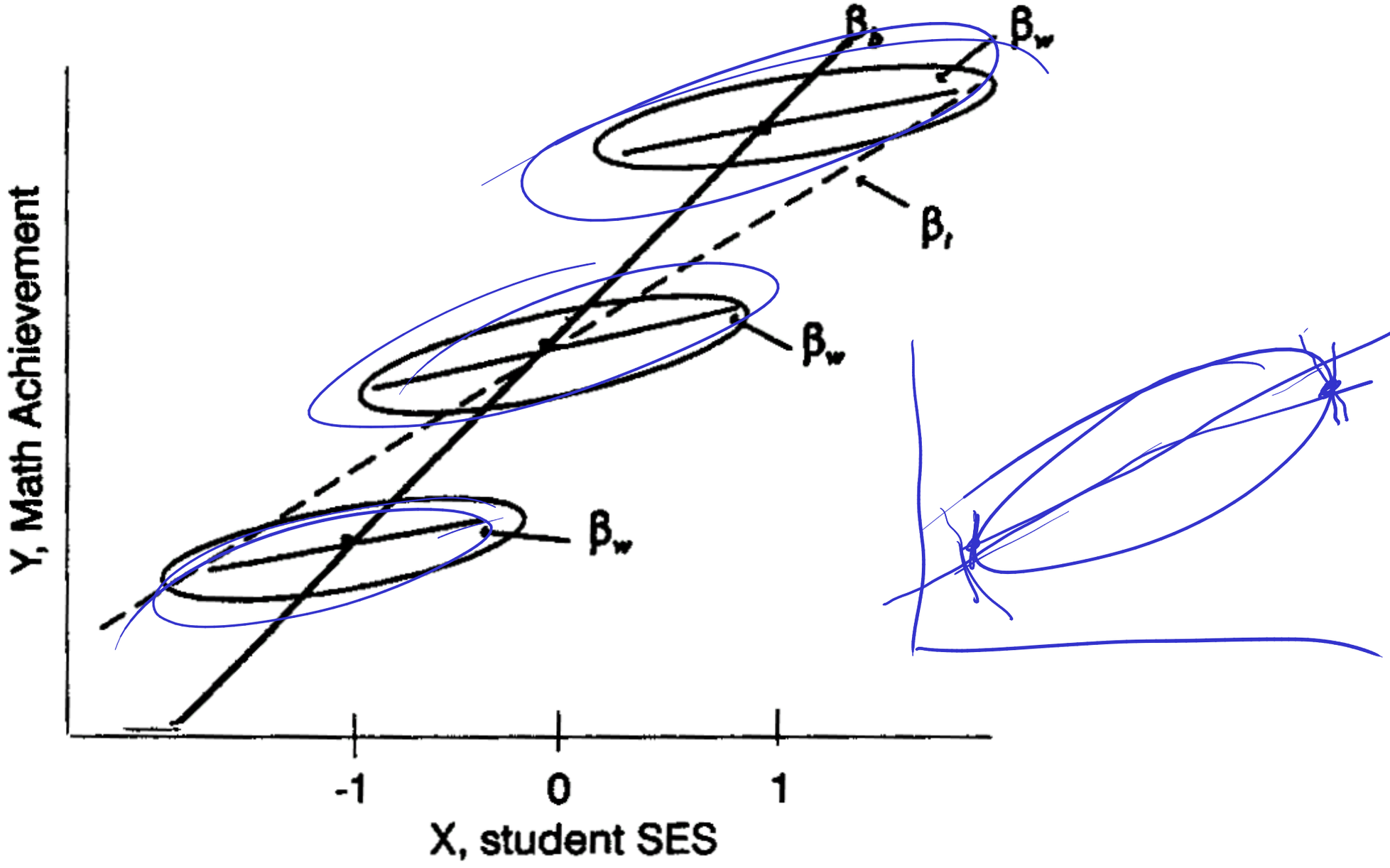
Statistical Reasoning with Ellipses: The Data Ellipse

Simple regression, the regression paradox, the anatomy of outliers: fit vs leverage, significance at a glance

Georges Monette

random@yorku.ca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β_w
Y, Math Achievement
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X, student SES

Example: Origins of regression

Galton and Pearson studied the inheritance of height from father to son

To make things simple we use simulated data with adult heights in two generations where there is no drift:

Fathers: mean 68″ sd 3″

Sons (adult) mean 68″ sd 3″

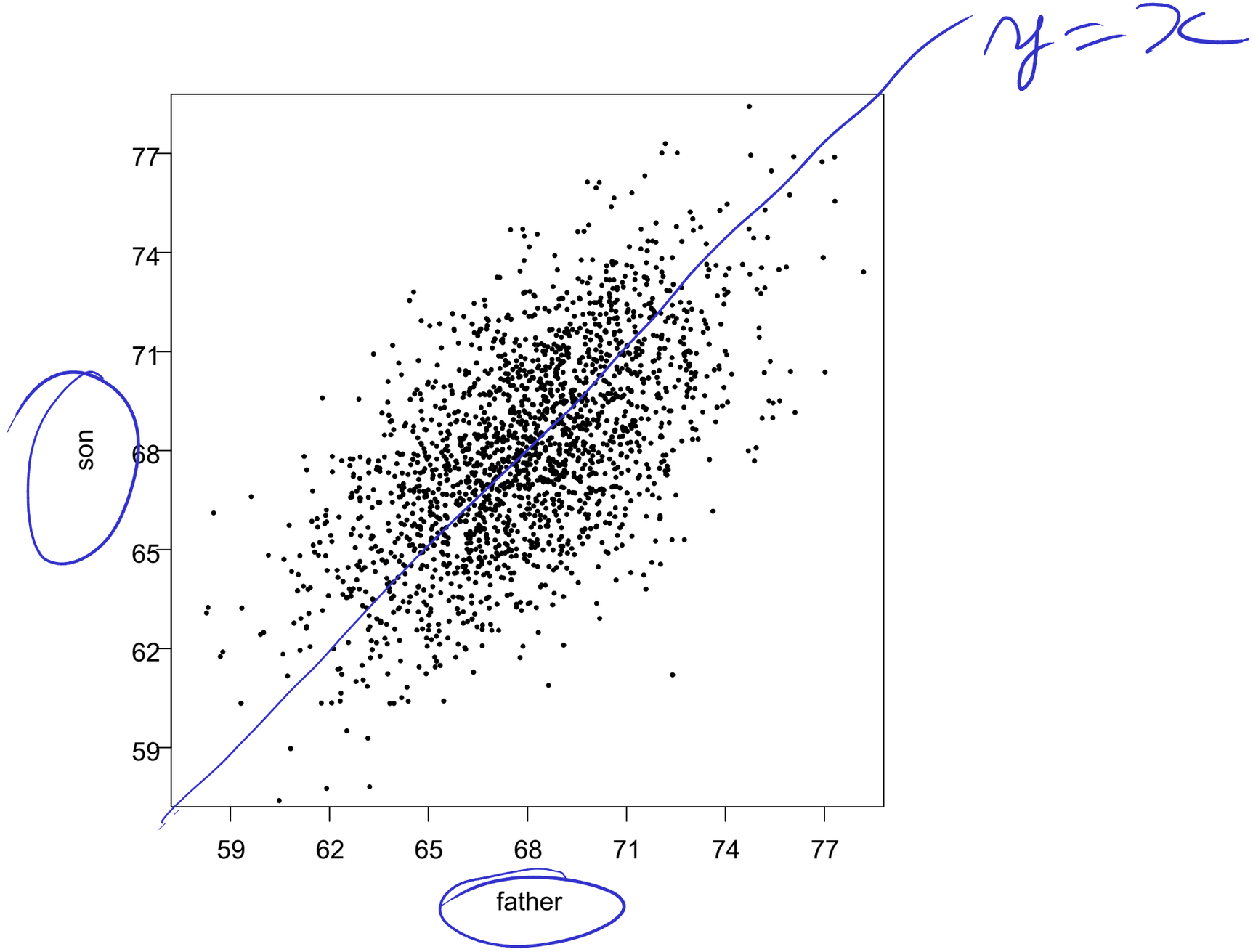
Father's height	Son's expected height
68	?

Father's height	Son's expected height
68	68

Father's height	Son's expected height
68	68
$71 = 68 + 3$	71

Father's height	Son's expected height
68	68
$71 = 68 + 3$	$71 = 68 + 3$

Father's height	Son's expected height
68	68
$71 = 68 + 3$	$71 = 68 + 3$
$74 = 68 + 6$	$74 = 68 + 6$
$65 = 68 - 3$	$65 = 68 - 3$
$62 = 68 - 6$	$62 = 68 - 6$


