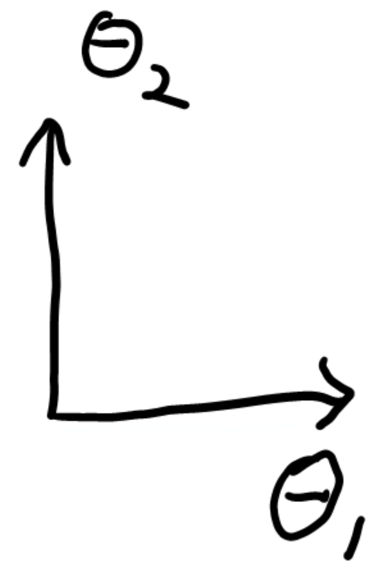

$-\log p(x,\theta)$
θ_2
θ_1

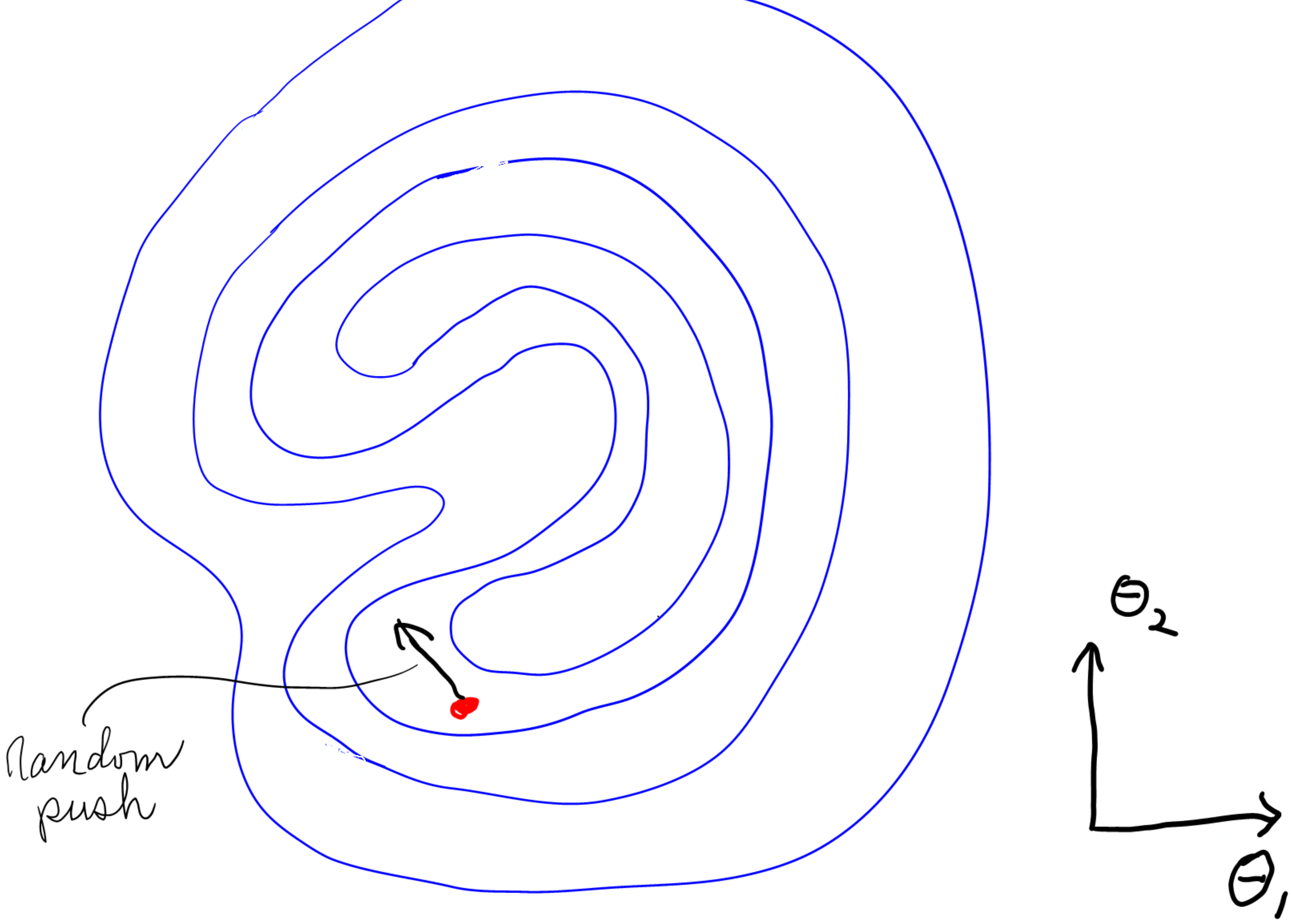



θ_2
θ_1

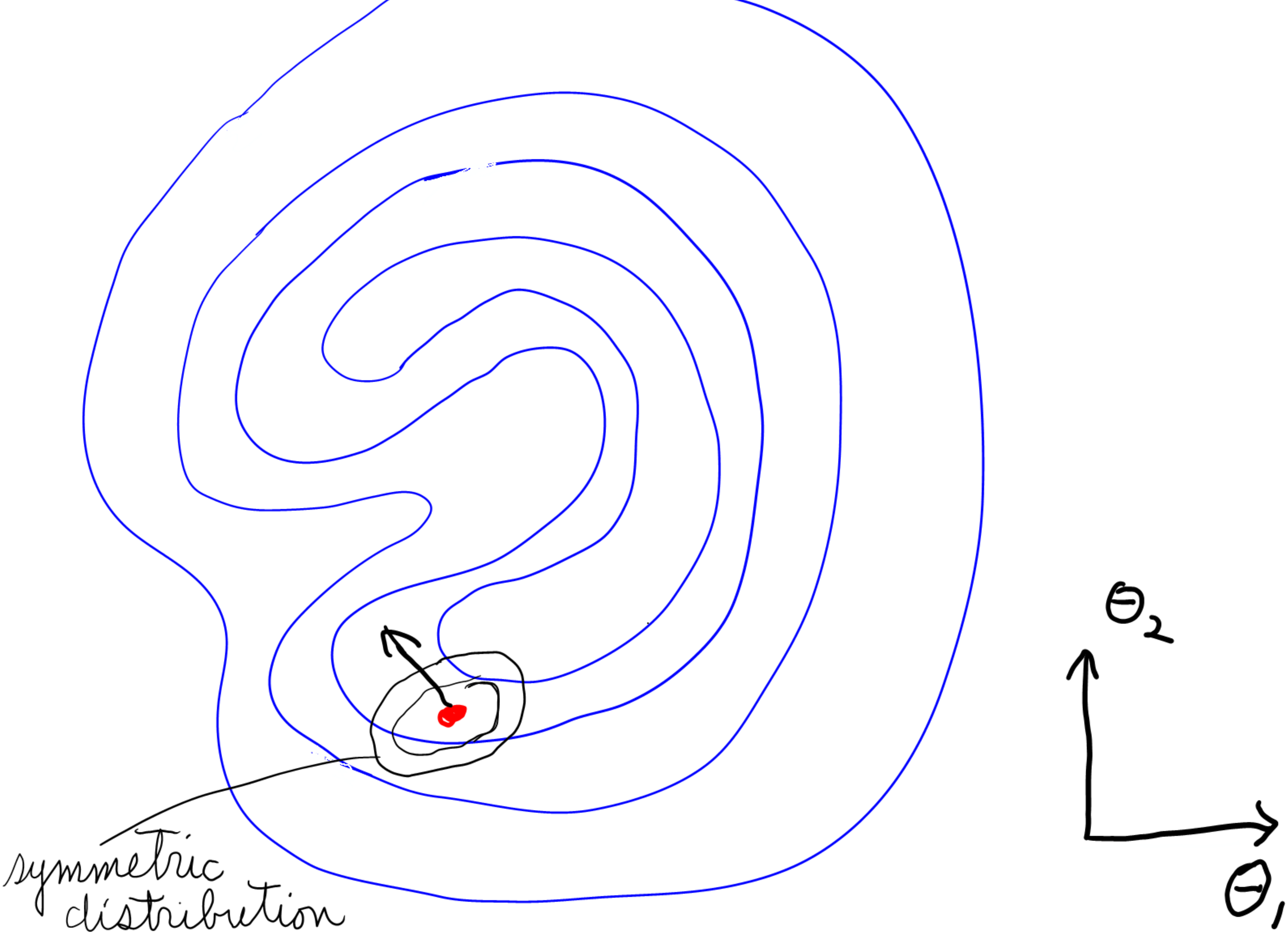
Hamiltonian monte Carlo



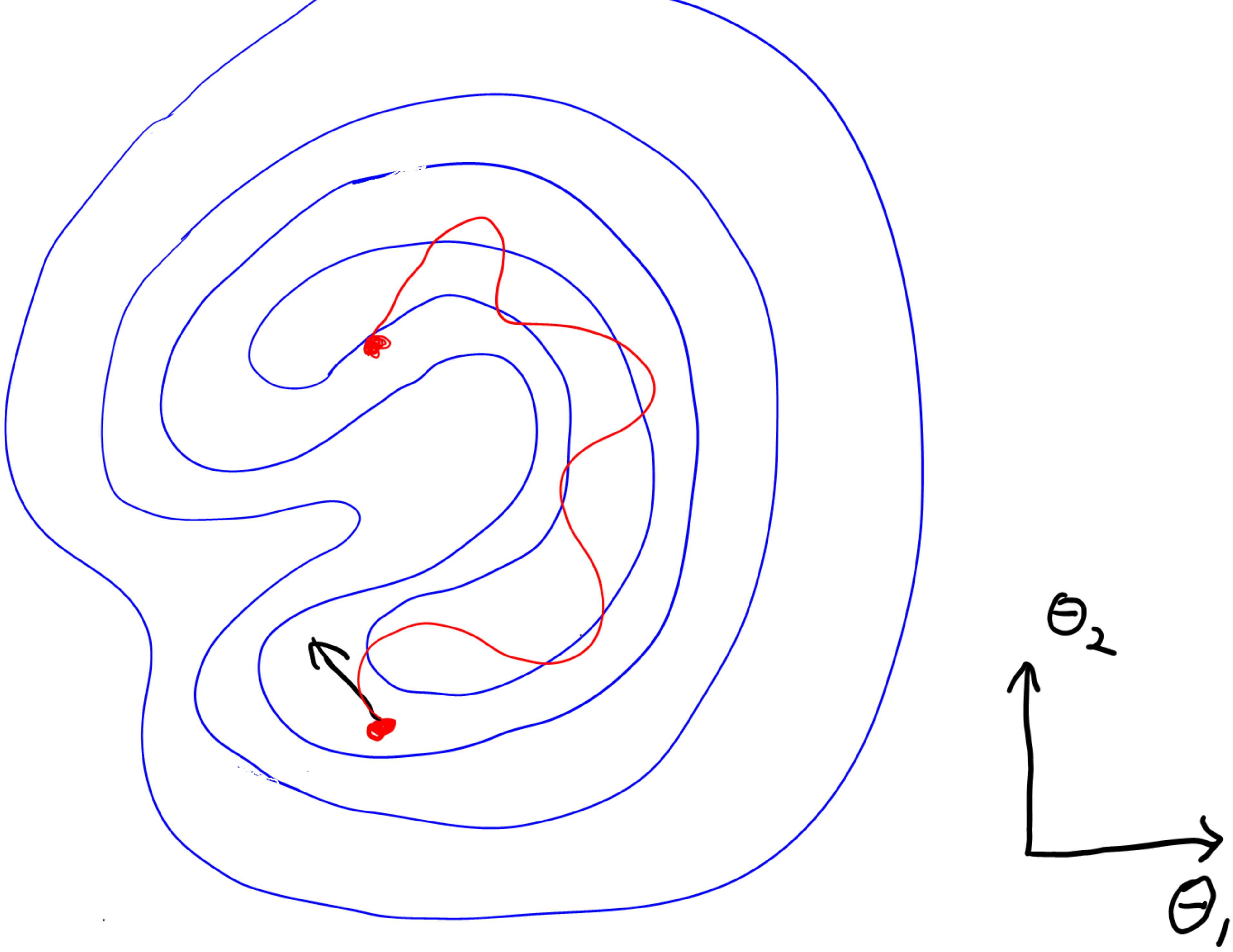




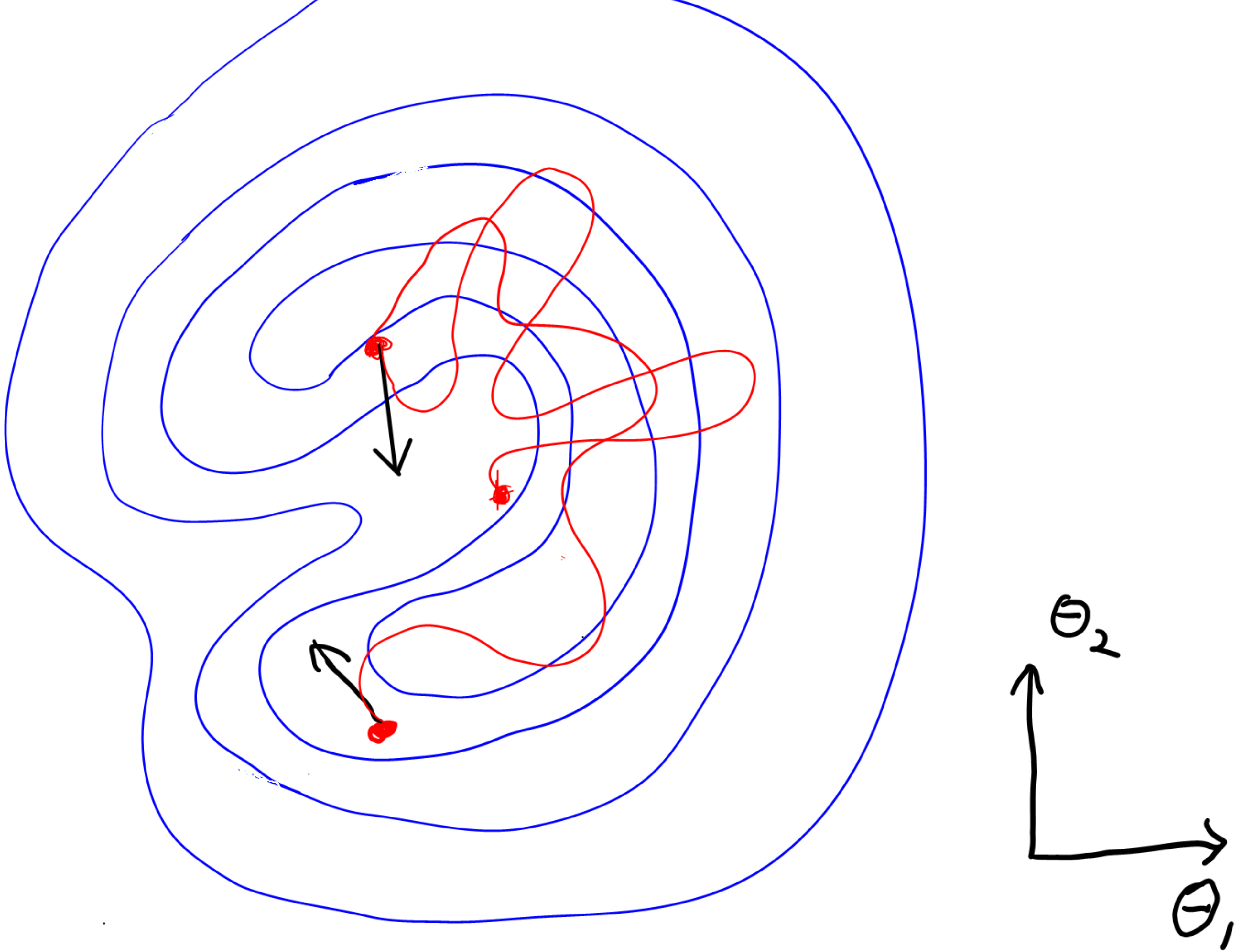
Random push
θ_2
θ_1