y=x
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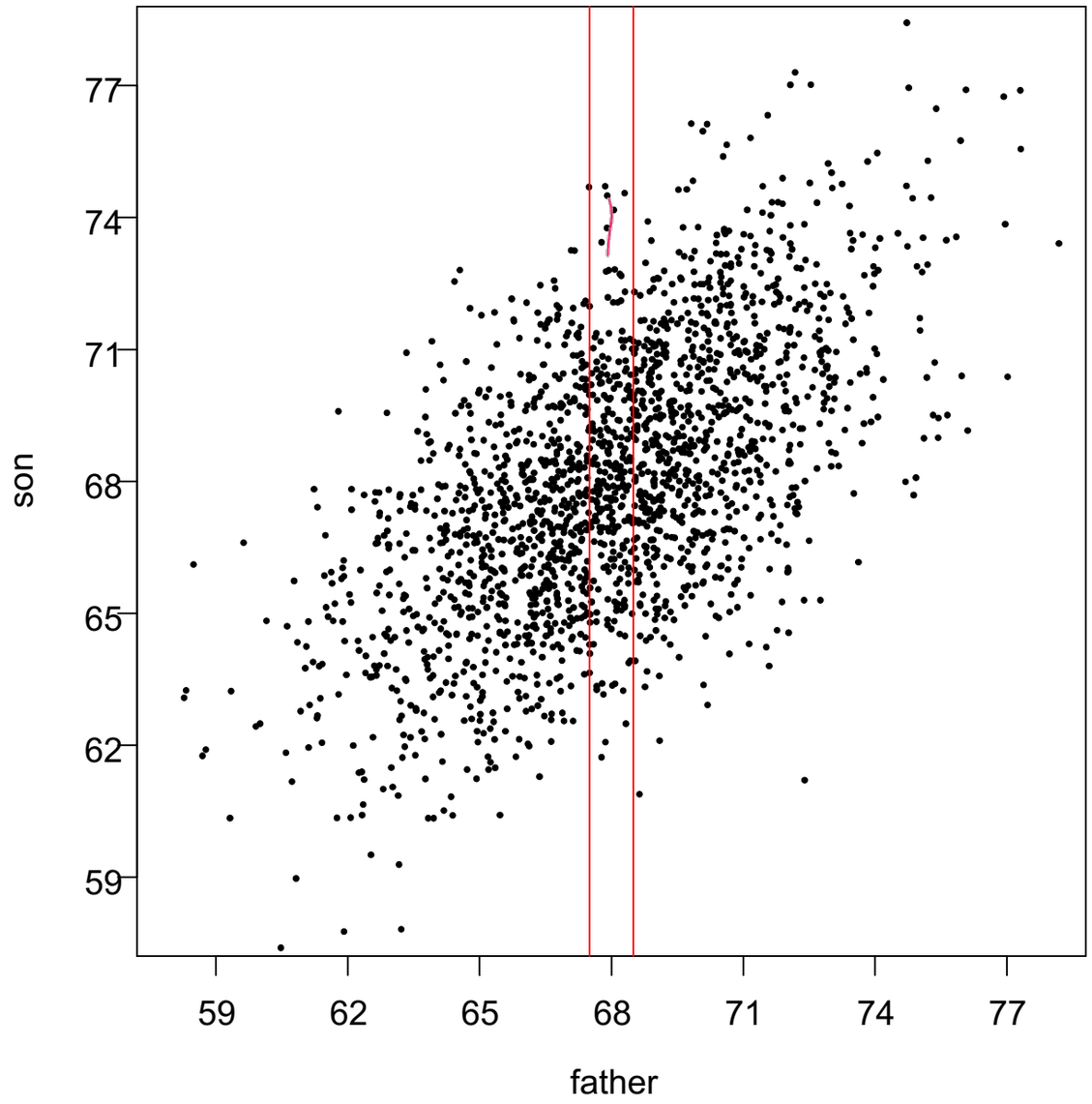
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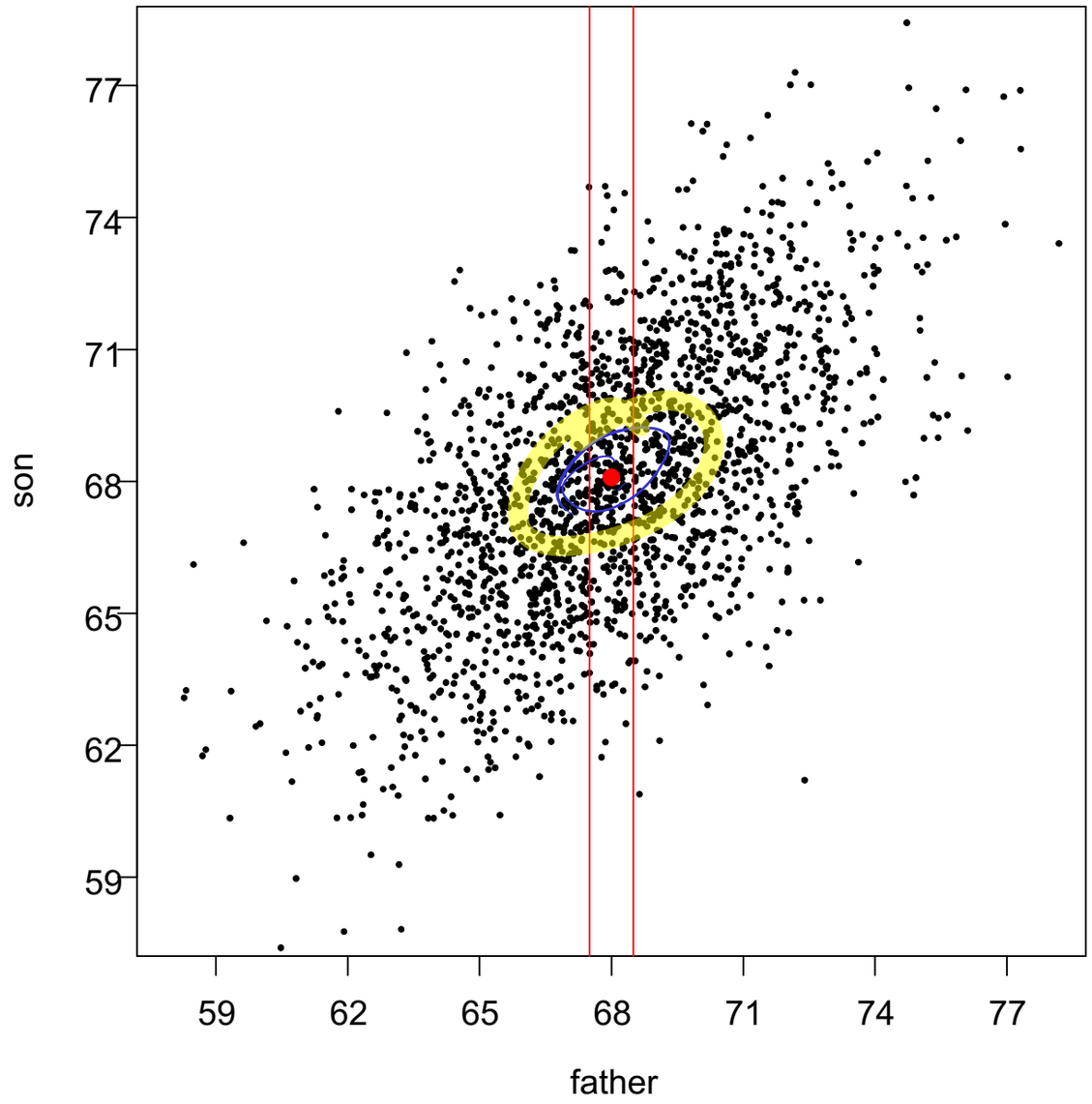
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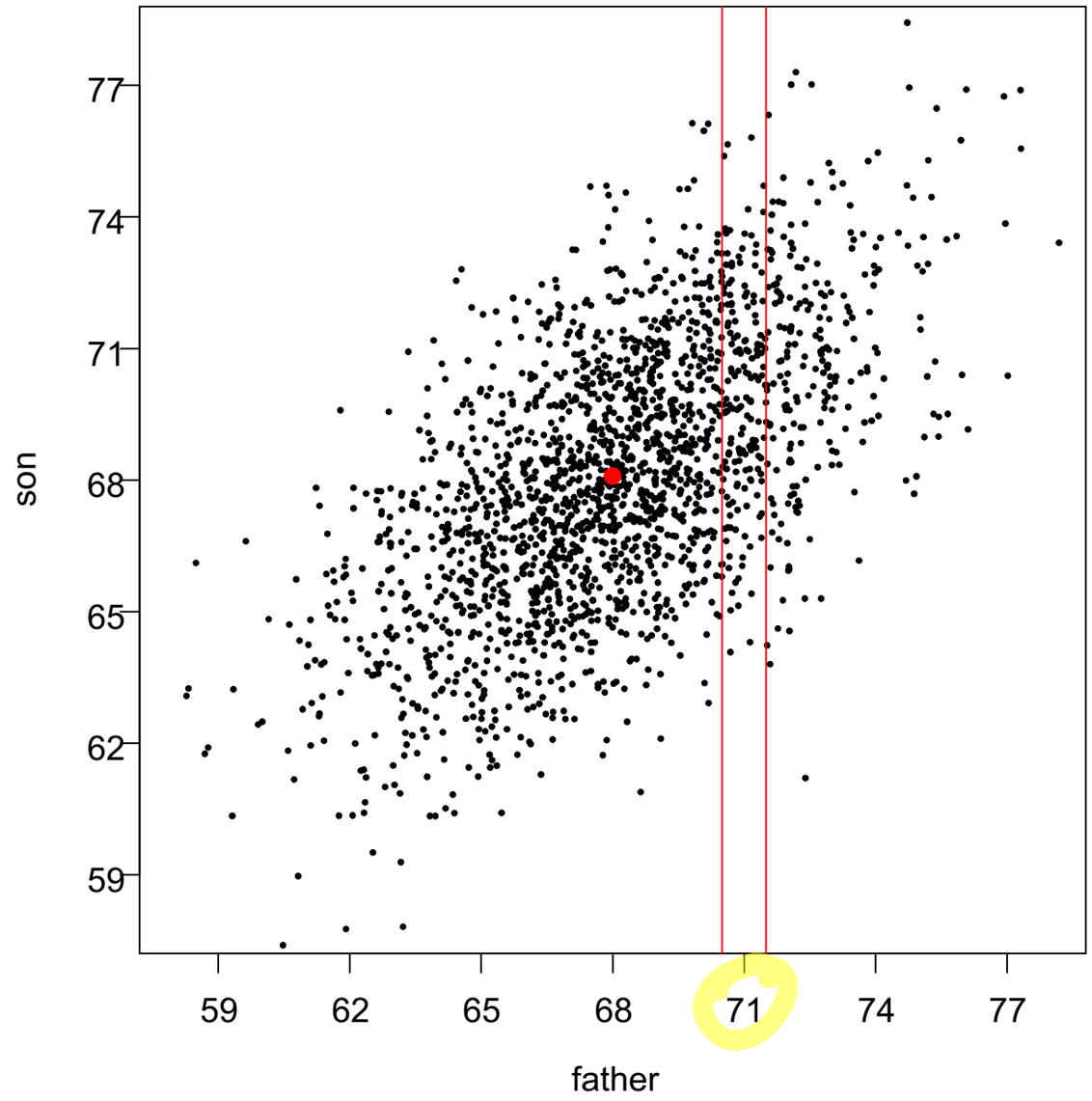
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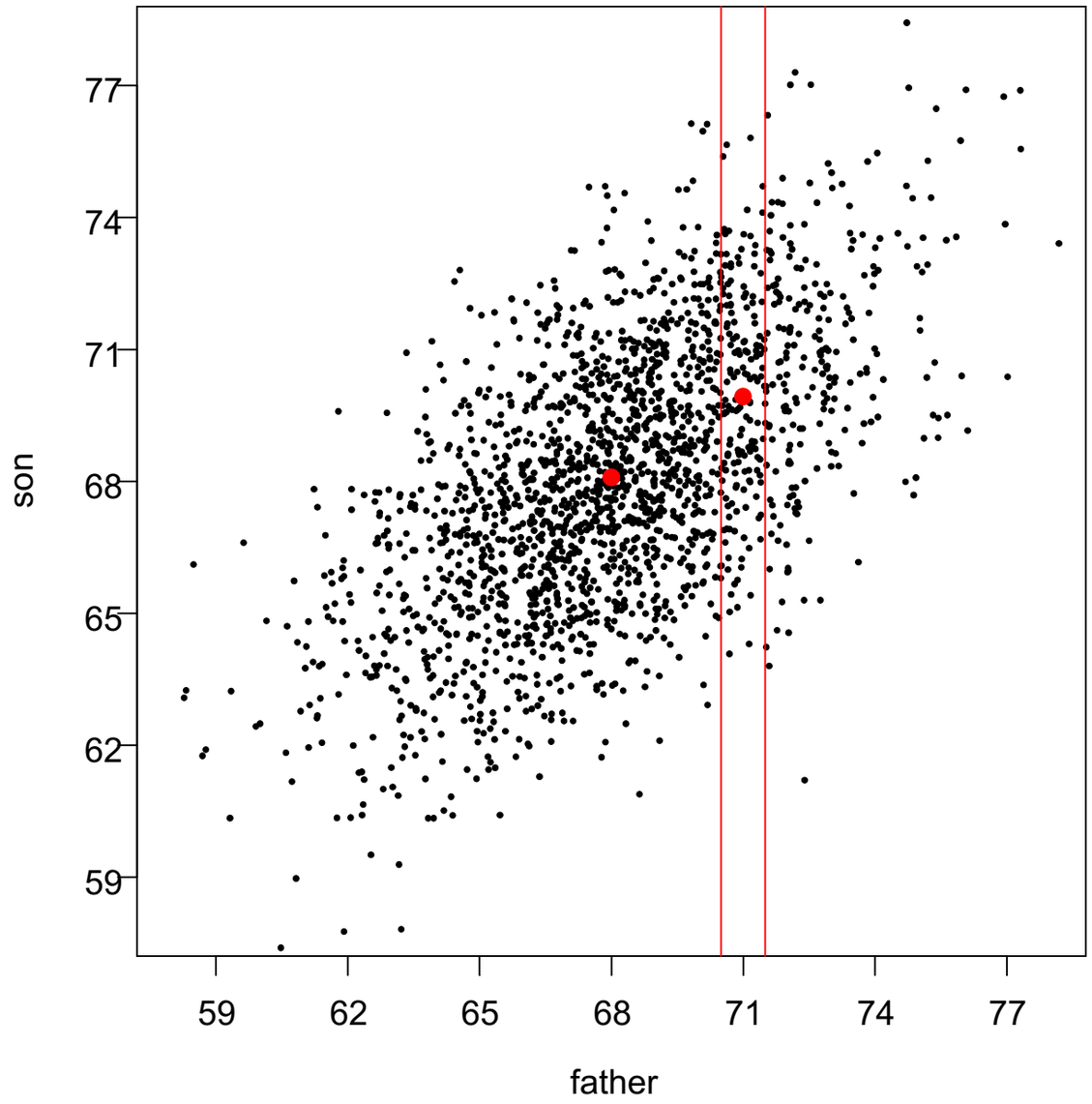