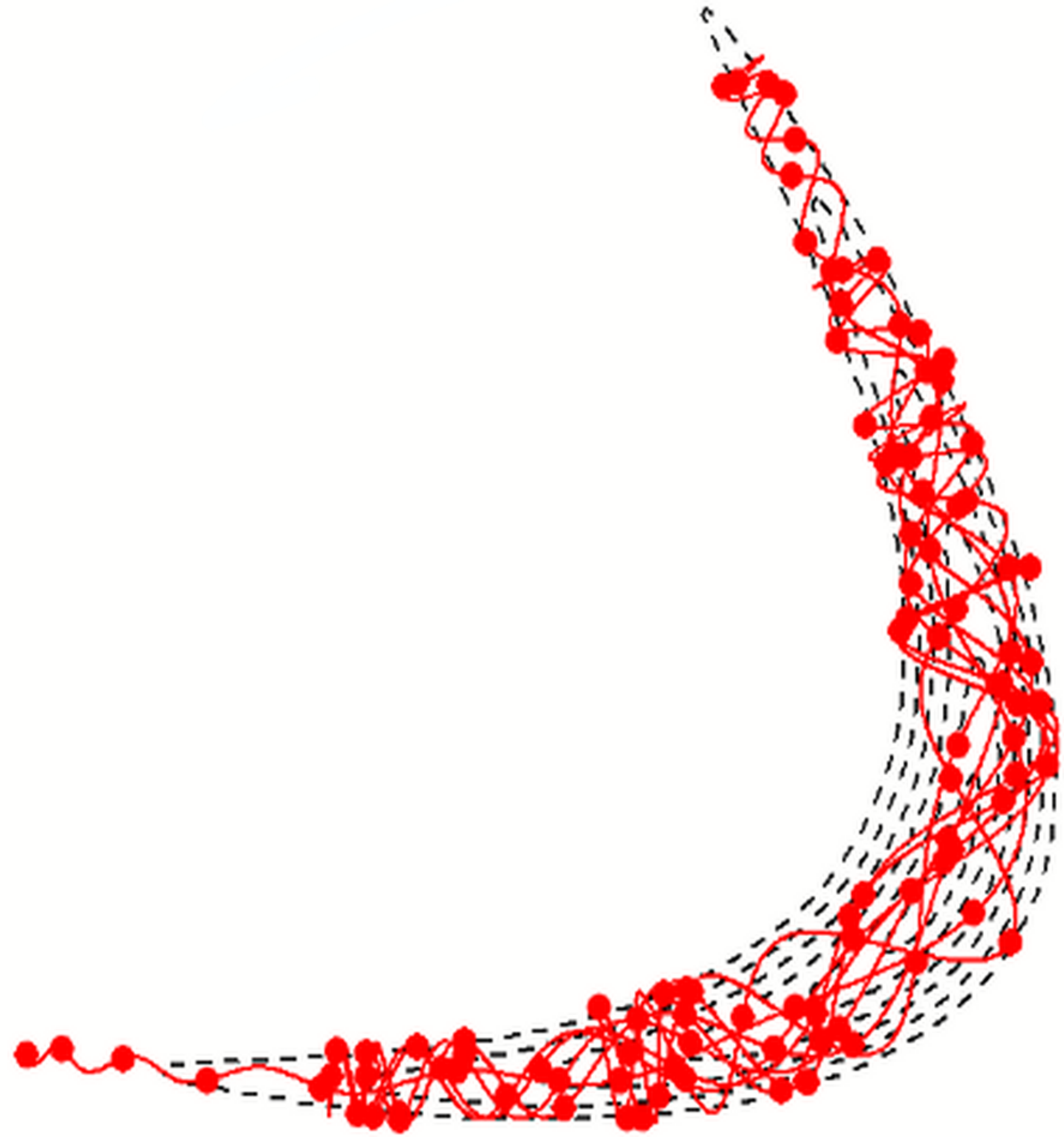
symmetric distribution
θ_2
θ_1



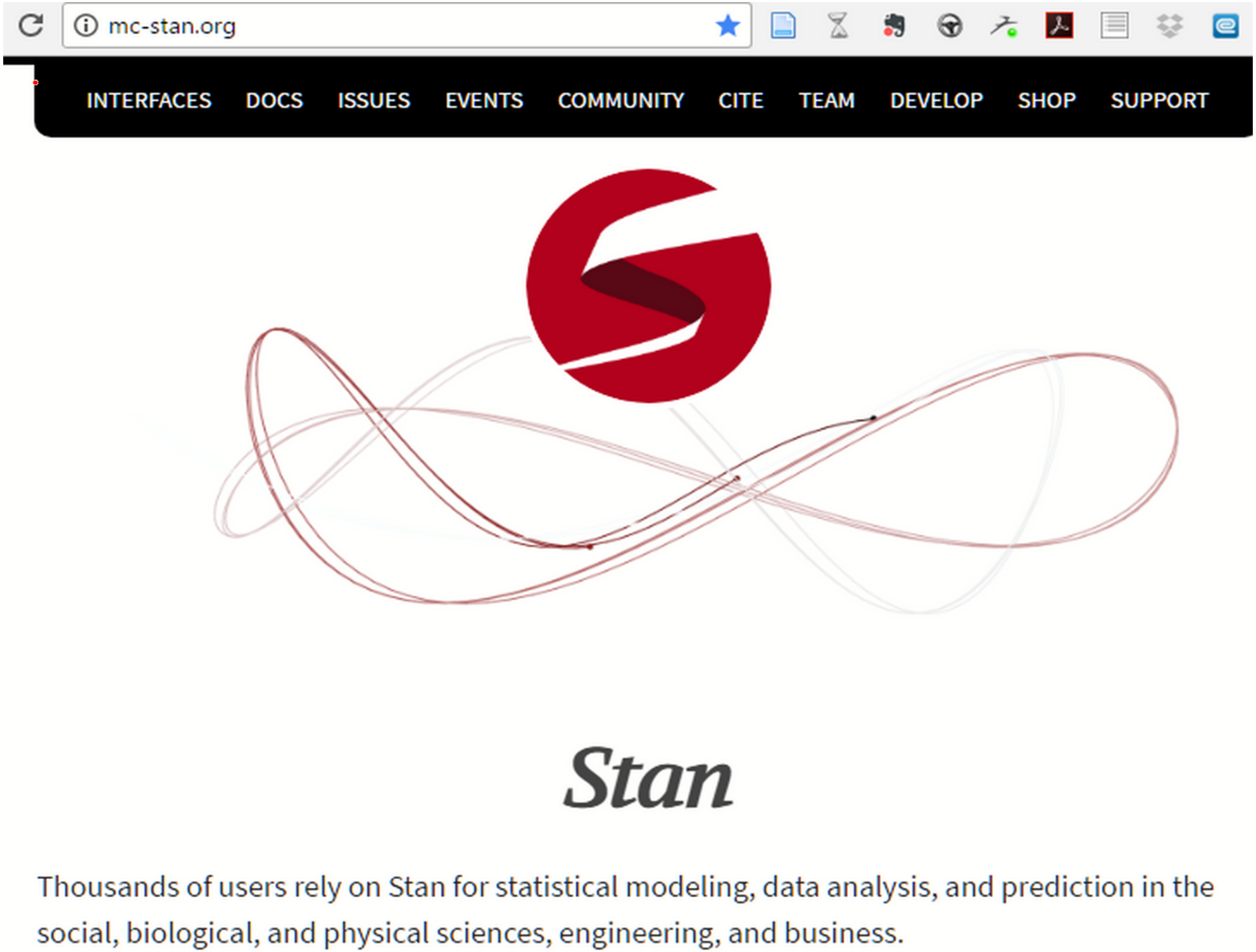
θ_2
θ_1

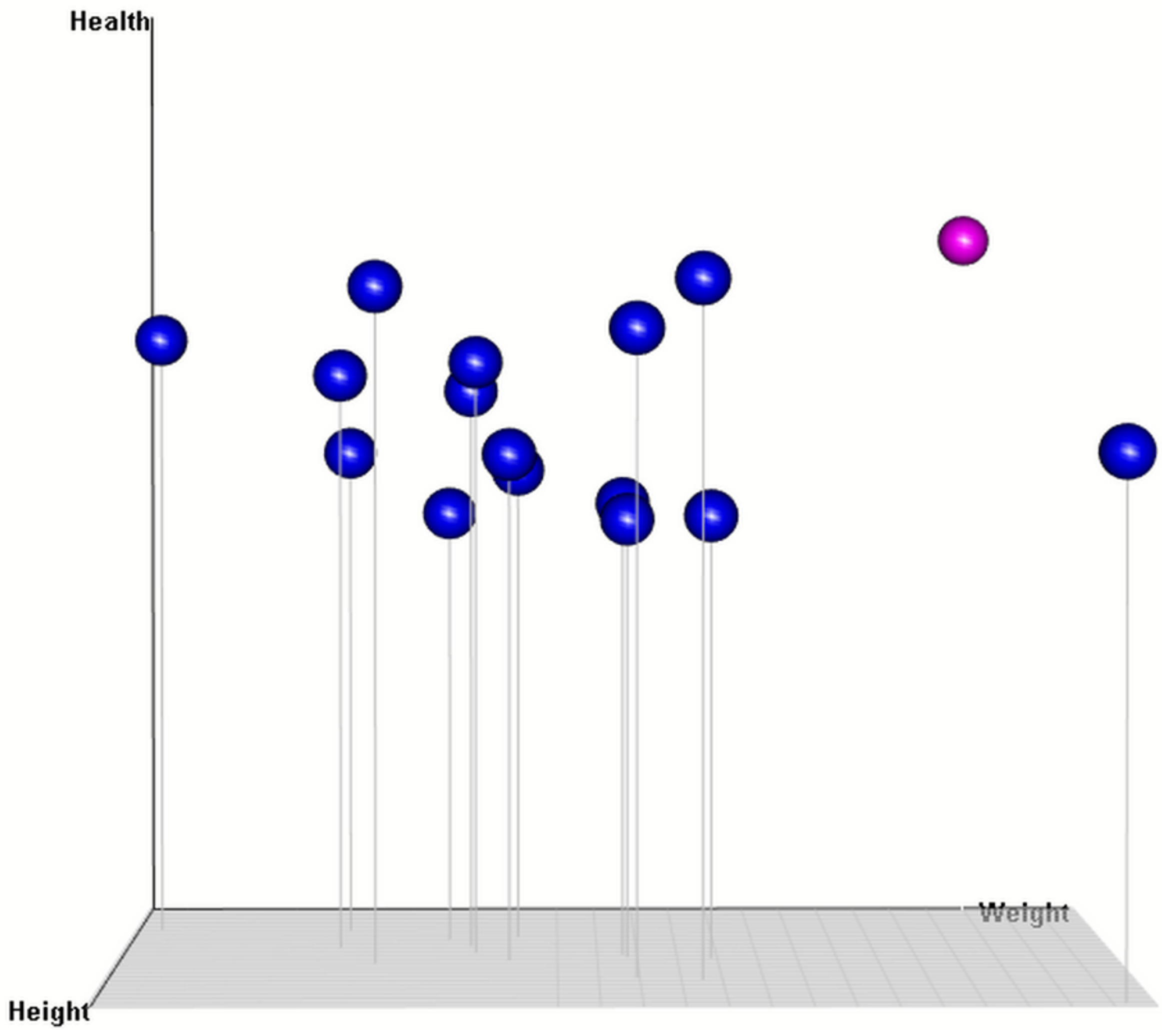


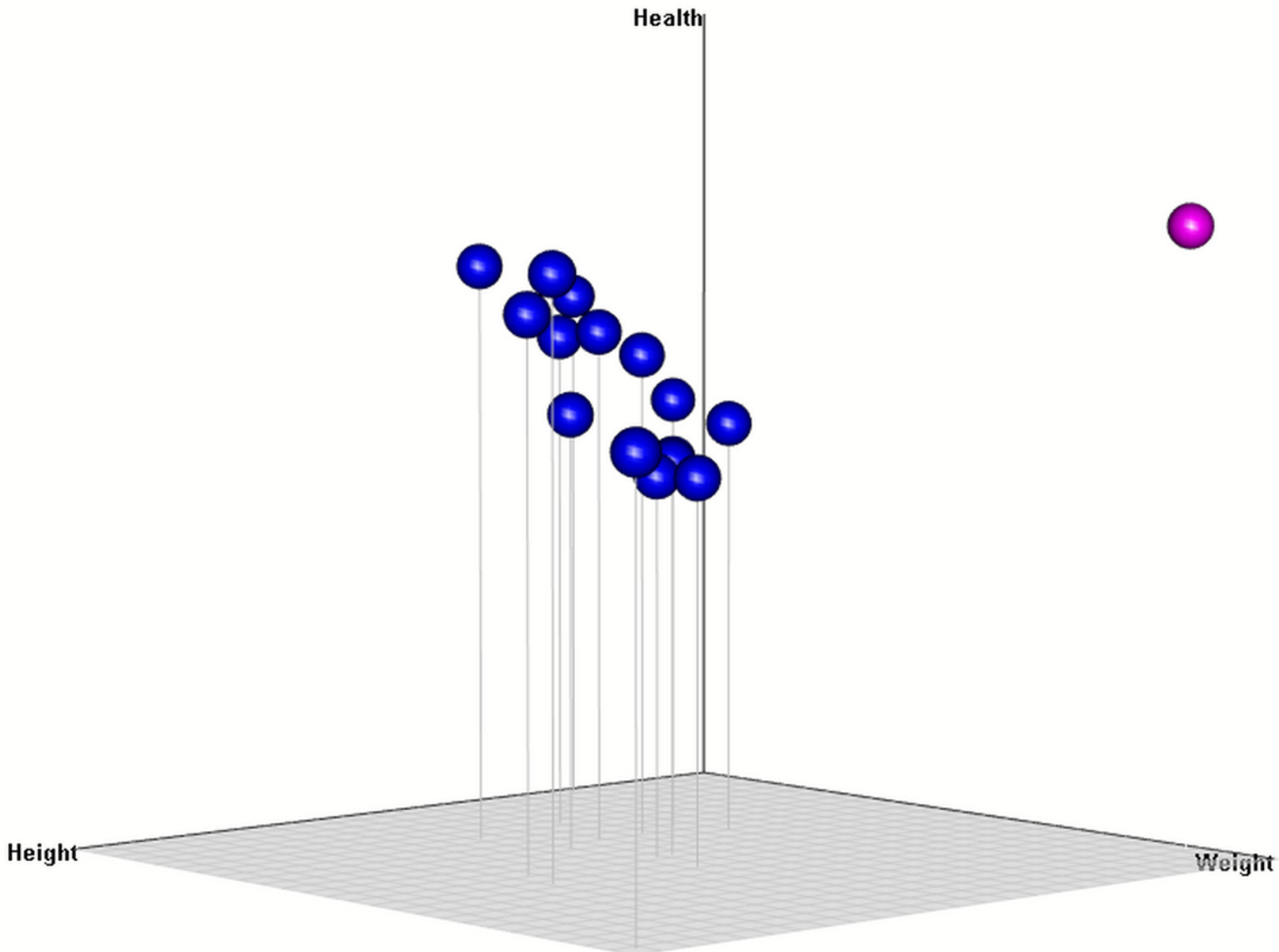
θ_2
θ_1

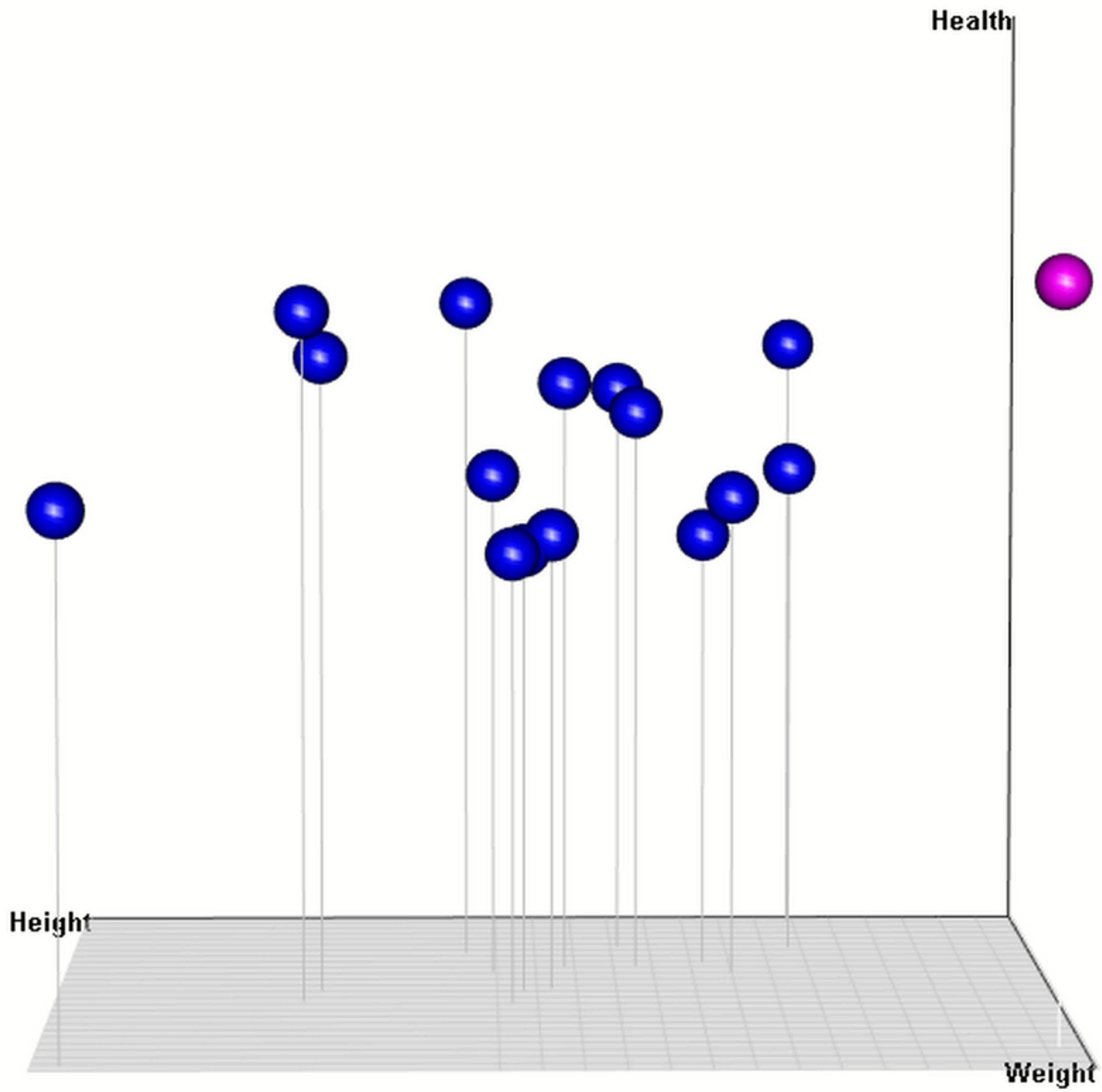


- Much less dependency between successive points in comparison with other MCMC methods
- However, each point is more expensive to generate
- Much faster with large # of parameters


mc-stan.org
INTERFACES
DOCS
ISSUES
EVENTS
COMMUNITY
CITE
TEAM
DEVELOP
SHOP
SUPPORT
Stan
Thousands of users rely on Stan for statistical modeling, data analysis, and prediction in the social, biological, and physical sciences, engineering, and business.


Health
Weight
Height


Health
Height
Weight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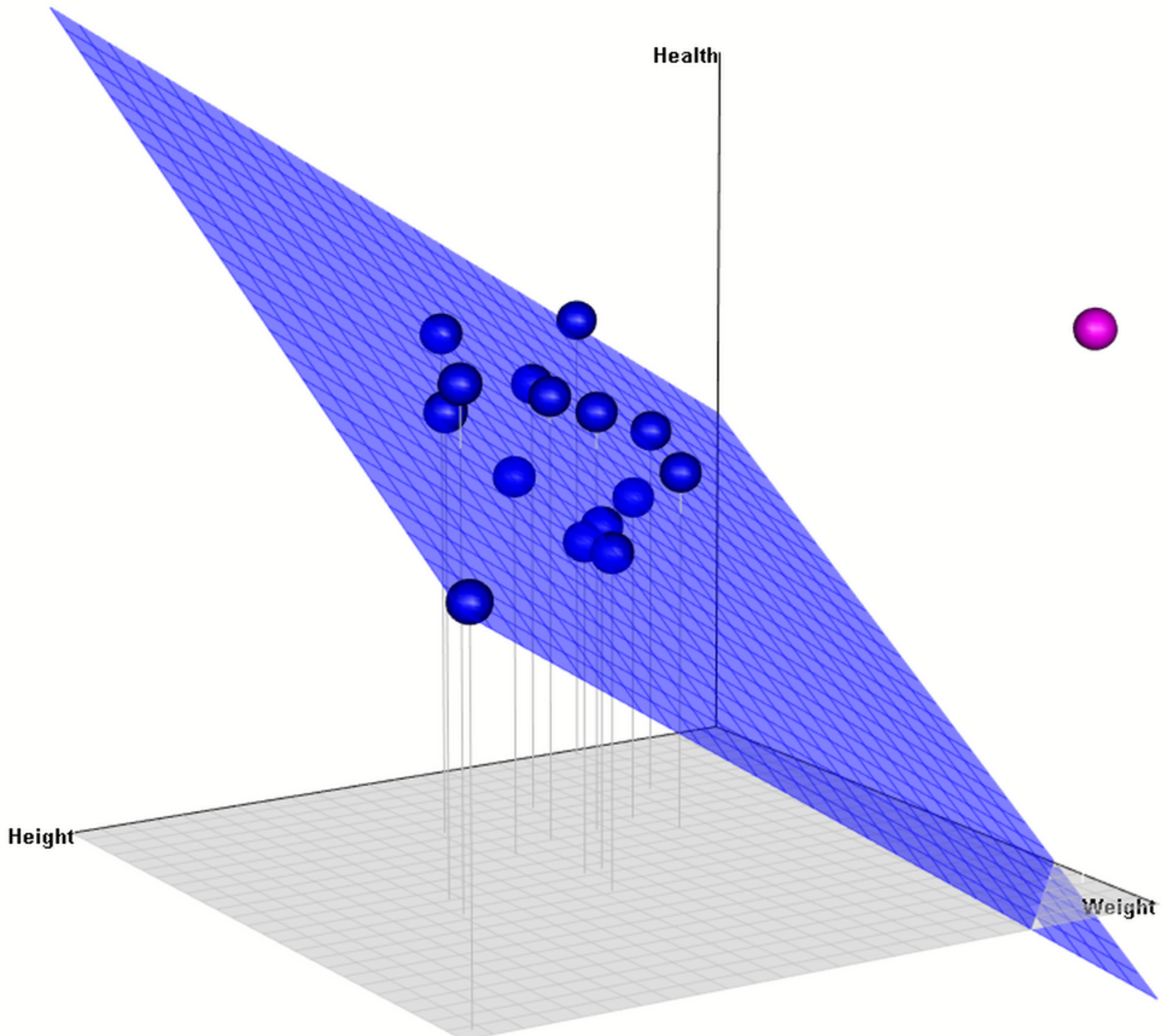
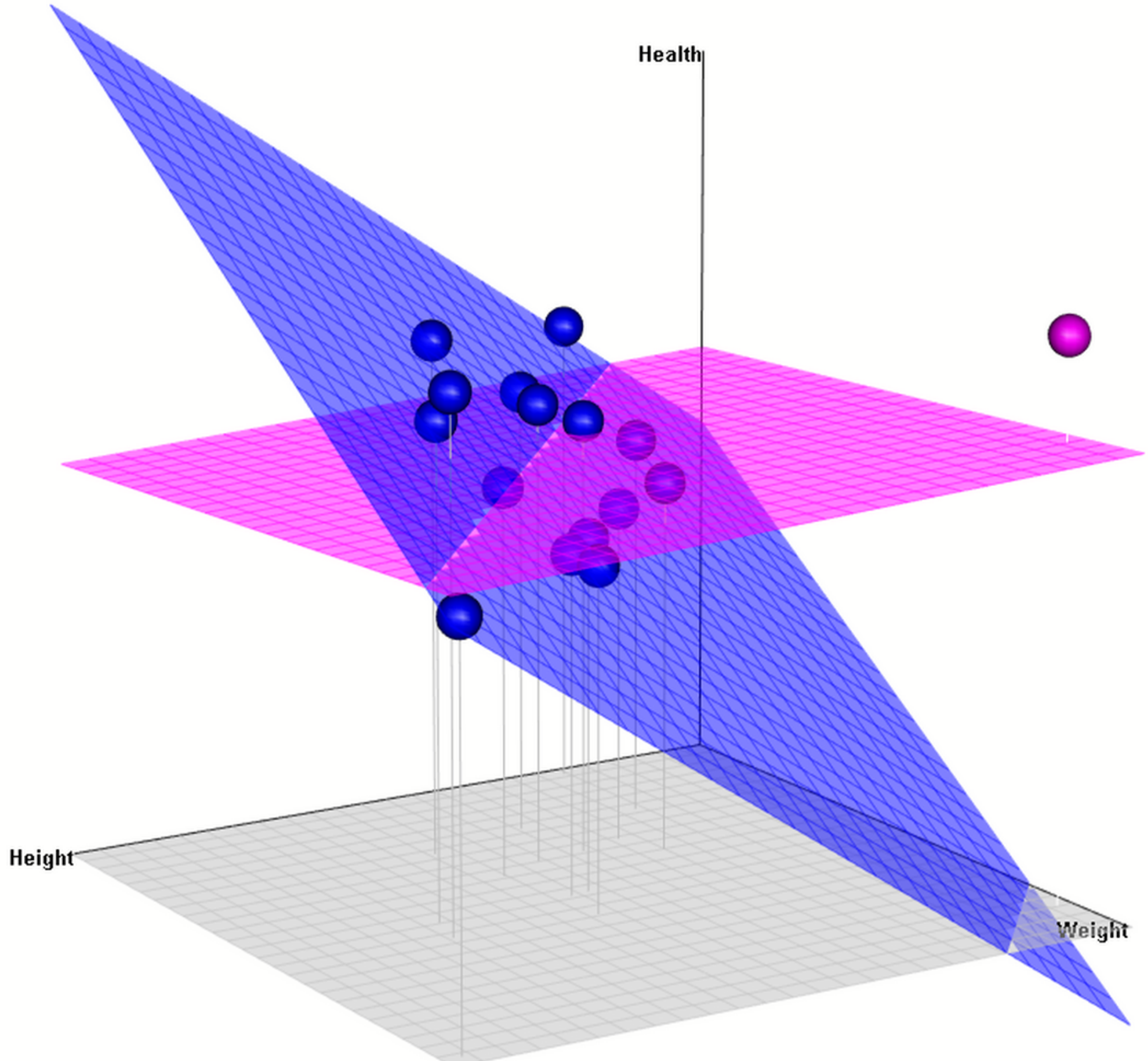
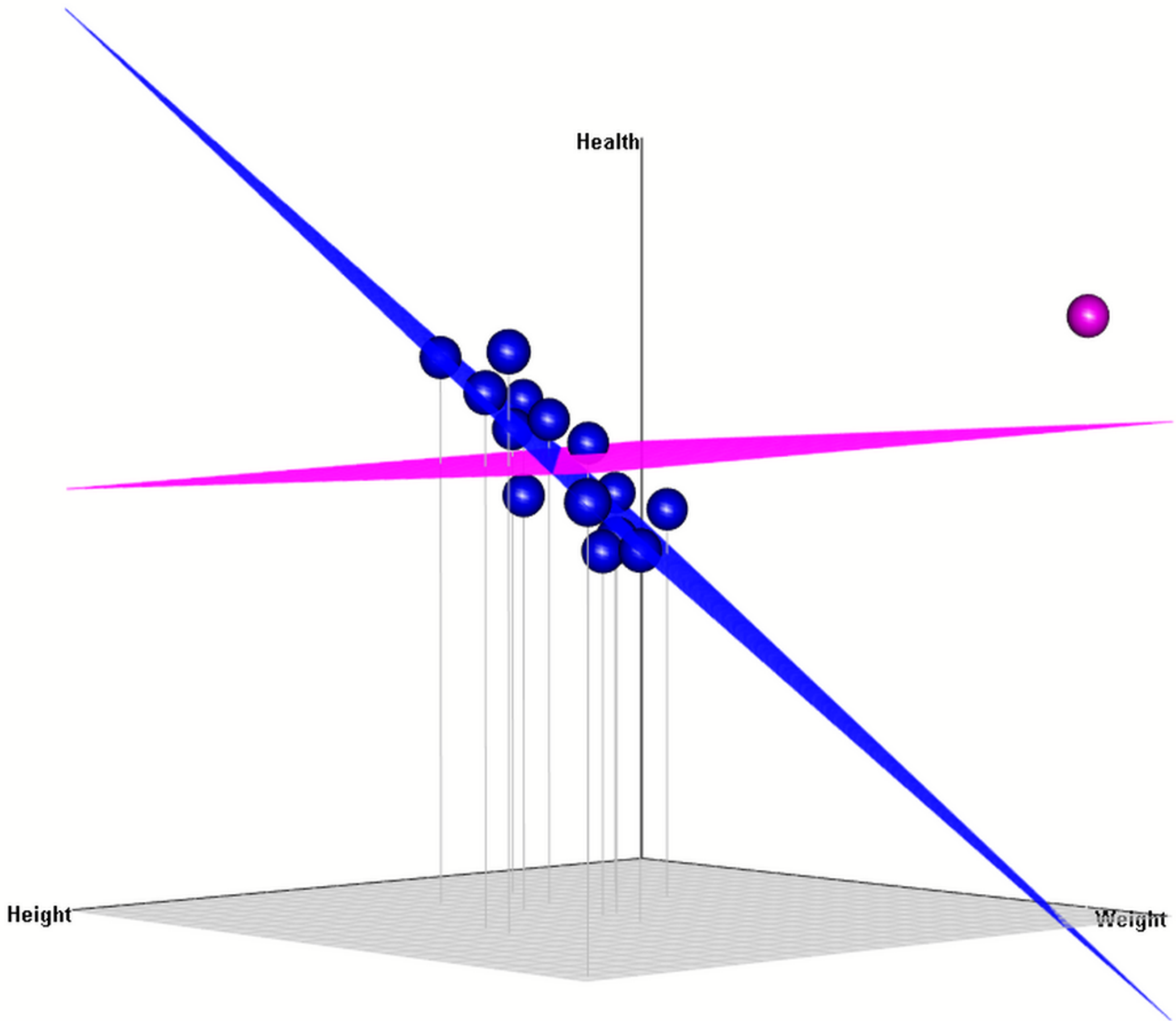
Height
Weight