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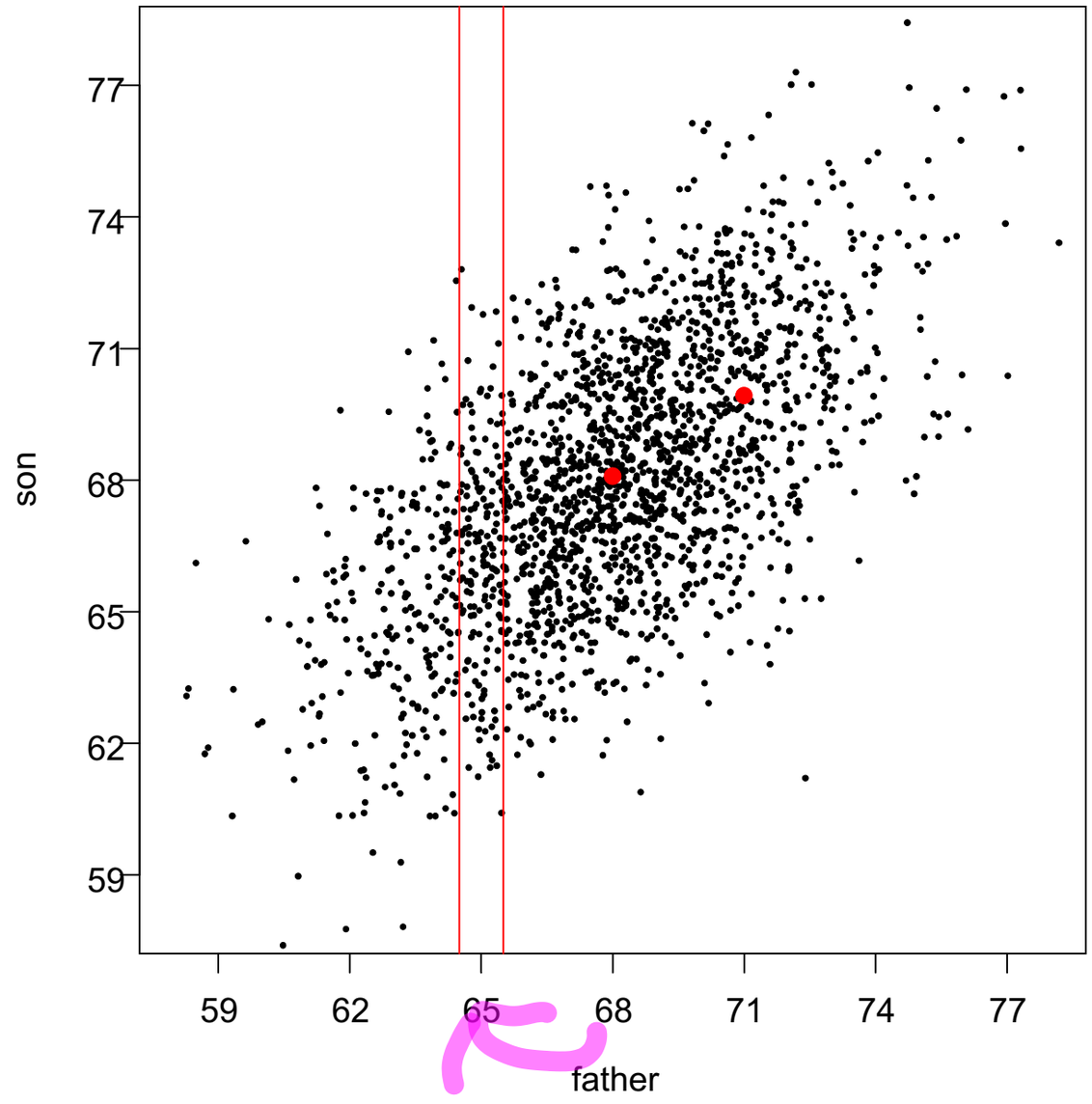
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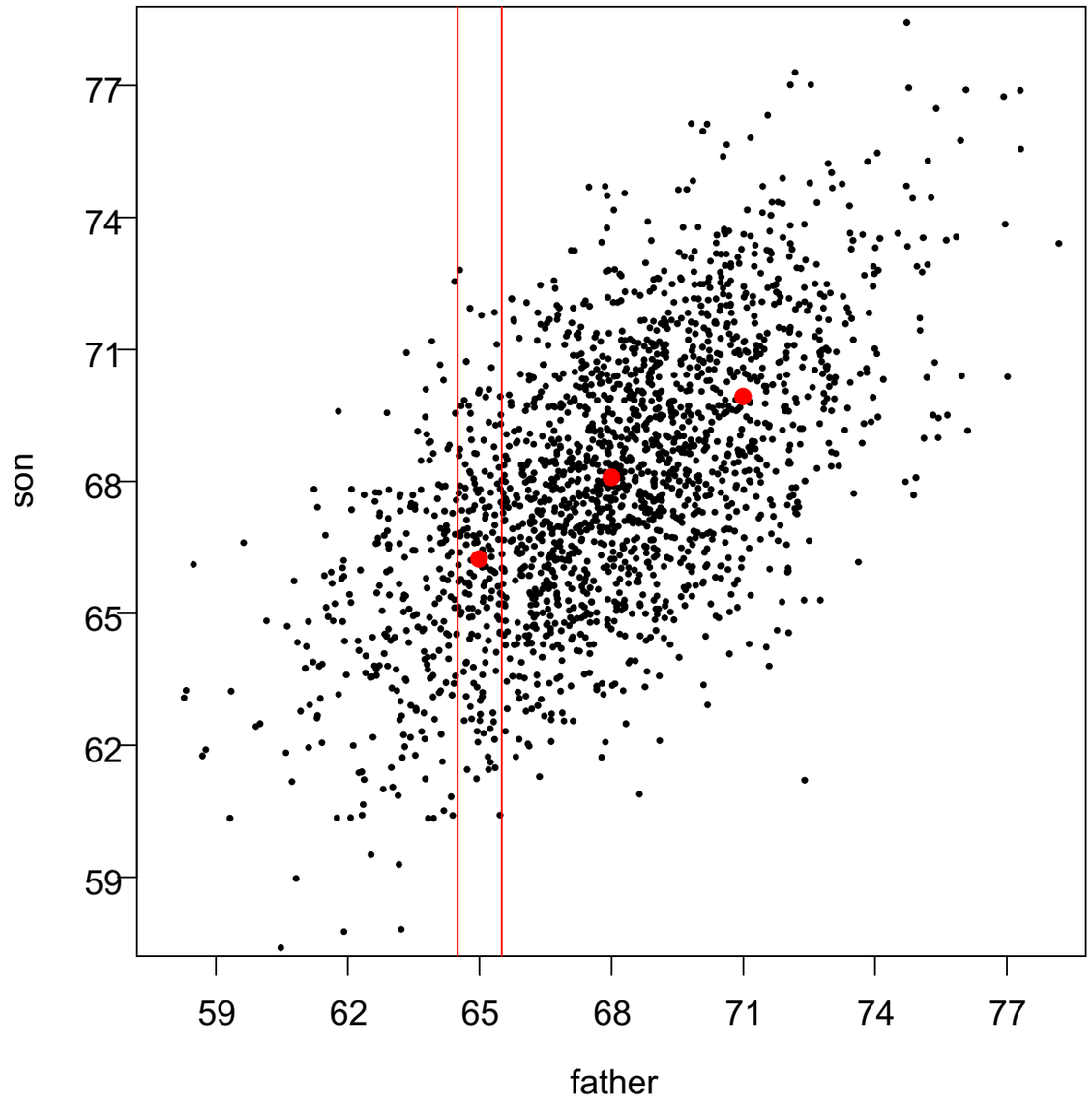
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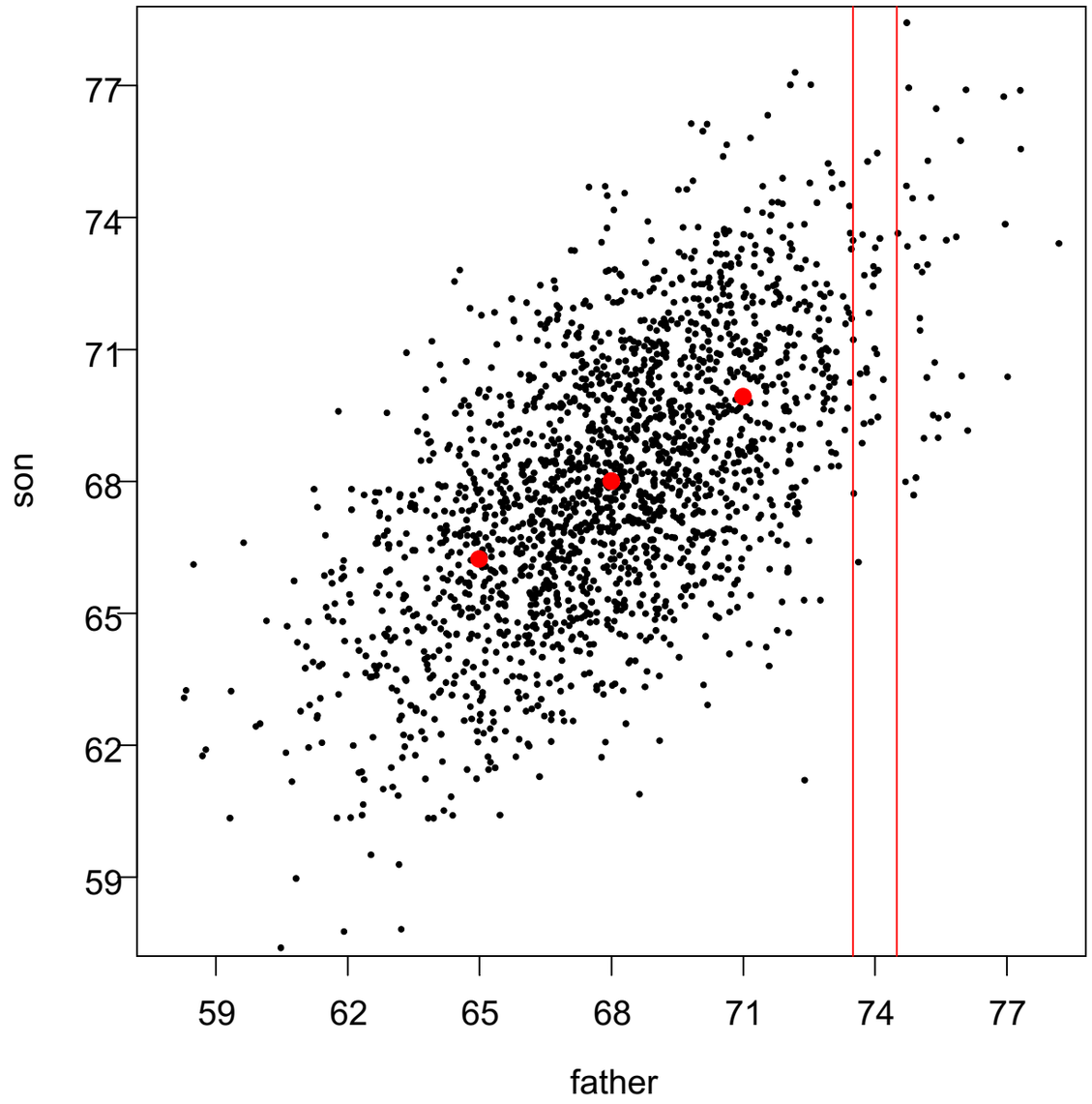
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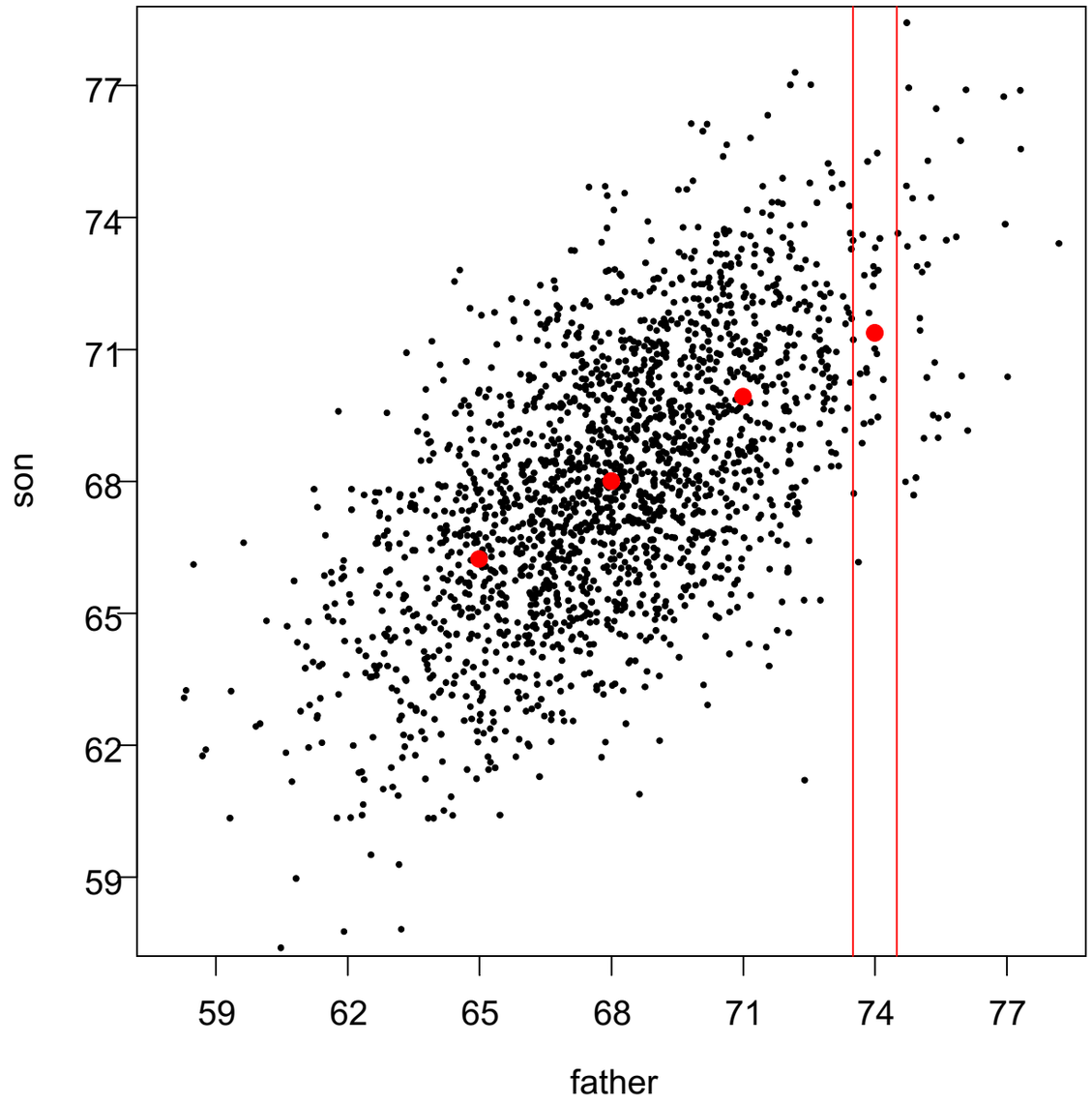