Health
Height
Weight


Health
Height
Weight


Health
Height
Weight

Defining a model in Stan

```
data {
  int N;   // number of observations
  int P;   // number of columns of X matrix (including intercept)
  matrix[N,P] X;   // X matrix including intercept
  vector[N] y;  // response
}

parameters {
  vector[P] beta;       // default uniform prior if nothing specied in model
  real <lower=0> sigma; // uniform on positive reals
}

model {
  y ~ normal( X * beta, sigma ); // note that * is matrix mult.
                                 // For elementwise multiplication use .*
}
```

Data

```
head( Xmat <- model.matrix(Health ~ Weight + Height, dd) )
```

```
  (Intercept) Weight Height
1           1 0.3355 0.6008
2           1 0.6890 0.9440
3           1 0.6980 0.6150
4           1 0.7617 1.2340
5           1 0.8910 0.7870
6           1 0.9330 0.9150
```

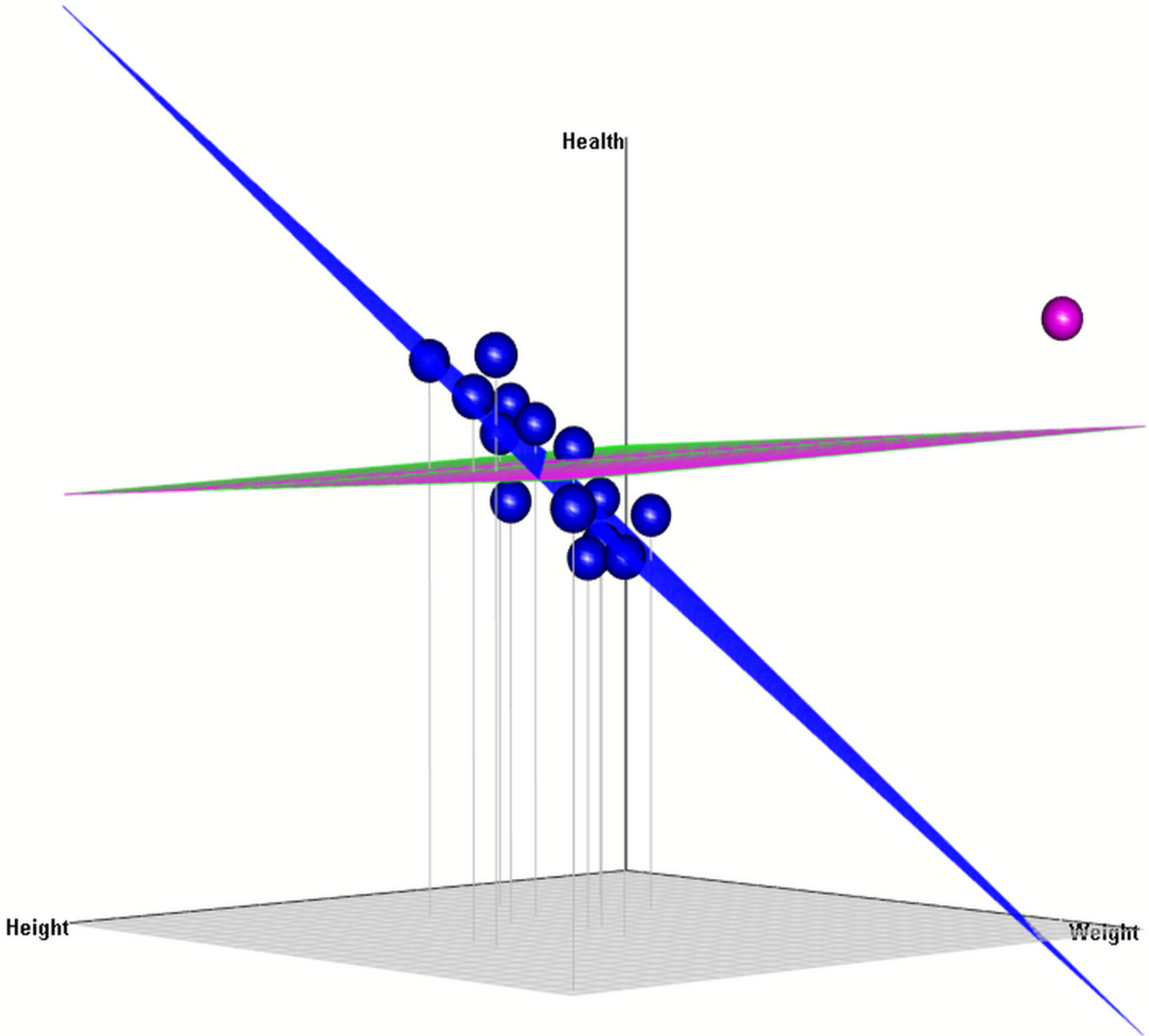
```
dat_list <- list(N = nrow(Xmat), P = ncol(Xmat),
                 X = Xmat, y = dd$Health)
```

```
$N
[1] 16

$P
[1] 3

$X
   (Intercept) Weight Height
1            1 0.3355 0.6008
2            1 0.6890 0.9440
3            1 0.6980 0.6150
4            1 0.7617 1.2340
5            1 0.8910 0.7870
6            1 0.9330 0.9150
7            1 0.9430 1.0490
8            1 1.0060 1.1840
9            1 1.0200 0.7370
10           1 1.2150 1.0770
11           1 1.2230 1.1280
12           1 1.2360 1.5000
13           1 1.3530 1.5310
14           1 1.3770 1.1500
15           1 2.0734 1.9340
18           1 1.9000 0.2000
attr(,"assign")
[1] 0 1 2

$y
 [1] 1.280 1.208 1.036 1.395 0.912 1.175 1.237 1.048 1.003 0.943 0.912
[12] 1.311 1.411 0.920 1.073 1.500
```