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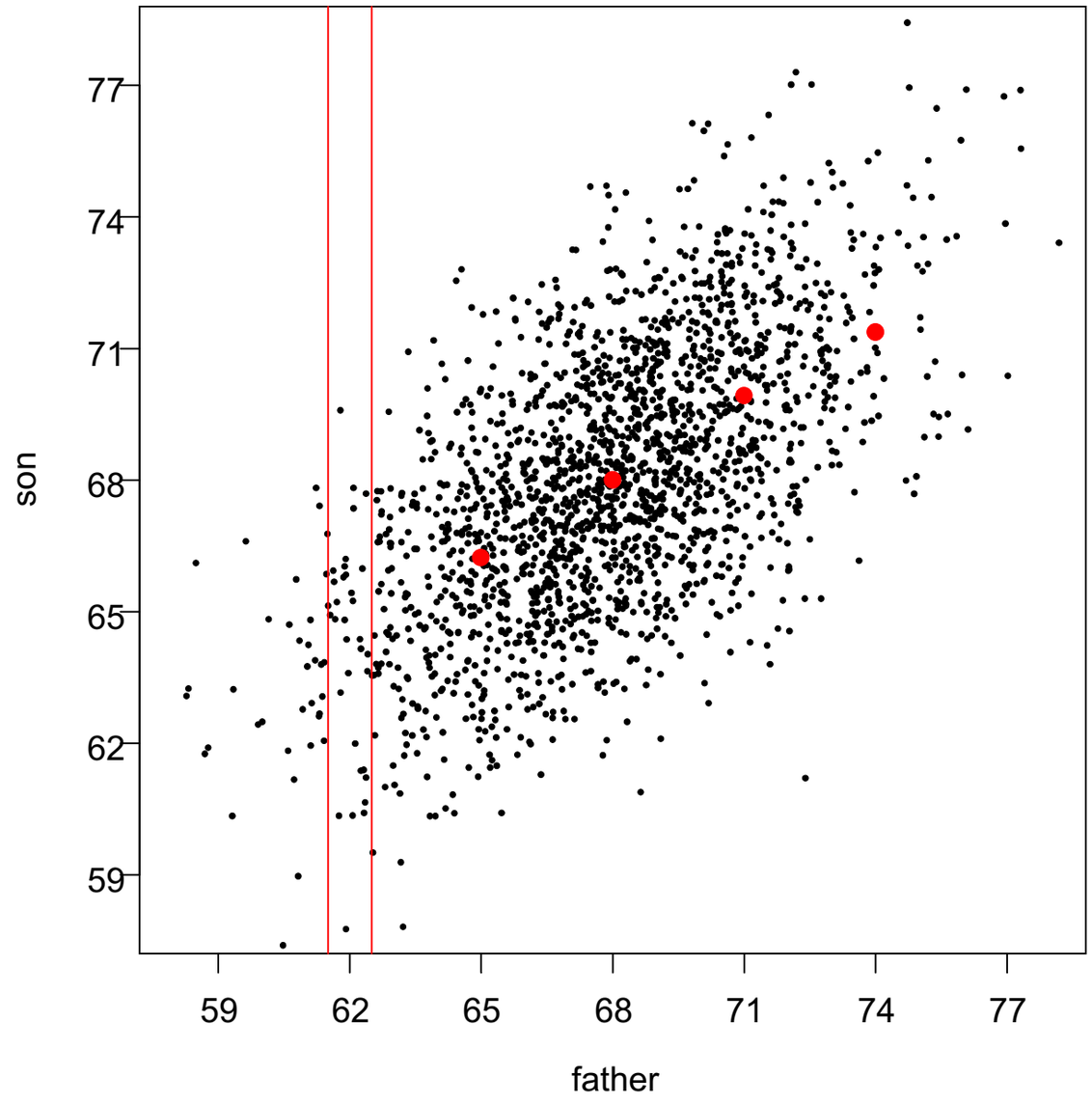
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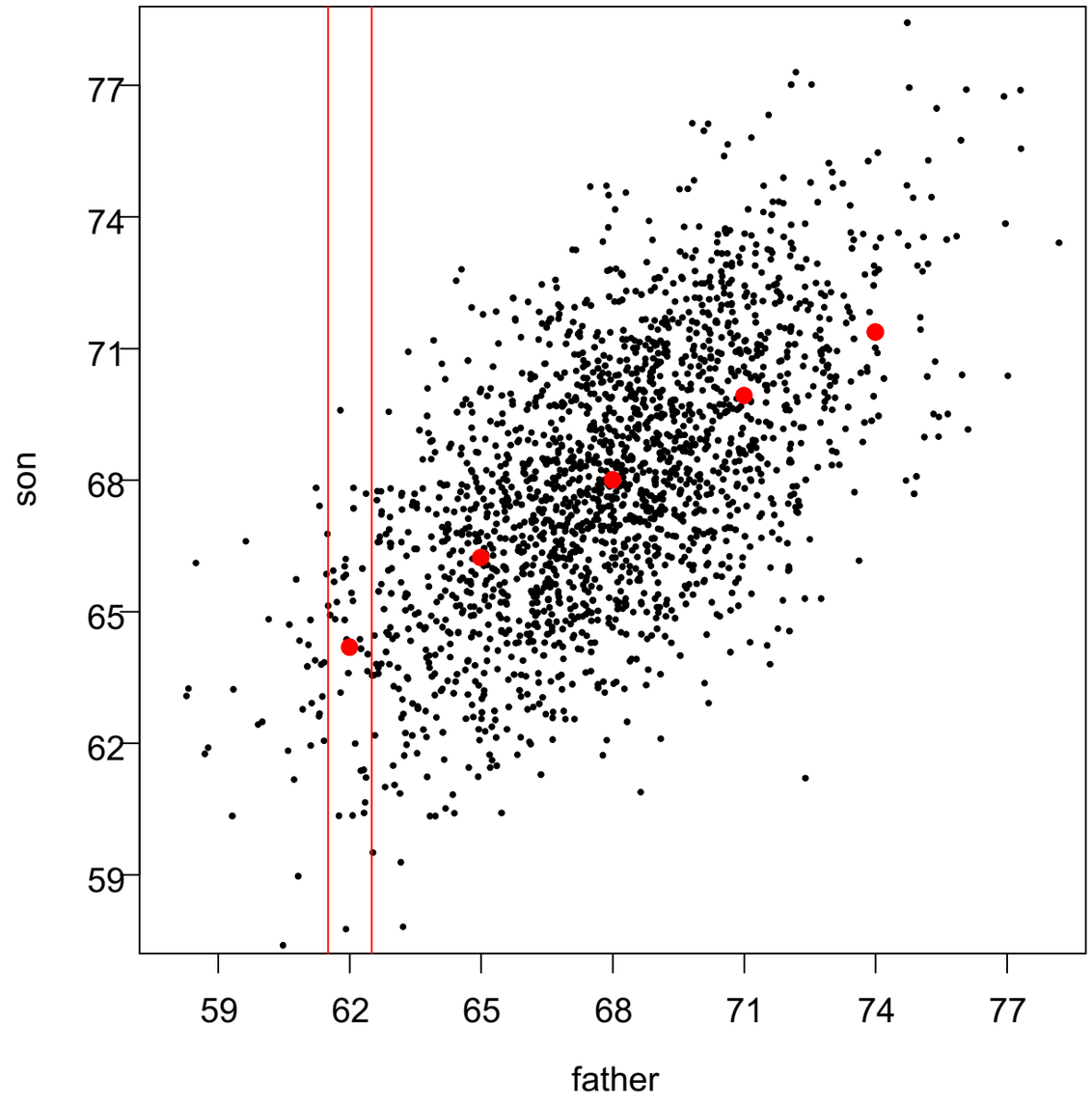
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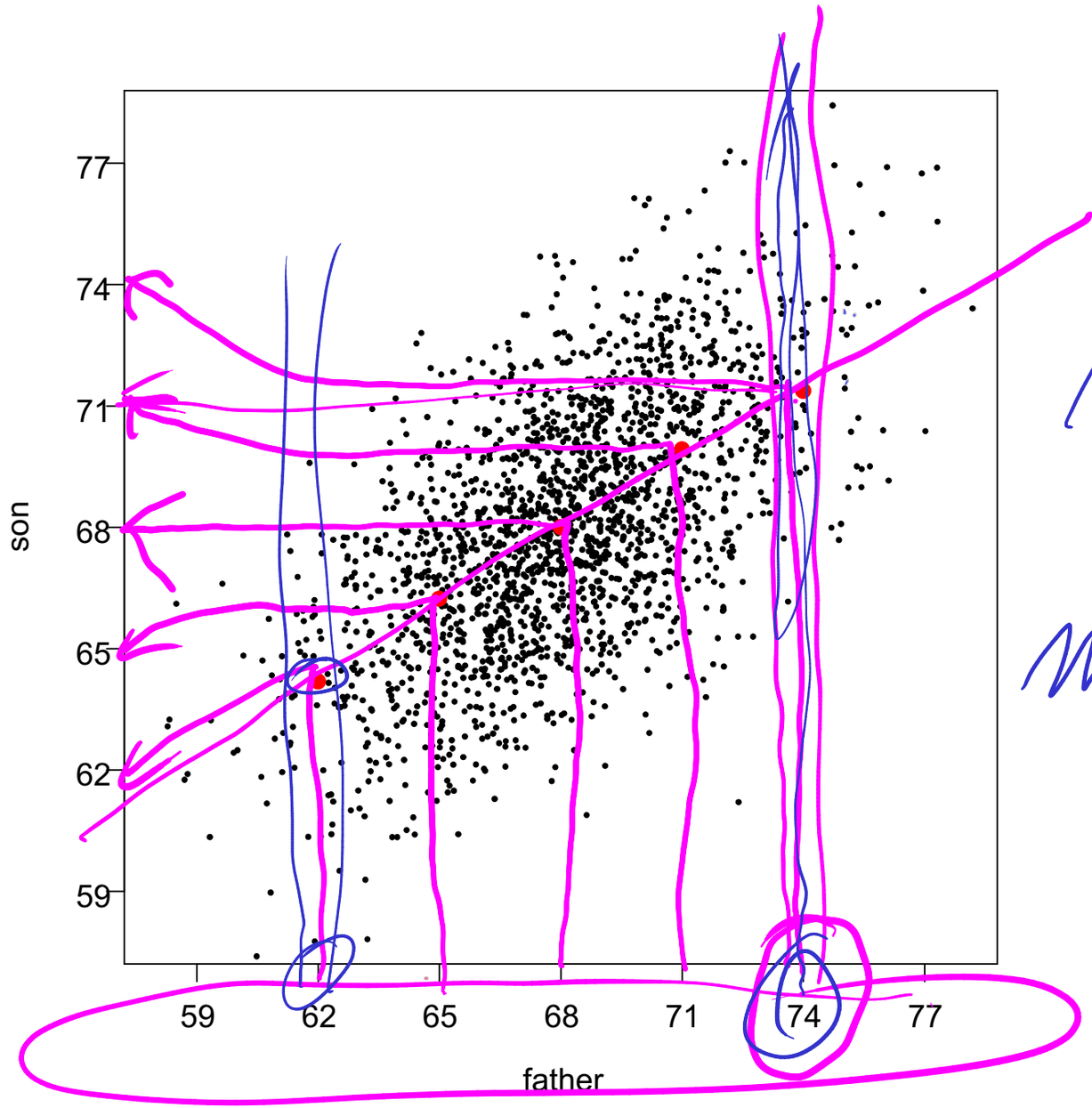
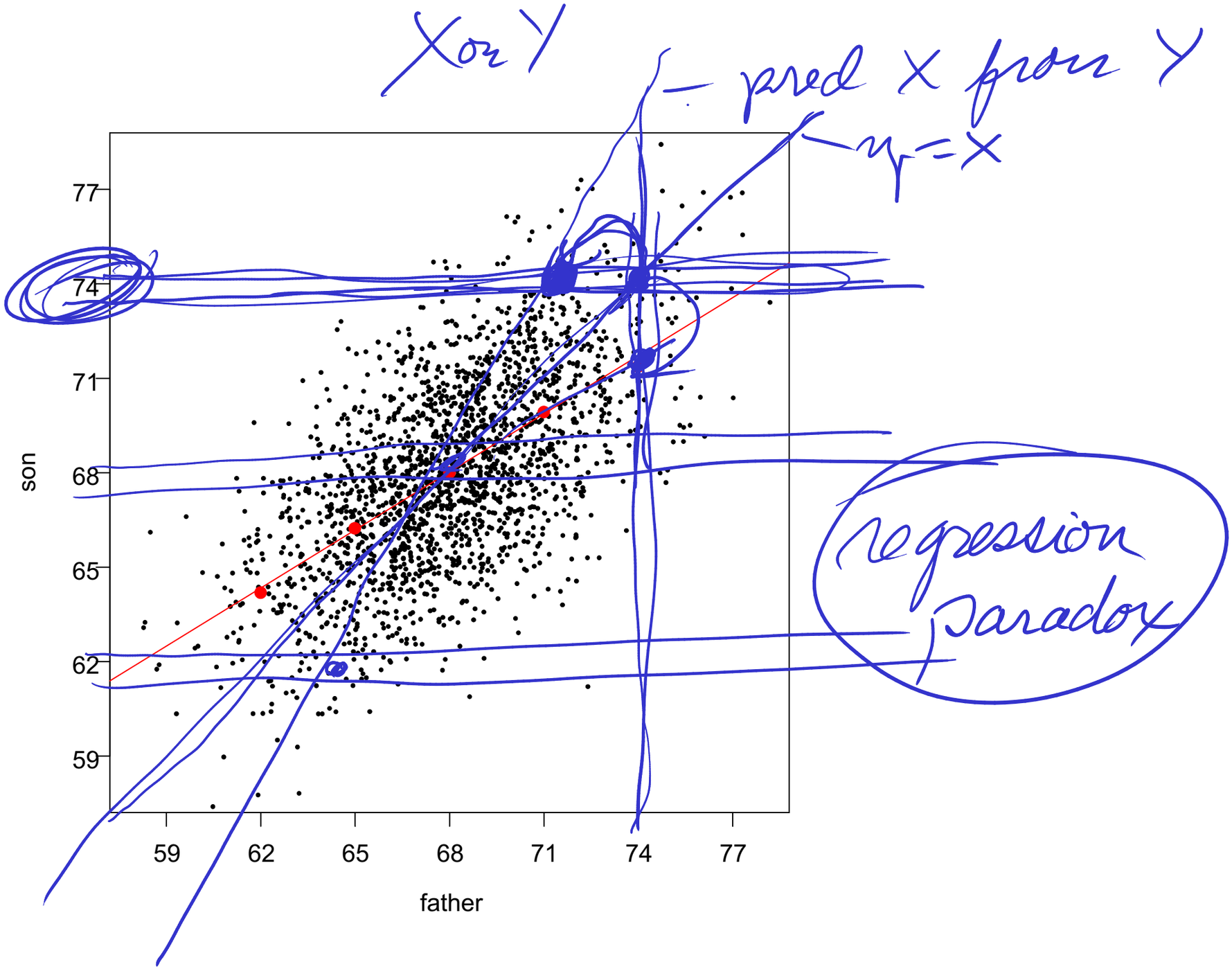