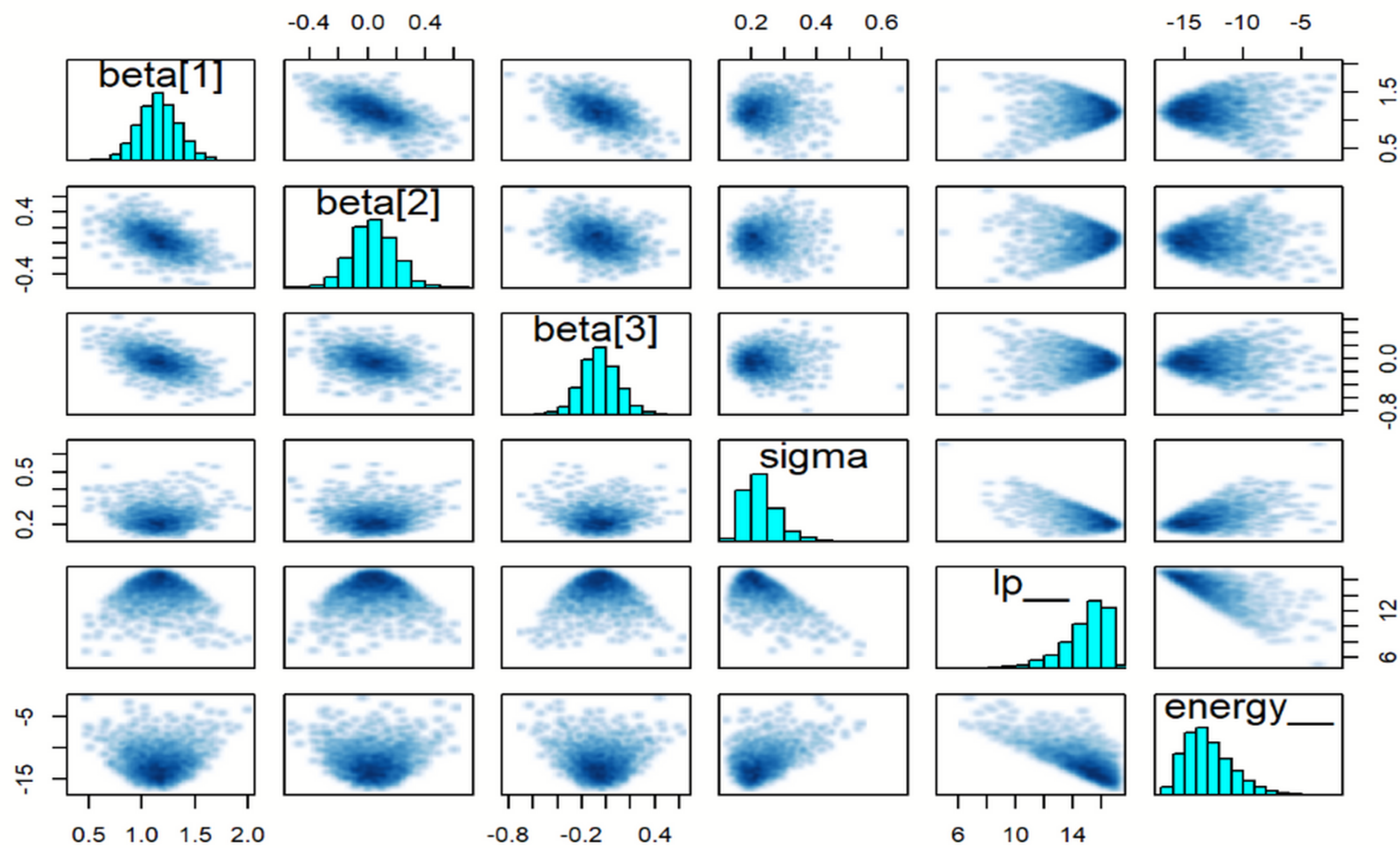
```r
fit_stan <- sampling(reg_model_dso, dat_list)
```

```r
fit_stan
```

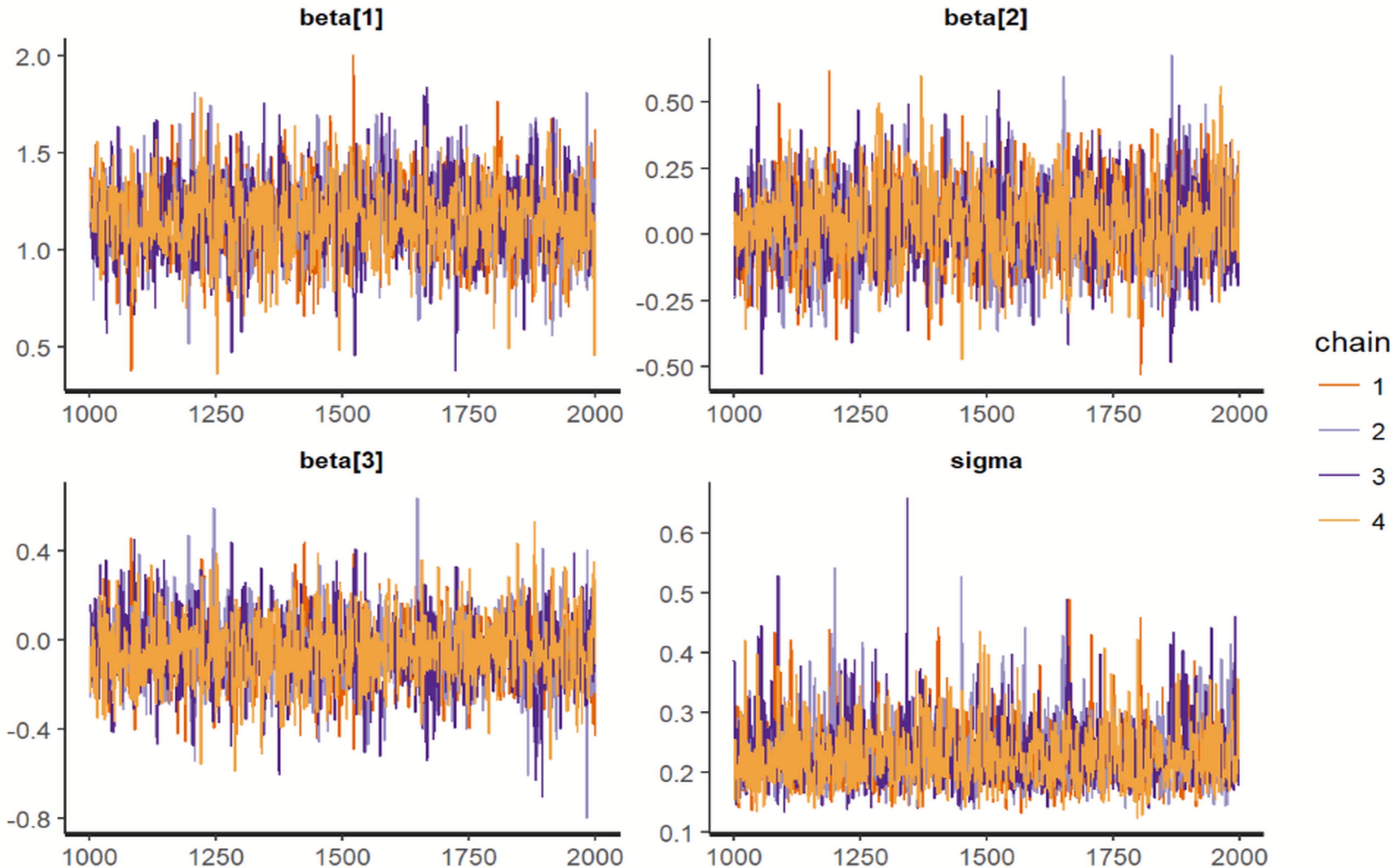
```
Inference for Stan model: 5a6361673e39acd797b5518d36e20615.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

         mean se_mean   sd  2.5%   25%   50%   75% 97.5% n_eff Rhat
beta[1]  1.16    0.00 0.20  0.77  1.03  1.16  1.29  1.55  2035    1
beta[2]  0.04    0.00 0.15 -0.26 -0.06  0.04  0.14  0.34  1980    1
beta[3] -0.05    0.00 0.16 -0.36 -0.15 -0.06  0.04  0.26  2135    1
sigma    0.23    0.00 0.05  0.15  0.19  0.22  0.26  0.36  1713    1
lp__    14.88    0.05 1.64 10.52 14.08 15.25 16.08 16.93  1084    1

Samples were drawn using NUTS(diag_e) at Mon Jun 05 22:52:12 2017.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```


Health
Height
Weight