Regression to mediocrity


X or Y
- pred X from Y
- y=x
regression paradox
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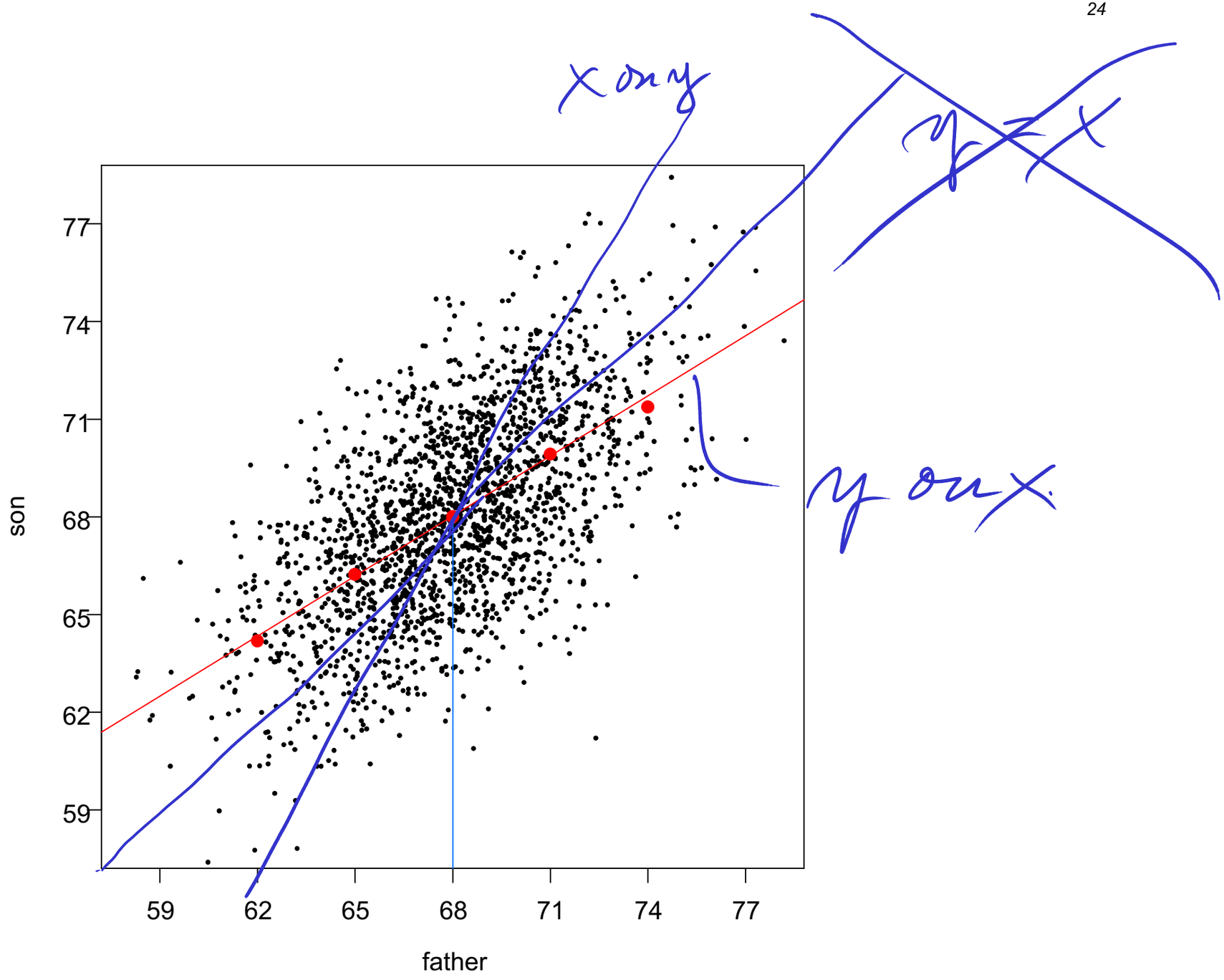
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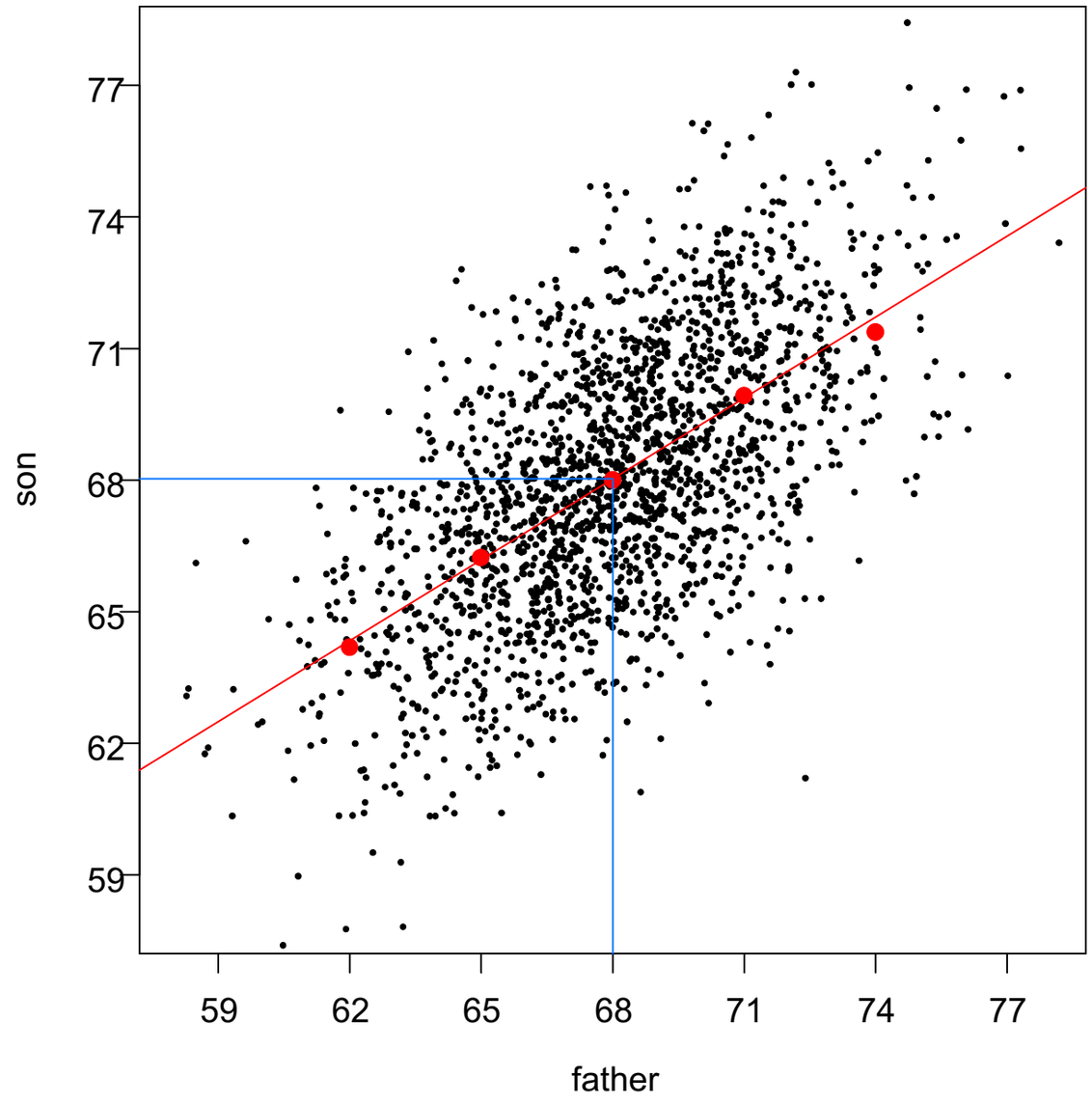
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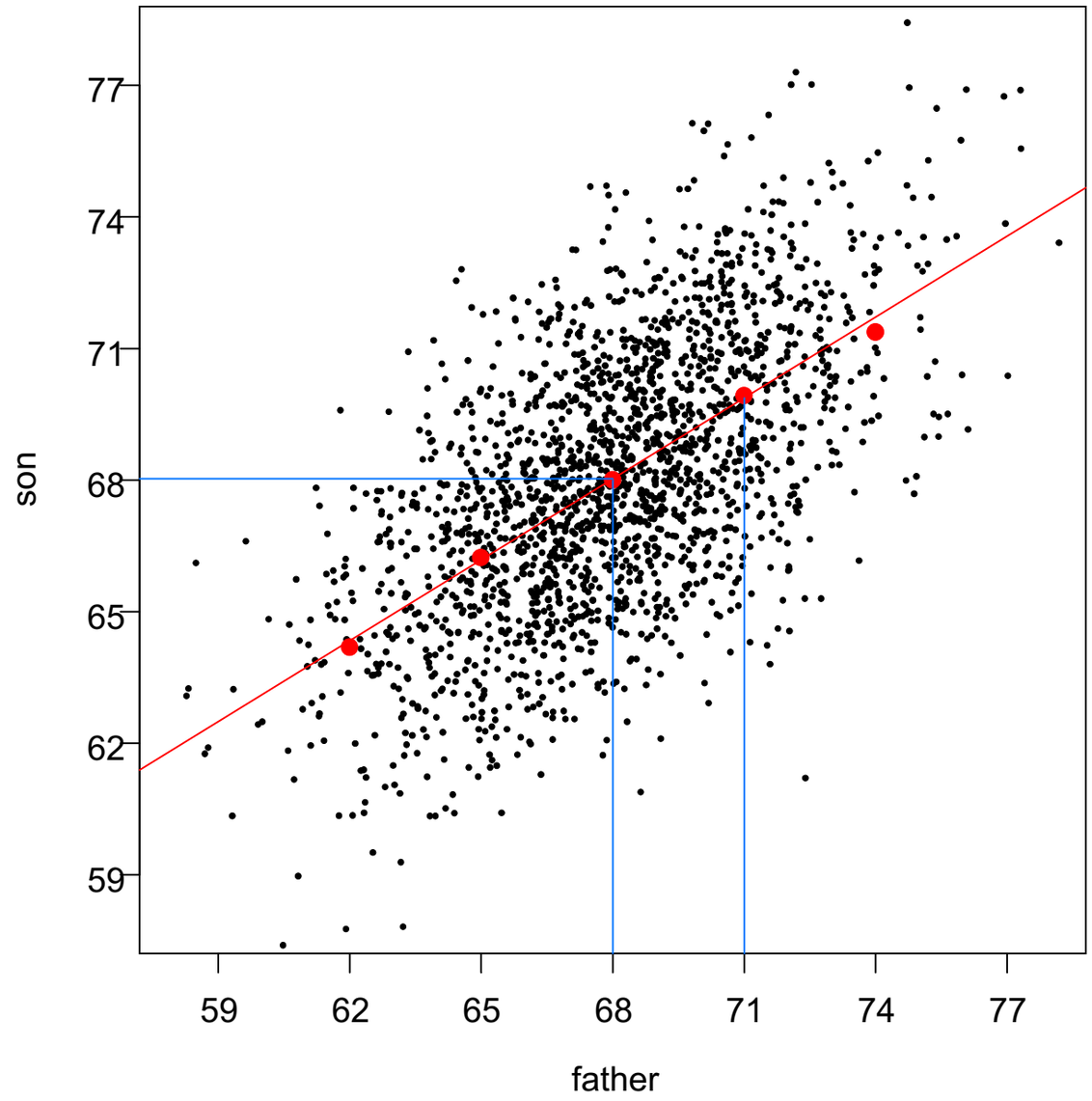
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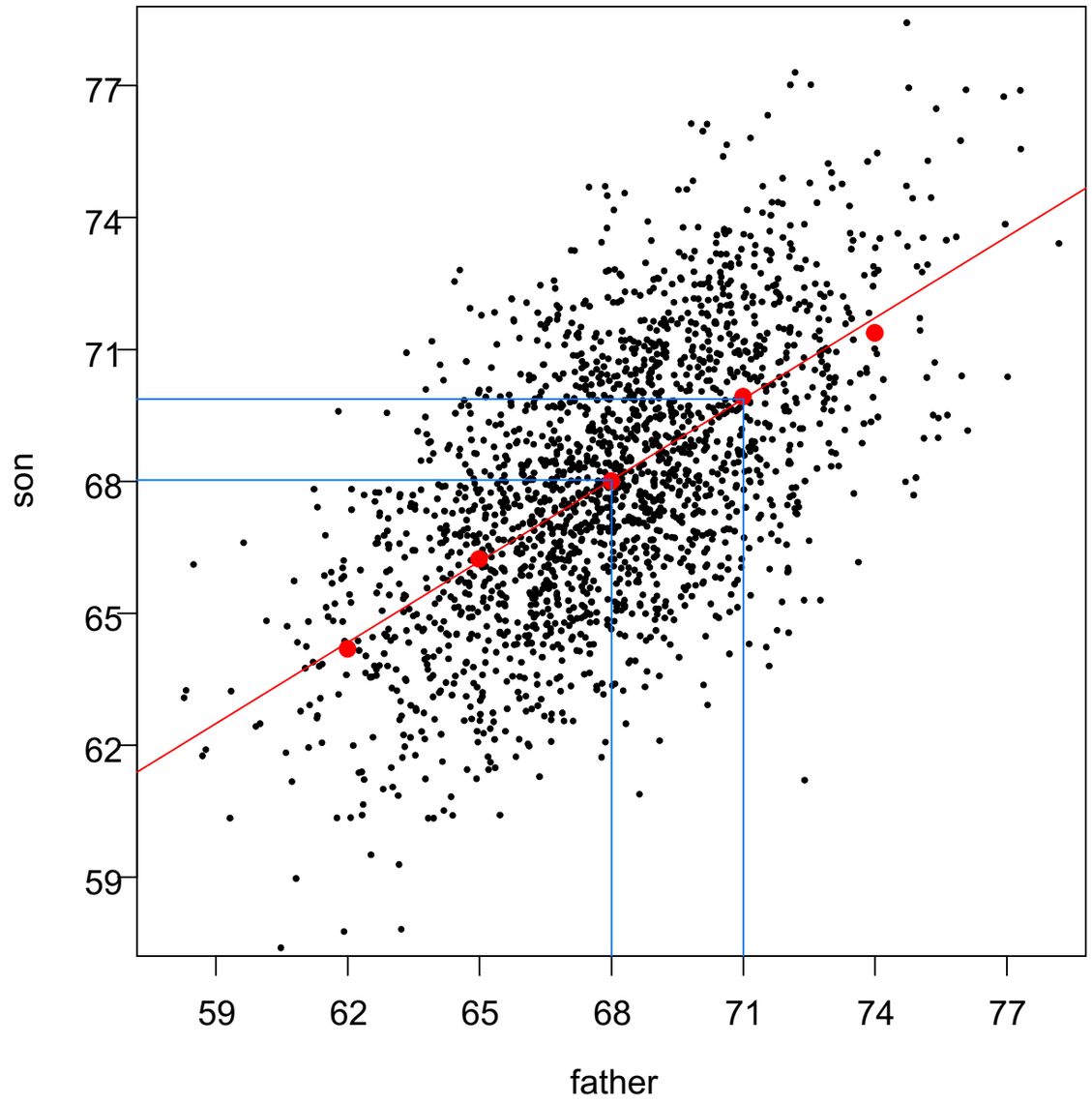
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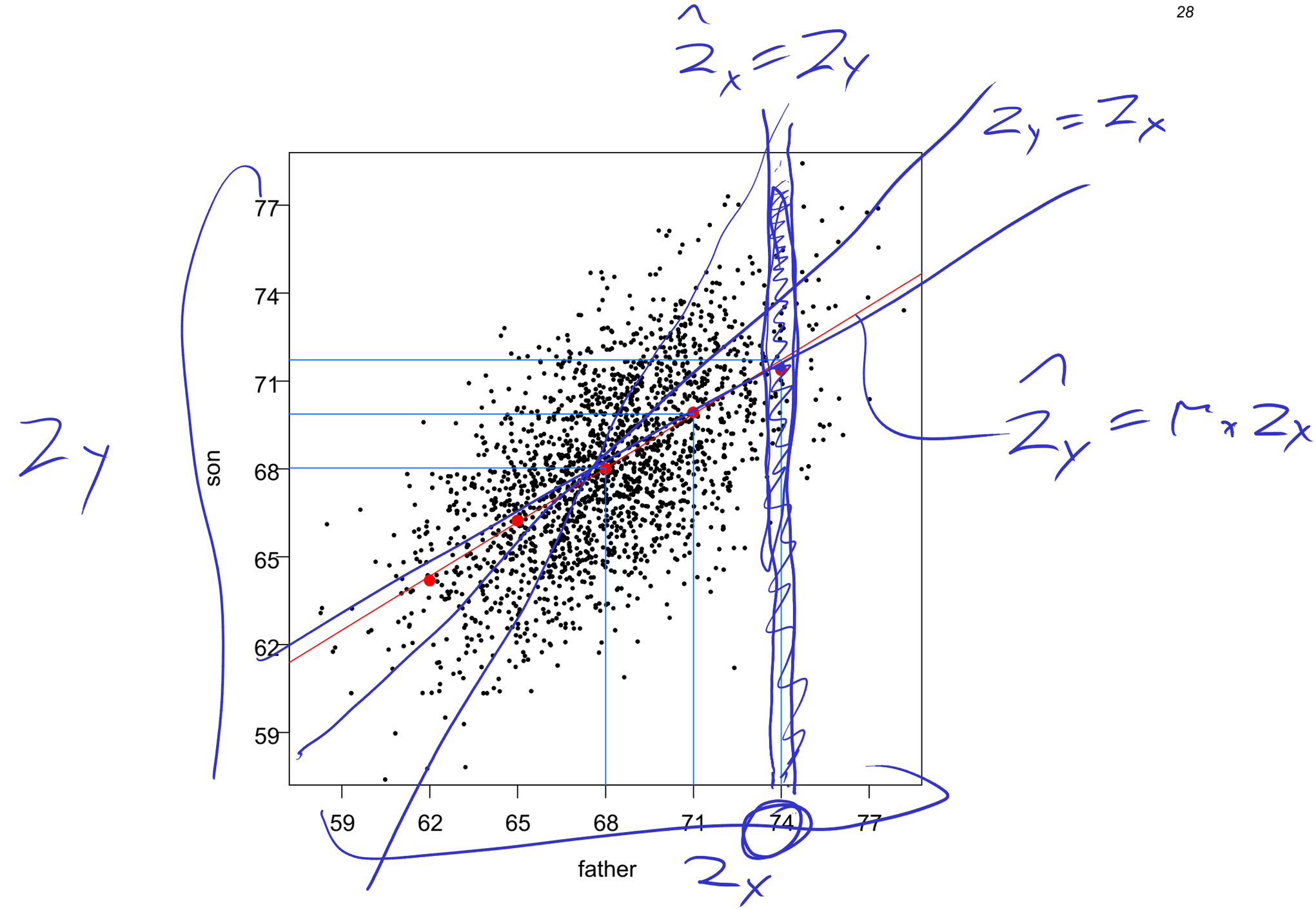
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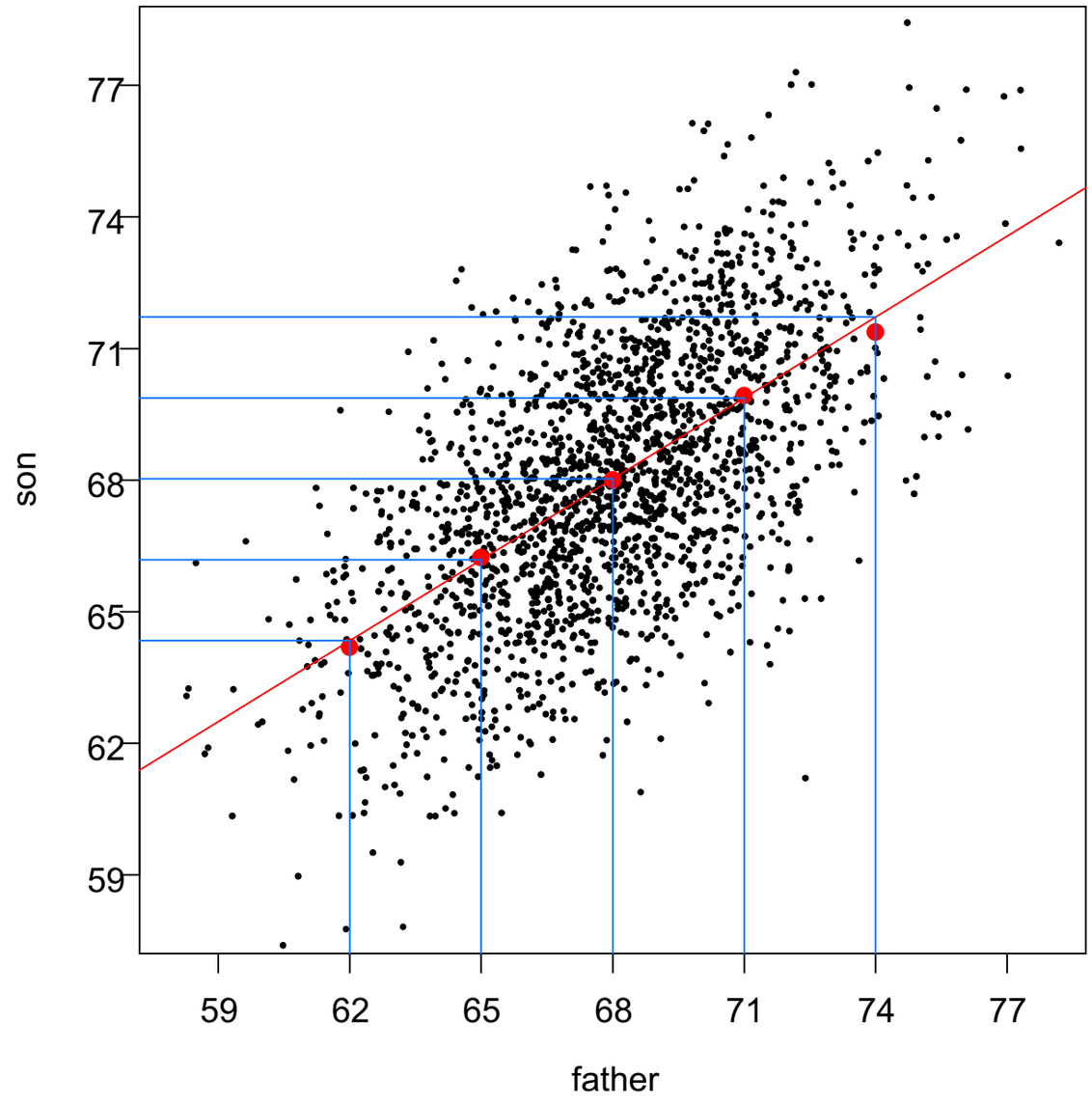
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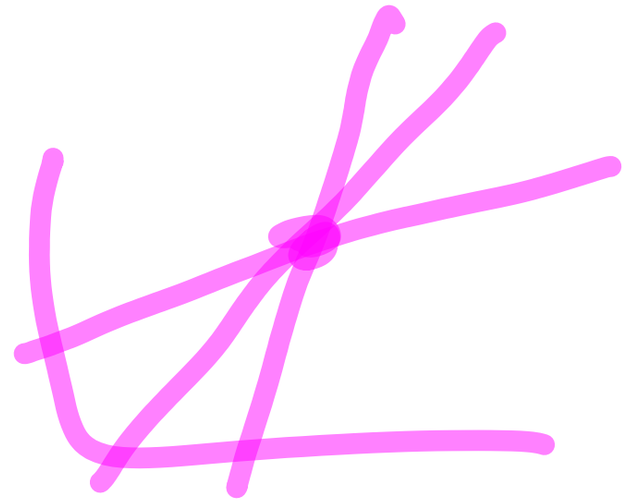
father

Father's height Son's expected height

68

Father's height	Son's expected height
68	68



Father's height	Son's expected height
68	68
$71 = 68 + 3$	

Father's height	Son's expected height
68	68
$71 = 68 + 3$	$69.8 = 68 + r \times 3$

$r = .6$

$r \times 3 = 1.8$

Father's height	Son's expected height
68	68
$71 = 68 + 3$	$69.8 = 68 + r \times 3$
$74 = 68 + 6$	$71.6 = 68 + r \times 6$
$65 = 68 - 3$	$66.2 = 68 - r \times 3$
$62 = 68 - 6$	$64.4 = 68 - r \times 6$

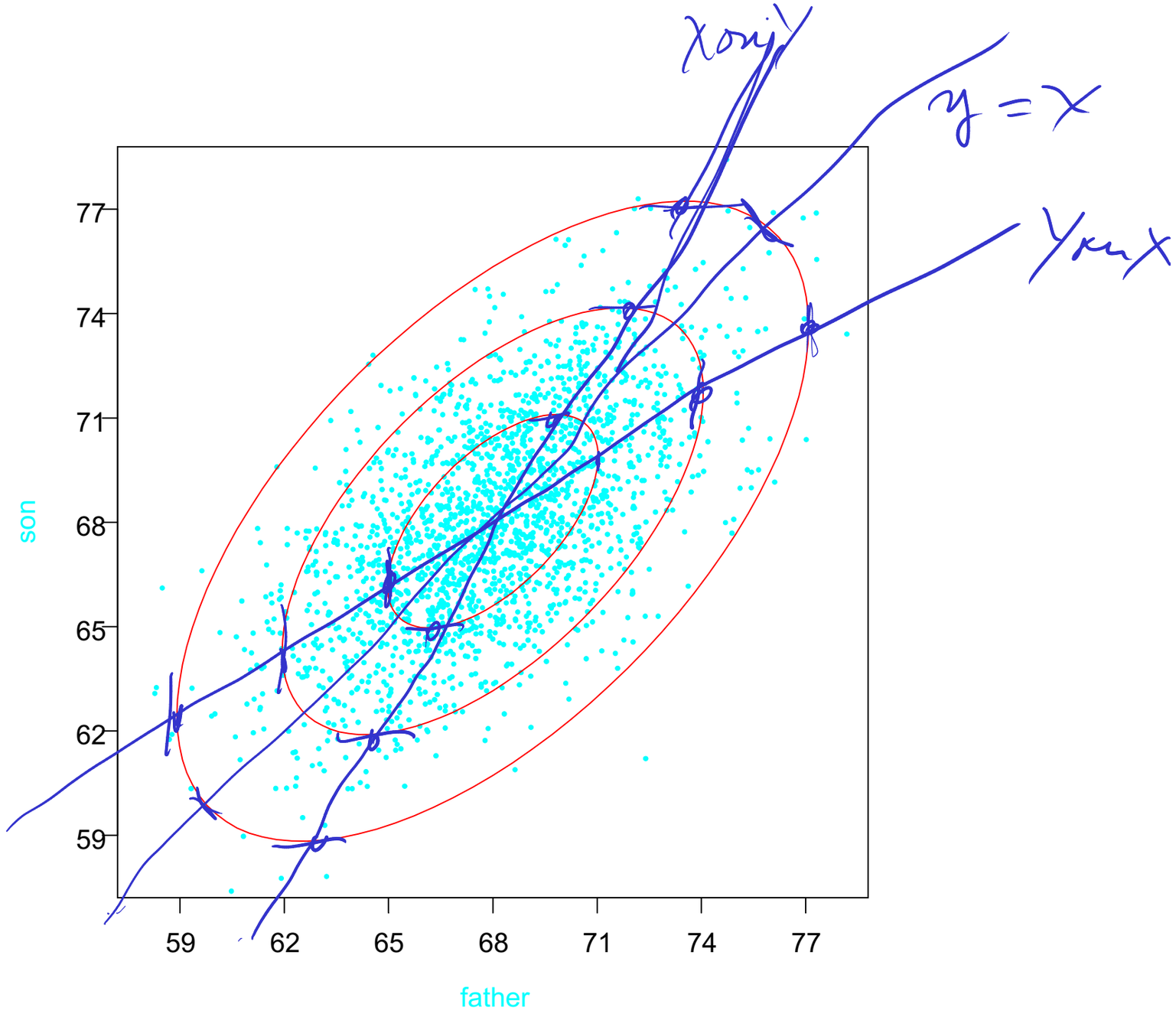
Correlation

$\hat{Y} = a + bX$

$z_y = \frac{Y - \bar{Y}}{s_y}$ $\quad z_x = \frac{X - \bar{X}}{s_x}$

$\hat{z}_y = 0 + r_{xy} z_x$

$\hat{z}_x = r_{xy} z_y$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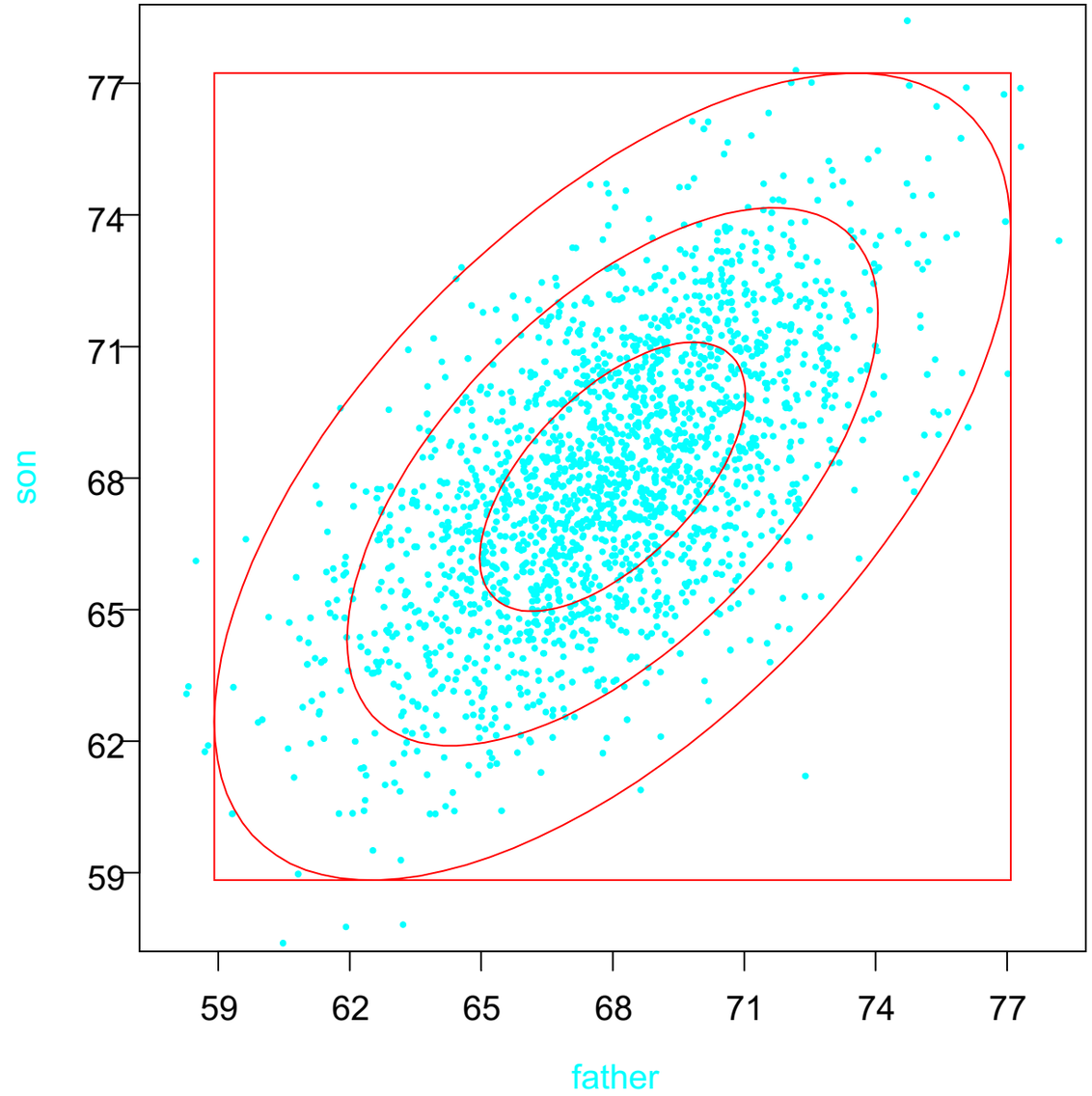
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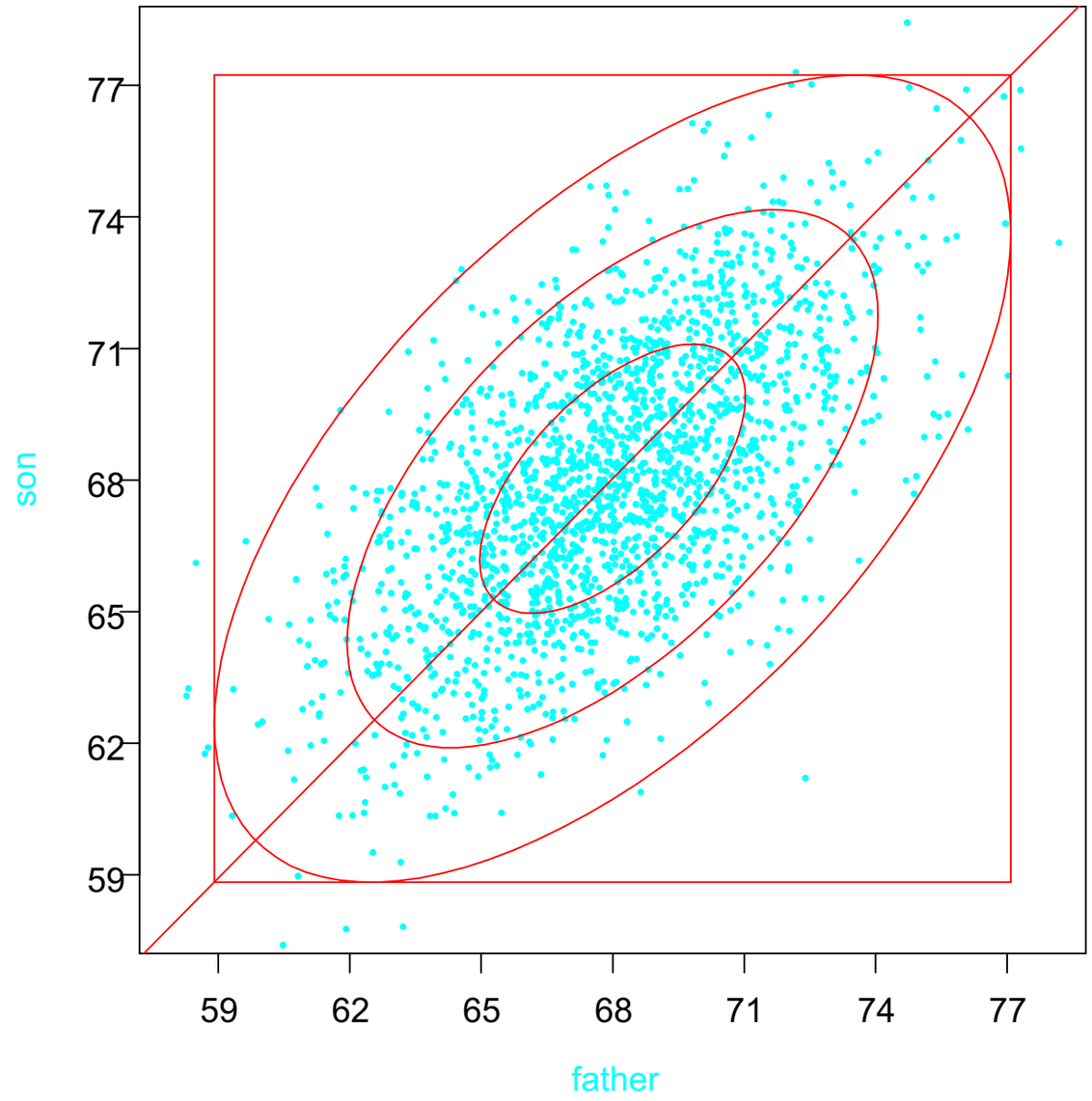
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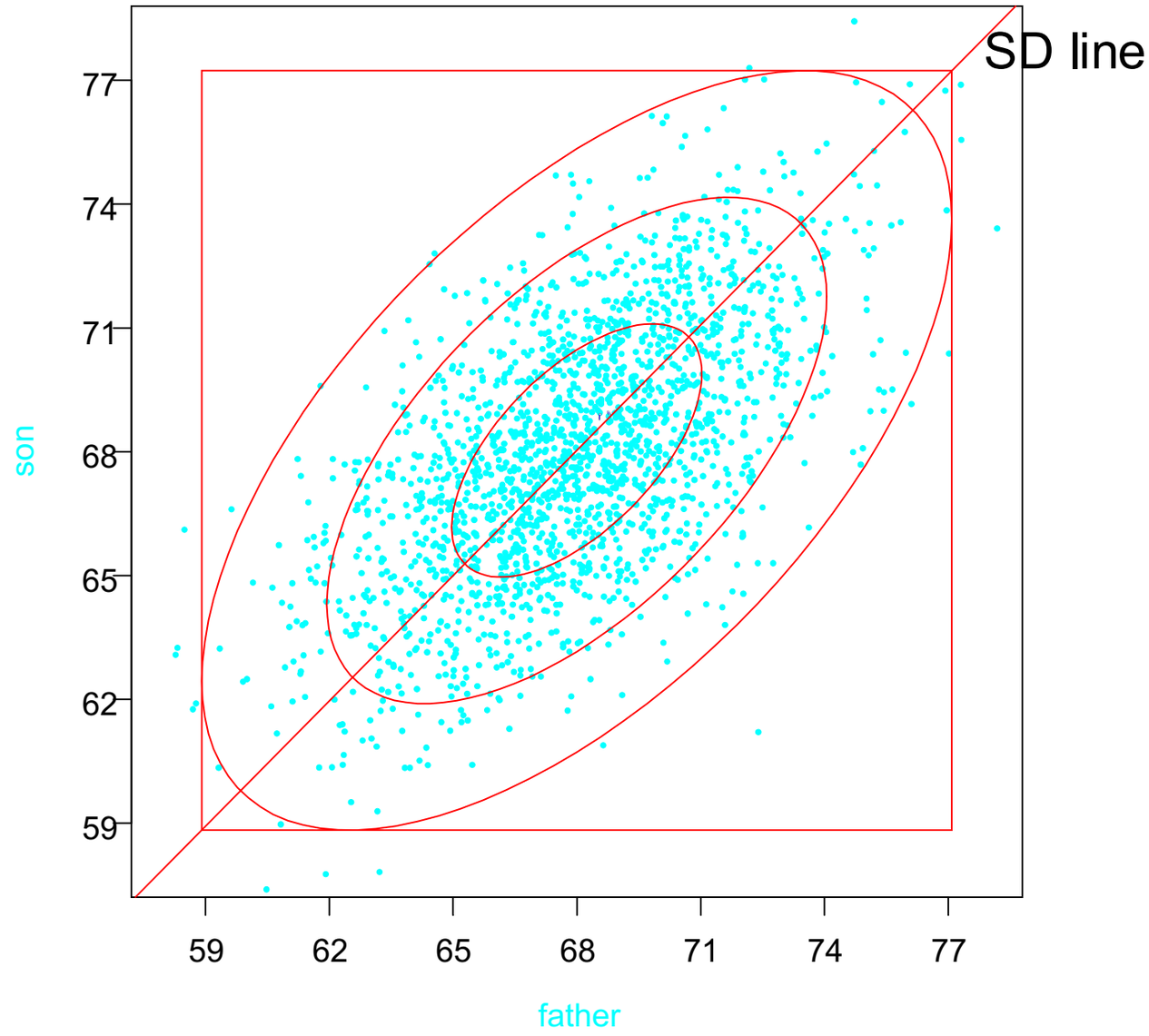
SD line
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71
74
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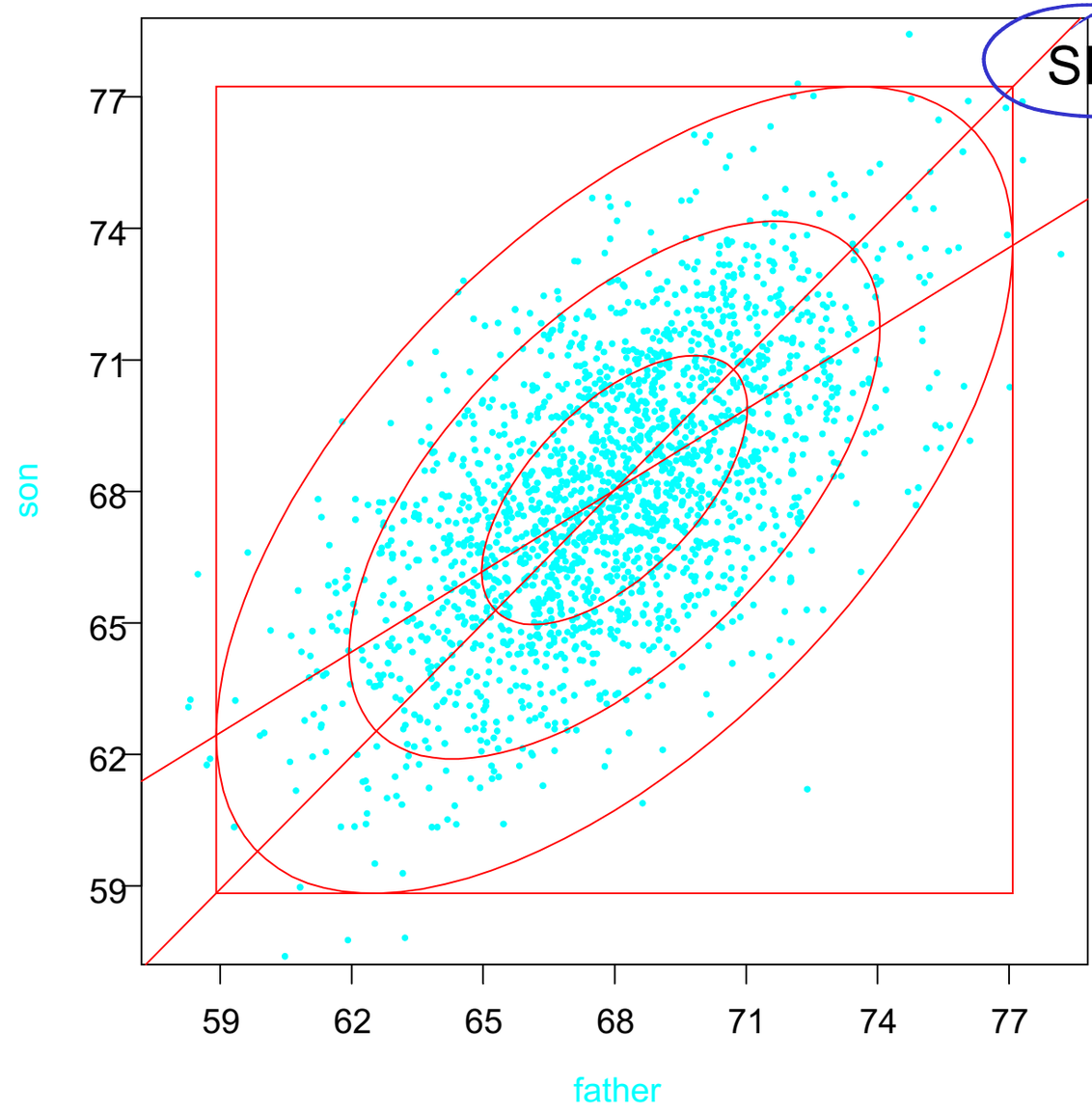
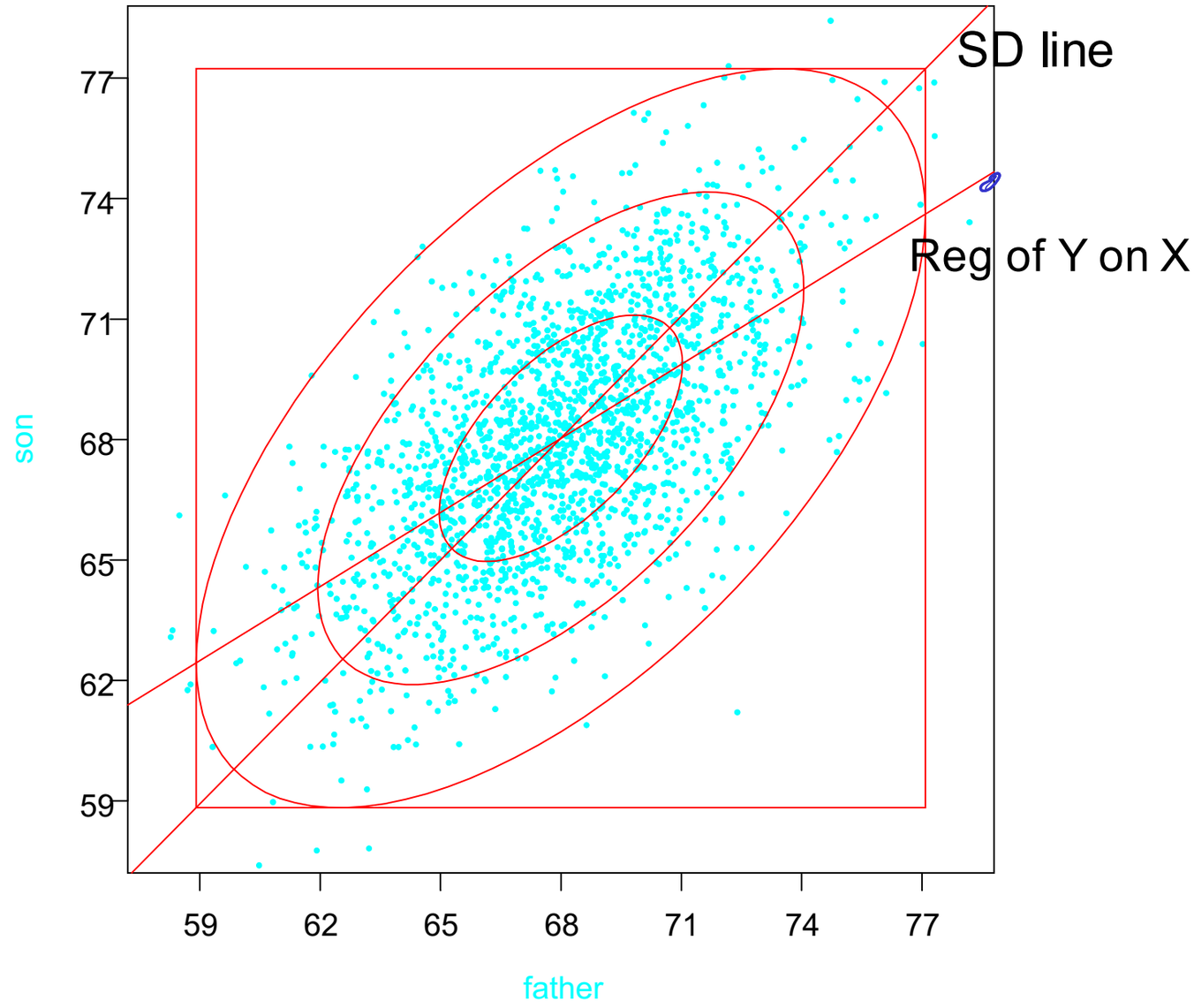
father