```
pairs(fit_stan)
```



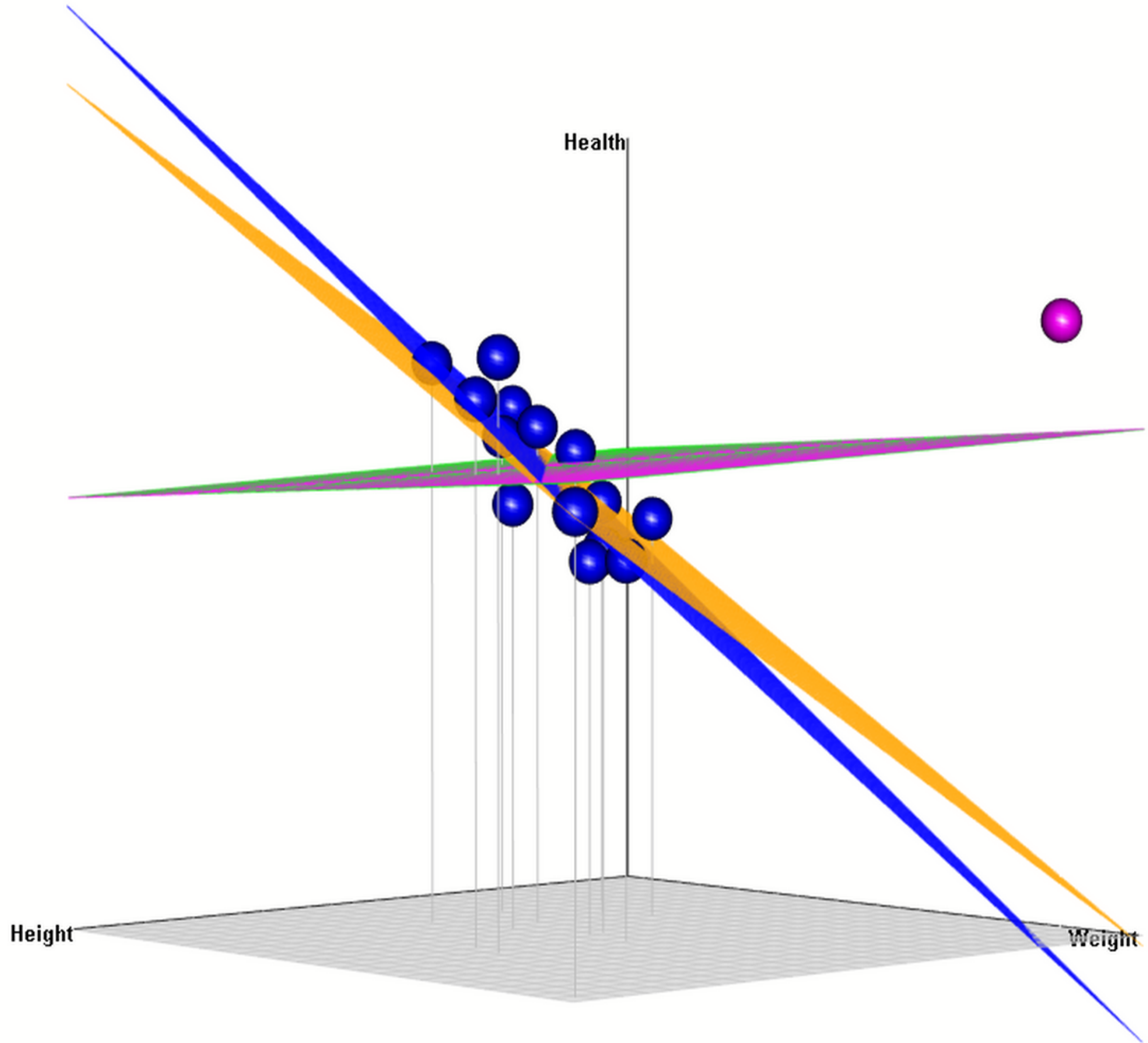
```
traceplot(fit_stan)
```



Robust model: Just change the distribution of Y

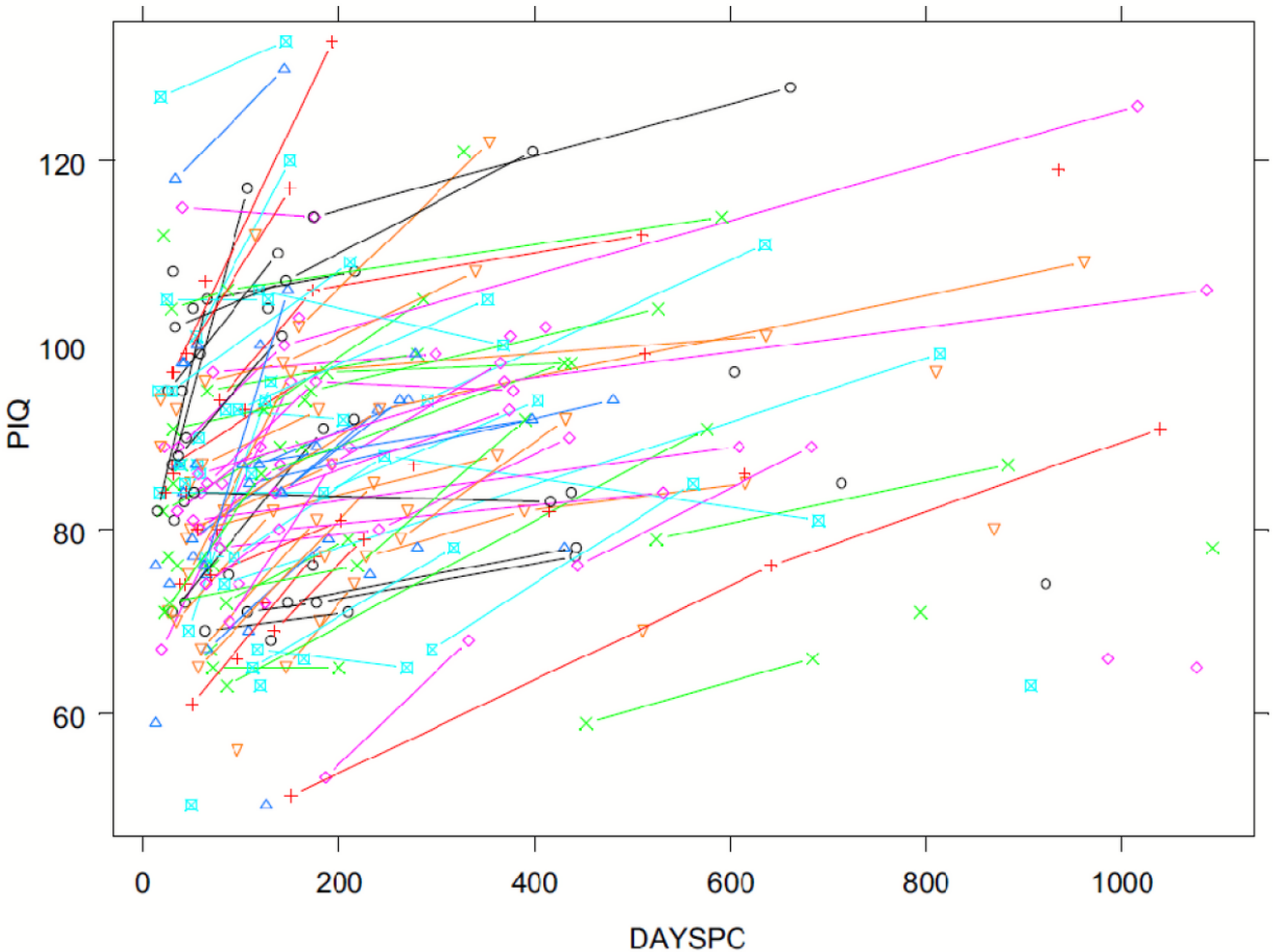
```
data {
  int N;    // number of observations
  int P;    // number of columns of X matrix (including intercept)
  matrix[N,P] X;    // X matrix including intercept
  vector[N] y;  // response
  int nu;    // degrees for freedom for student_t
}

parameters {
  vector[P] beta;    // default uniform prior if nothing specied in model
  real <lower=0> sigma;
}