↑ on SD_y per SD_x


SD line
Reg of Y on X
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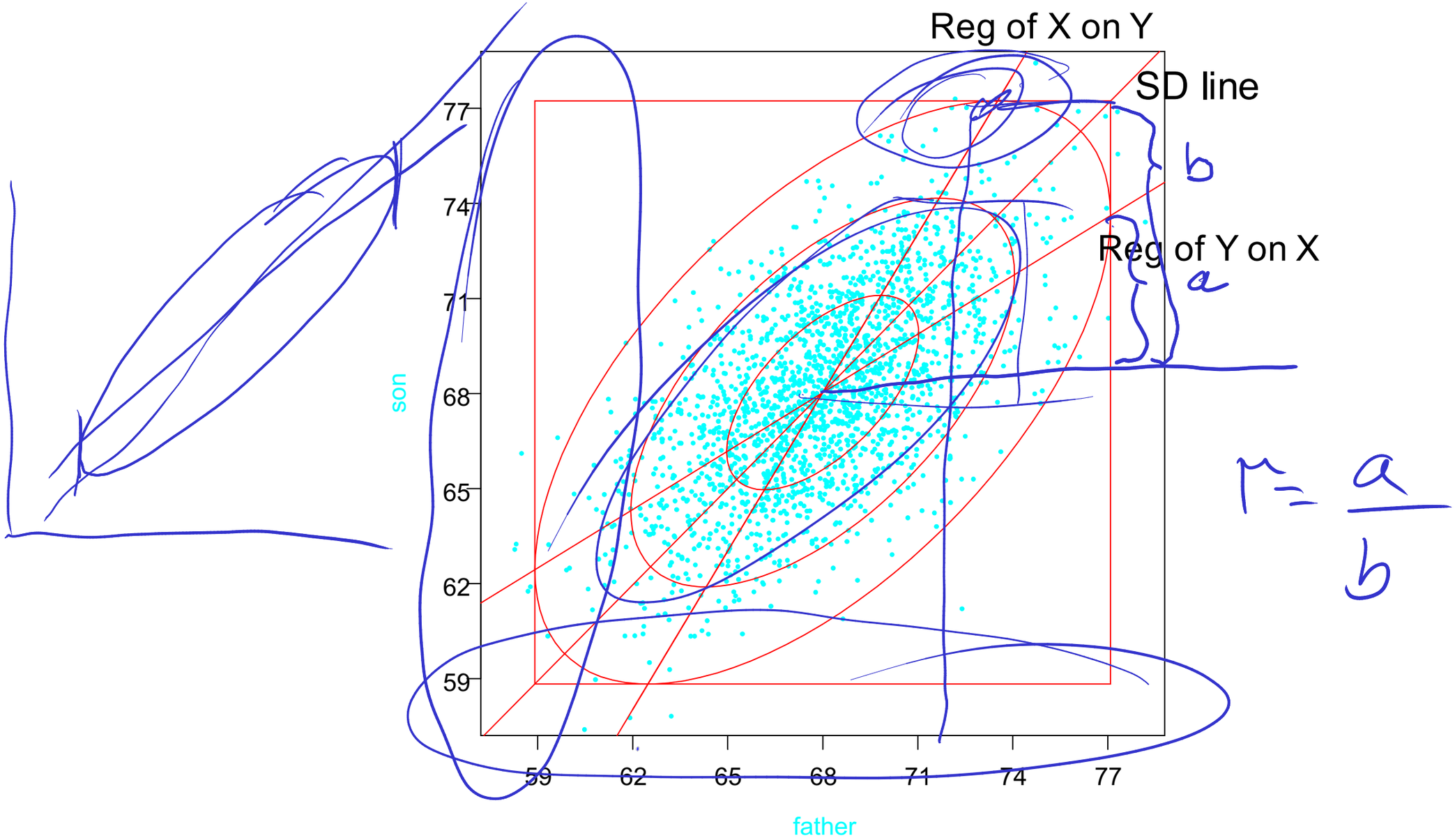
Reg of X on Y
SD line
b
Reg of Y on X
a
son
father
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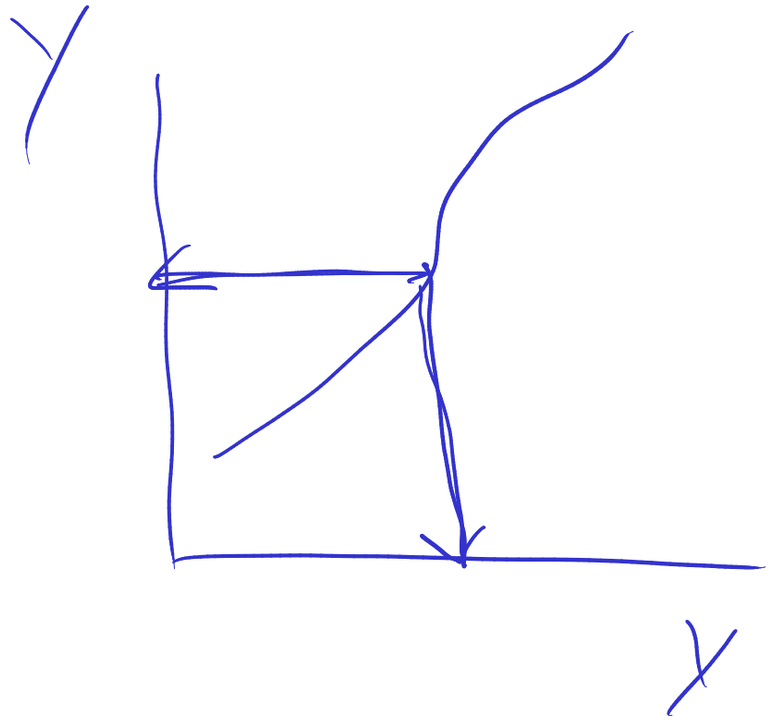
$$r = \frac{a}{b}$$

Regression Paradox:



Mathematics:

$$f(X|Y) = f^{-1}(Y|X)$$

Statistics:

$$E(X|Y) \neq E^{-1}(Y|X)$$

unless X and Y are exact functions of each other.

In this example, the overall distribution of heights remains the same from generation to generation.

But, following individuals, it seems that heights are *regressing* to the overall mean.

Moreover, this works both forward and backward in time!

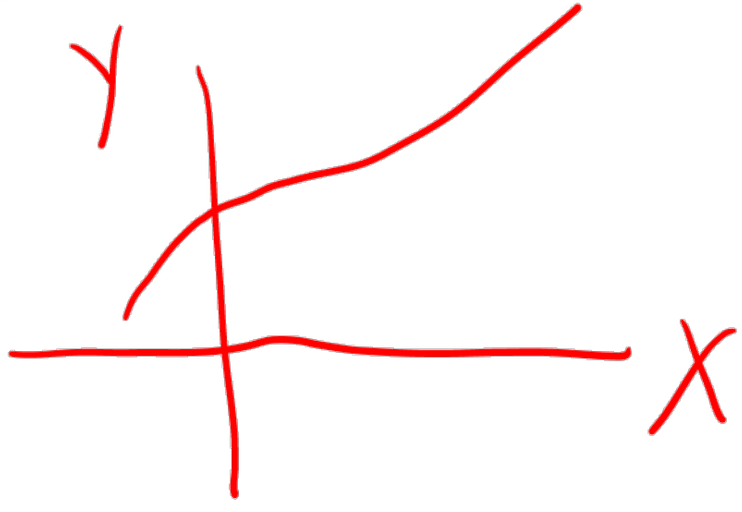
Regression Paradox

- The overall distribution of heights stays the same from generation to generation
- But when you follow individuals it looks like heights are being compressed towards the mean.
- Moreover, the same thing happens whether you go forward or backward in time.

Simply

The regression of Y on X, $E(Y|X)$ is not the mathematical inverse of the regression of X on Y, $E(X|Y)$

Mathematical inverse



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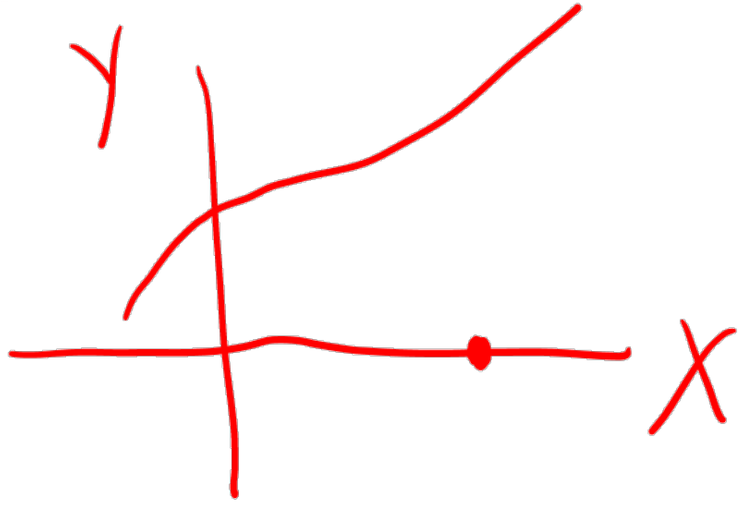

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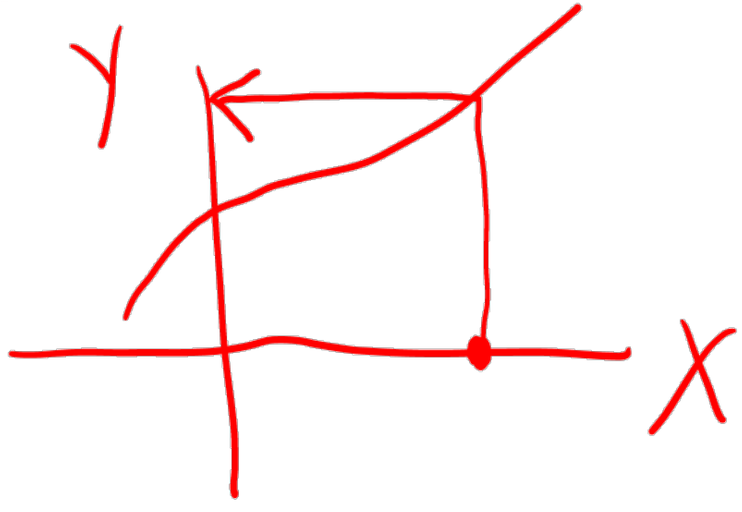

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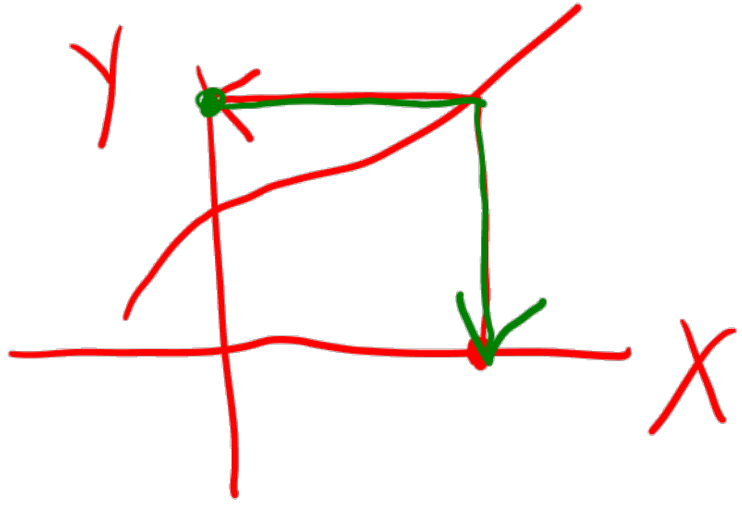

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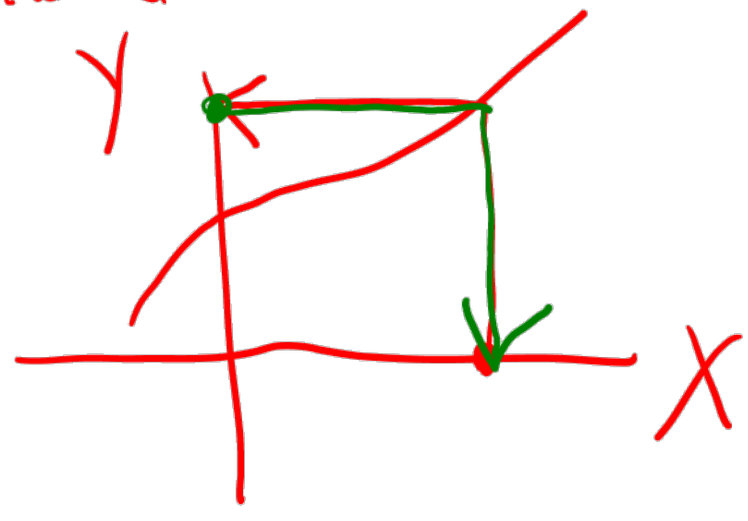
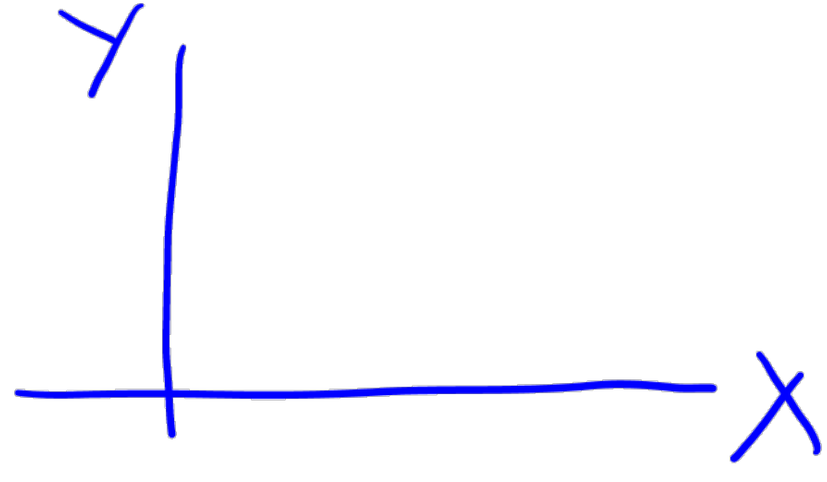
Mathematical inverse