model {
  y ~ student_t(nu, X * beta, sigma );
}
```

```
fit3_stan_2 <- sampling(robust_model_dso, c(dat_list, nu = 2))
```

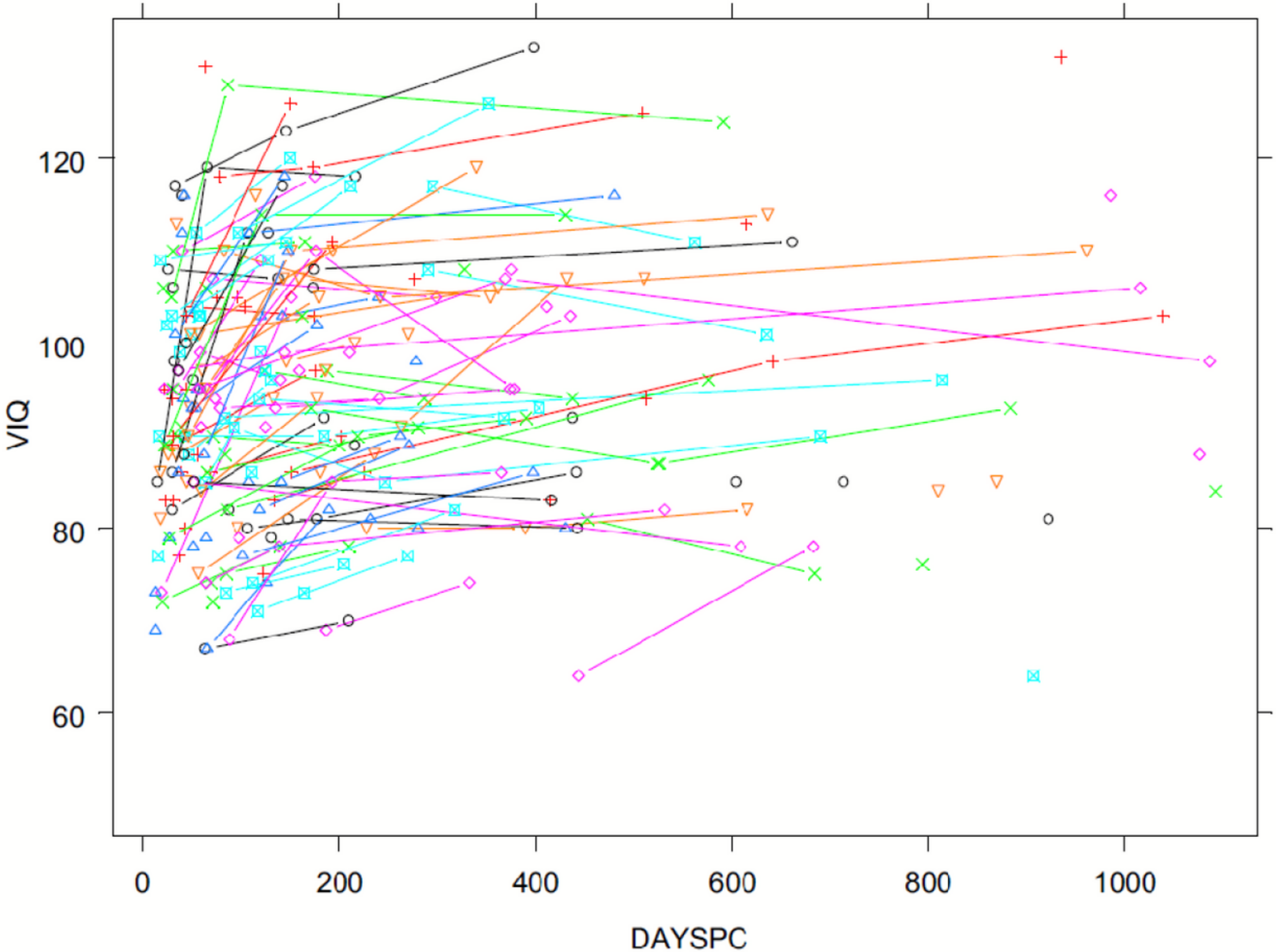

Health
Height
Weight

Traumatic Brain Injury

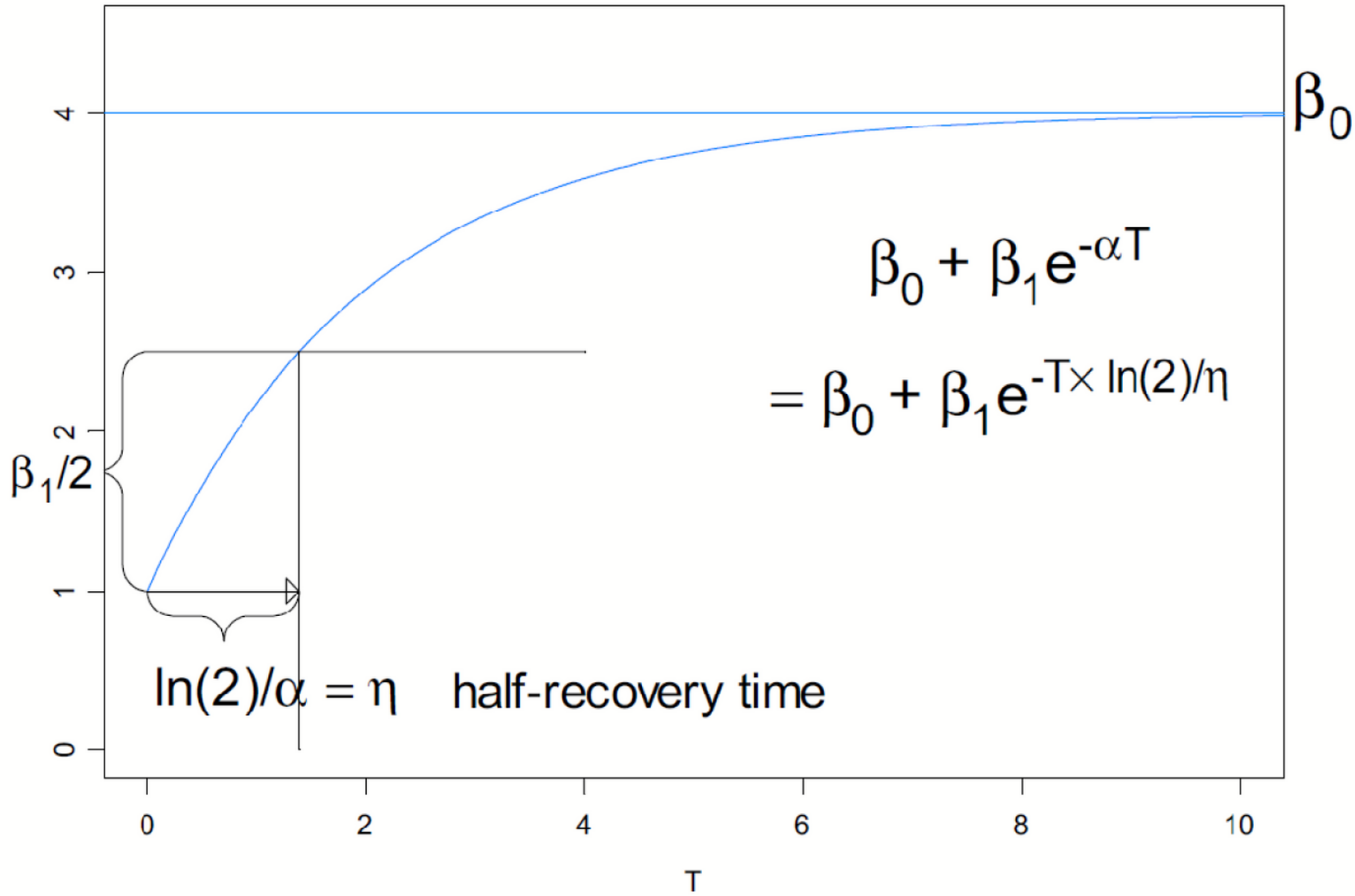
- Recovery after coma
- Non-linear asymptotic recovery curves



PIQ
DAYSPC
0
200
400
600
800
1000
60
80
100
120



VIQ
120
100
80
60
0
200
400
600
800
1000
DAYSPC

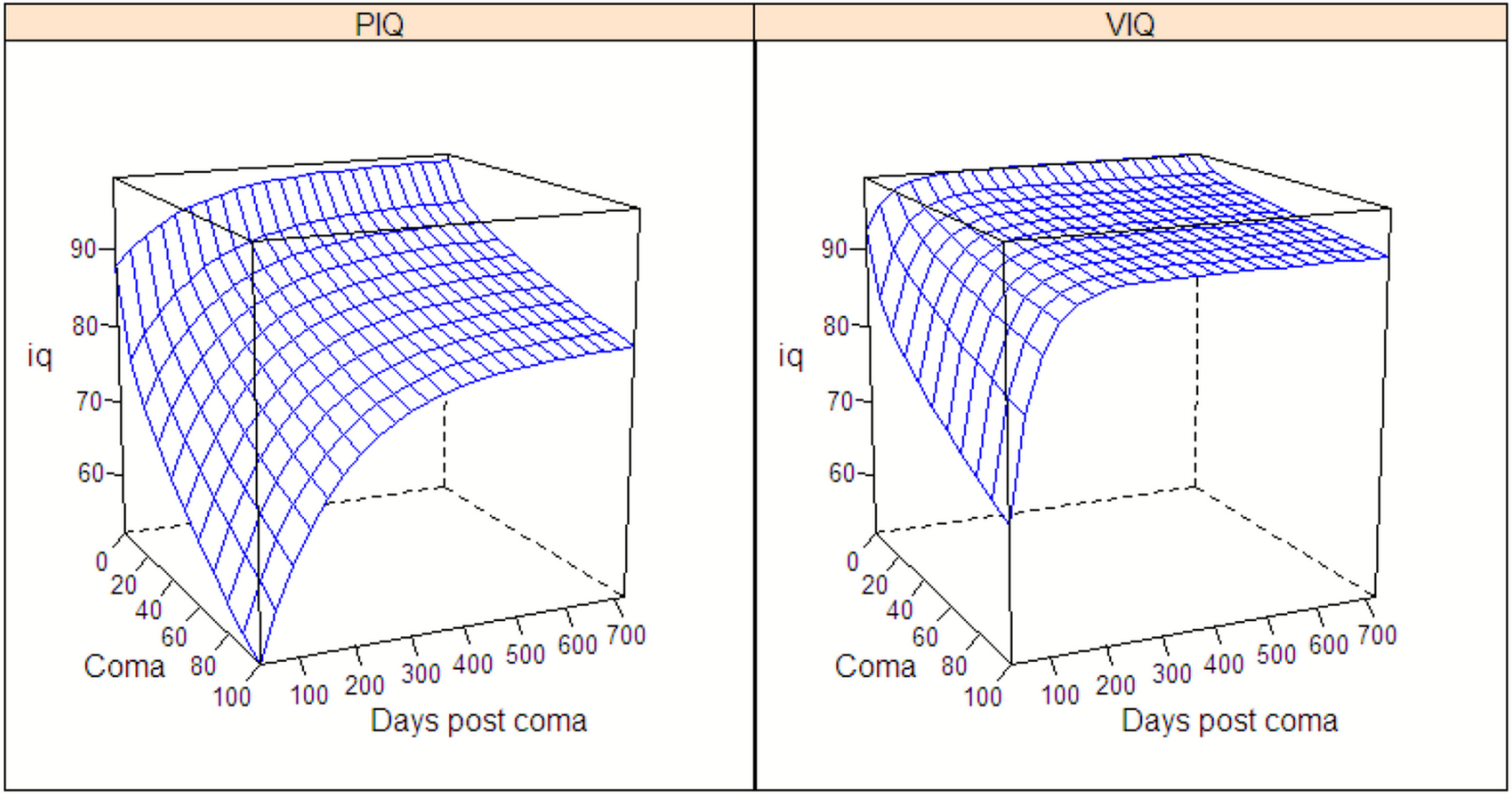


β_0
$\beta_0 + \beta_1 e^{-\alpha T}$
$= \beta_0 + \beta_1 e^{-T\times \ln(2)/\eta}$
$\beta_1/2$
$\ln(2)/\alpha = \eta$ half-recovery time
0
1
2
3
4
0
2
4
6
8
10
T

```
//
//  Multivariate model for VIQ and PIQ
//

data {
  int N;
  int J;
  matrix[N,2] iq;
  vector[N] time;
  vector[N] coma;
  int id[N];
}
```

```
parameters {
  vector <lower=1,upper=10000>[2] hrt;
  vector <lower=0,upper=200>[2] asymp;
  vector <lower=-100,upper=100>[2] init_def;
  vector [2] bcoma;
  vector[2] u[J];
  cov_matrix[2] Sigma;
  cov_matrix[2] Sigma_u;
}
transformed parameters {
  real hrt_diff;
  real bcoma_diff;
  hrt_diff = hrt[2] - hrt[1];
  bcoma_diff = bcoma[2] - bcoma[1];
}
```

```
model {
  vector[2] eta;
  // for the multinormal distribution we need to loop over observations
  for(j in 1:J) u[j] ~ multi_normal(zero, Sigma_u);
  for(n in 1:N) {
    eta[1] = asymp[1] + u[id[n],1] + bcoma[1] * coma[n] +
            init_def[1] * exp(-time[n]/(hrt[1]*ln2));
    eta[2] = asymp[2] + u[id[n],2] + bcoma[2] * coma[n] +
            init_def[2] * exp(-time[n]/(hrt[2]*ln2));
    iq[n,] ~ multi_normal(eta, Sigma);
  }
}
```


PIQ
VIQ
iq
90
80
70
60
Coma
0
20
40
60
80
100
Days post coma
100
200
300
400
500
600
700

```
Inference for Stan model: asymp_model_4.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

                 mean se_mean    sd     2.5%      25%      50%      75%    97.5% n_eff Rhat
hrt[1]          65.18    0.42 18.23    36.67    52.36    62.78    75.12   108.37  1867 1.00
hrt[2]         249.32    1.02 53.51   160.66   211.31   244.72   279.99   373.35  2741 1.00
asymp[1]        99.79    0.06  1.61    96.57    98.73    99.80   100.87   102.86   639 1.01
asymp[2]       100.53    0.07  1.97    96.62    99.23   100.52   101.86   104.38   818 1.00
init_def[1]    -23.34    0.12  4.88   -34.91   -25.94   -22.77   -20.00   -15.66  1633 1.00
init_def[2]    -19.46    0.04  1.95   -23.17   -20.77   -19.52   -18.15   -15.57  1916 1.00
bcoma[1]        -0.72    0.02  0.40    -1.50    -0.99    -0.72    -0.44     0.05   565 1.00
bcoma[2]        -1.93    0.02  0.42    -2.78    -2.22    -1.94    -1.64    -1.11   629 1.00
Sigma[1,1]      33.16    0.10  4.19    25.82    30.24    32.82    35.79    42.37  1632 1.00
Sigma[2,1]      20.38    0.12  4.22    13.02    17.47    20.00    22.96    29.64  1298 1.00
Sigma[1,2]      20.38    0.12  4.22    13.02    17.47    20.00    22.96    29.64  1298 1.00
Sigma[2,2]      49.72    0.17  6.55    38.23    45.07    49.34    53.68    63.89  1446 1.00
Sigma_u[1,1]   162.62    0.31 19.63   128.29   148.57   160.99   175.41   202.97  4000 1.00
Sigma_u[2,1]   119.42    0.34 18.19    86.67   106.59   118.38   131.11   158.27  2916 1.00
Sigma_u[1,2]   119.42    0.34 18.19    86.67   106.59   118.38   131.11   158.27  2916 1.00
Sigma_u[2,2]   176.27    0.43 22.67   135.45   160.26   174.76   191.17   224.59  2766 1.00
hrt_diff       184.14    0.93 52.12    99.58   147.66   178.34   213.67   303.20  3125 1.00
bcoma_diff      -1.22    0.01  0.33    -1.88    -1.44    -1.22    -1.01    -0.56  3120 1.00
lp__         -2604.98    0.66 20.41 -2644.44 -2619.18 -2605.04 -2590.67 -2565.84   945 1.00
```

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