Statistical Inverse



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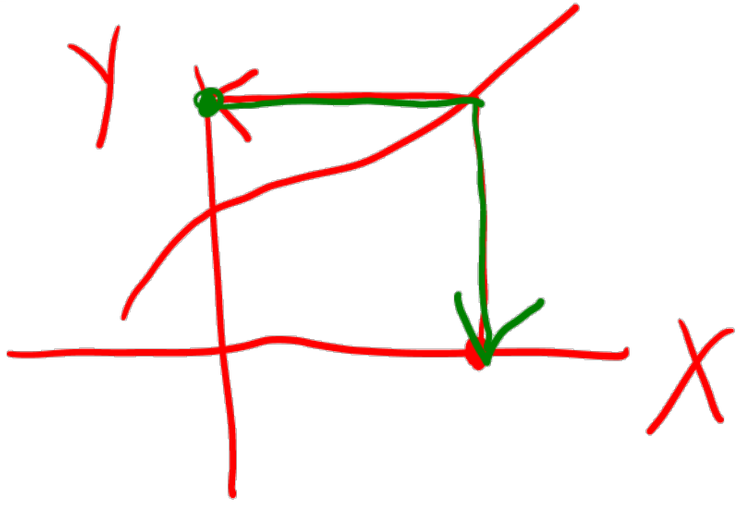
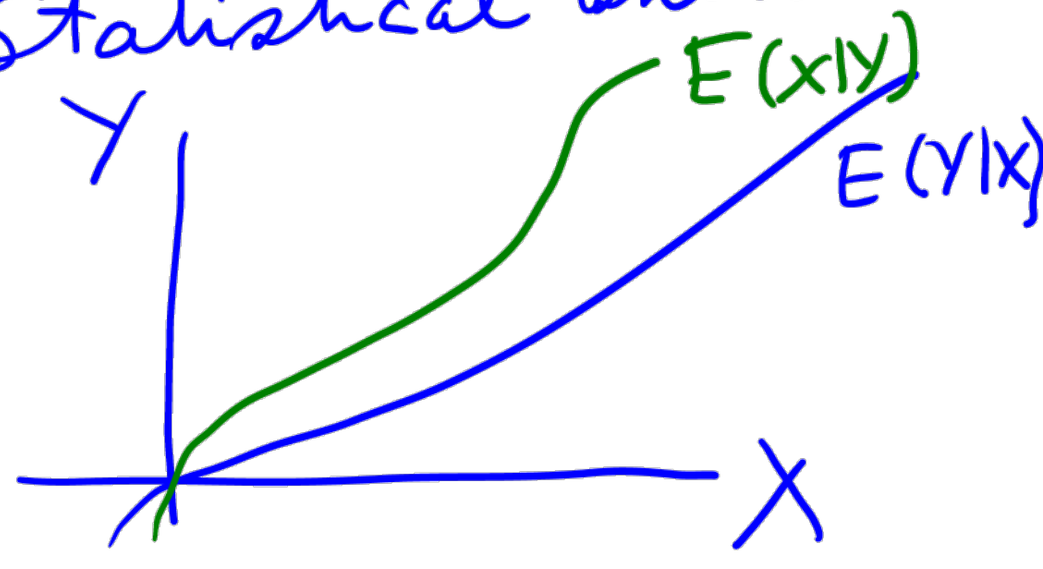
Mathematical inverse



Statistical Inverse



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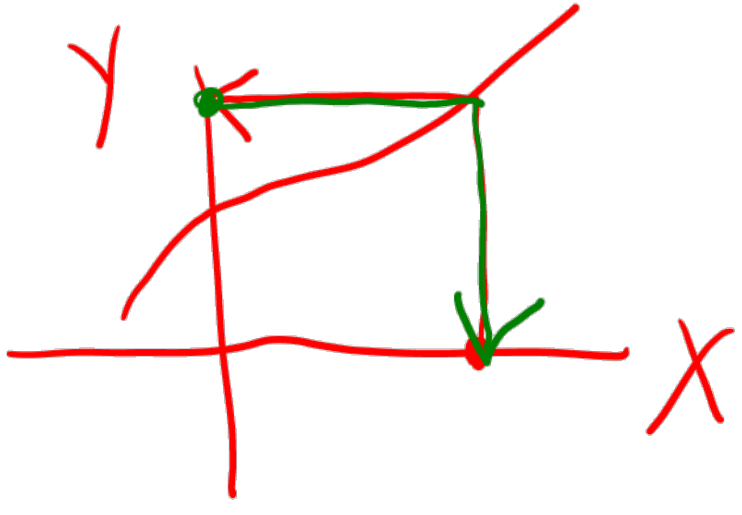
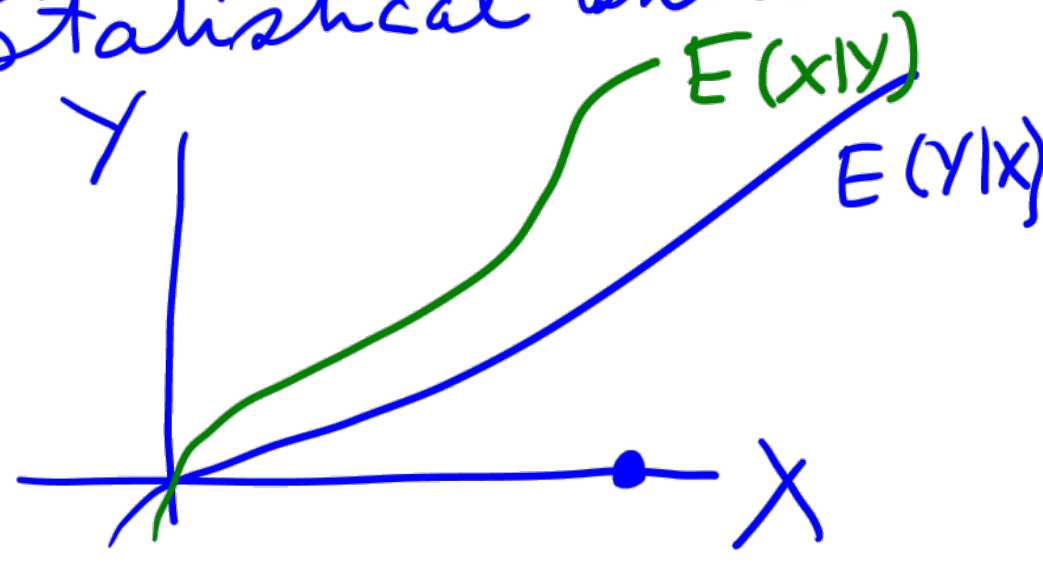
Mathematical inverse



Statistical Inverse



Simply

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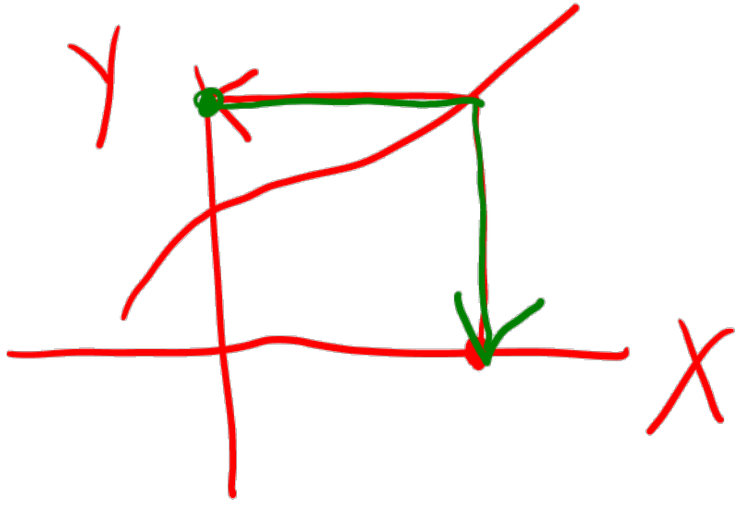
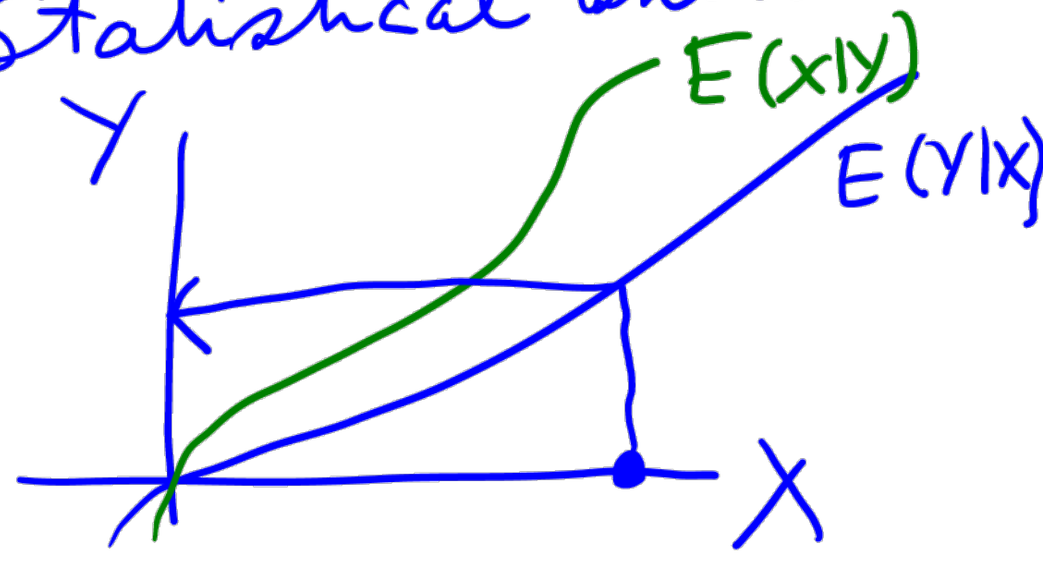
Mathematical inverse



Statistical Inverse



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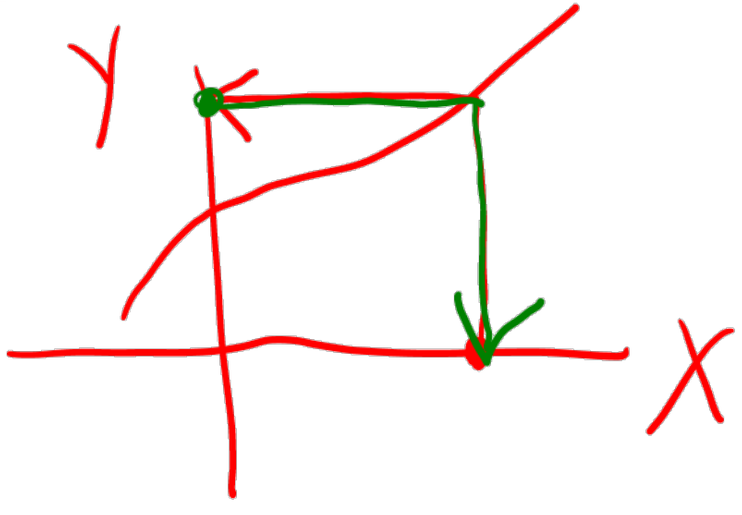
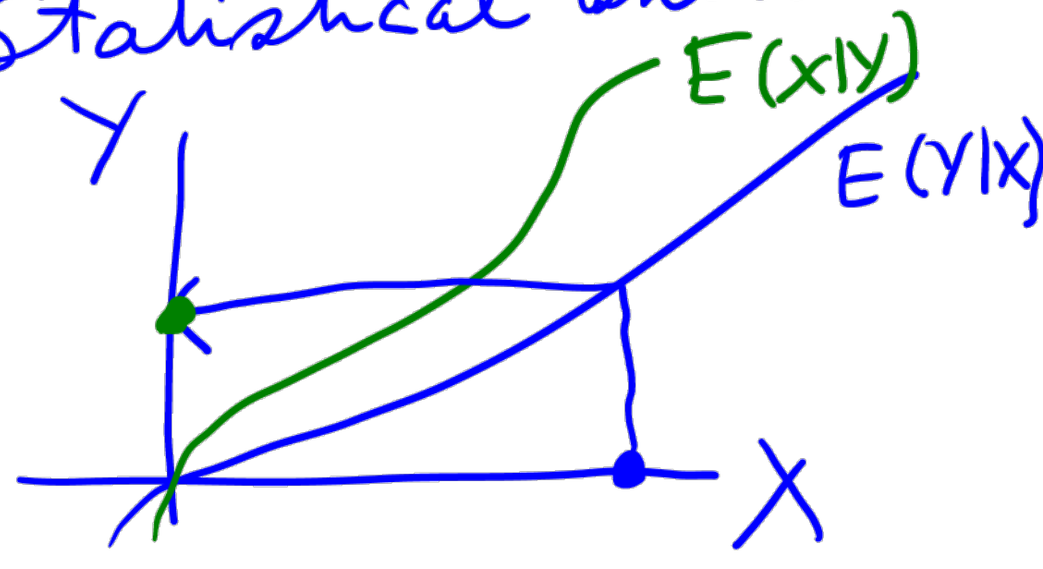


Statistical Inverse



Simply

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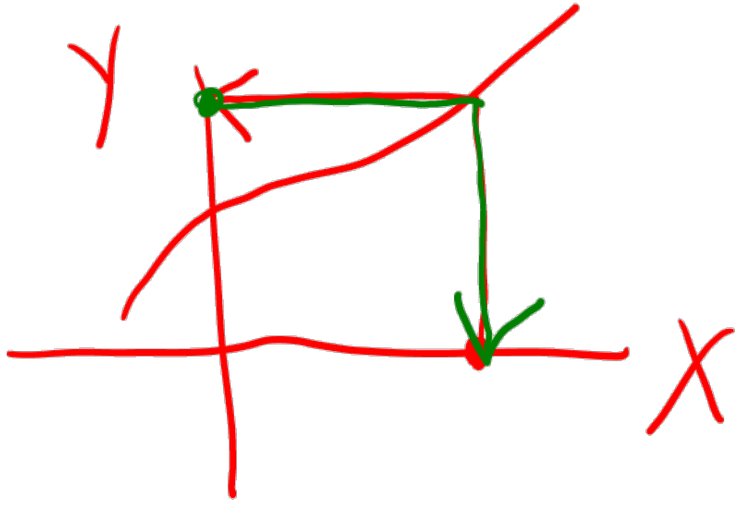
Mathematical inverse



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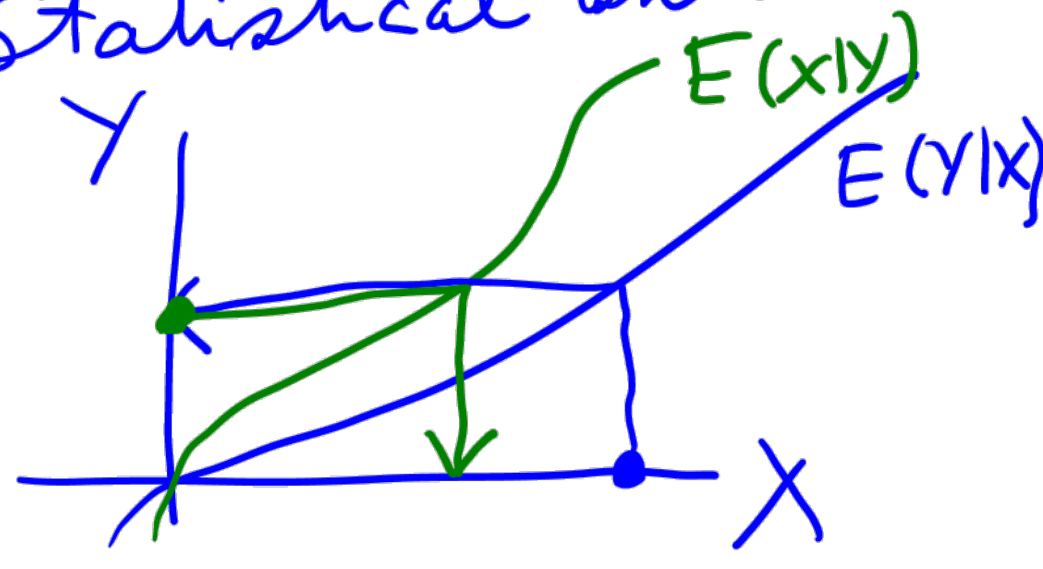

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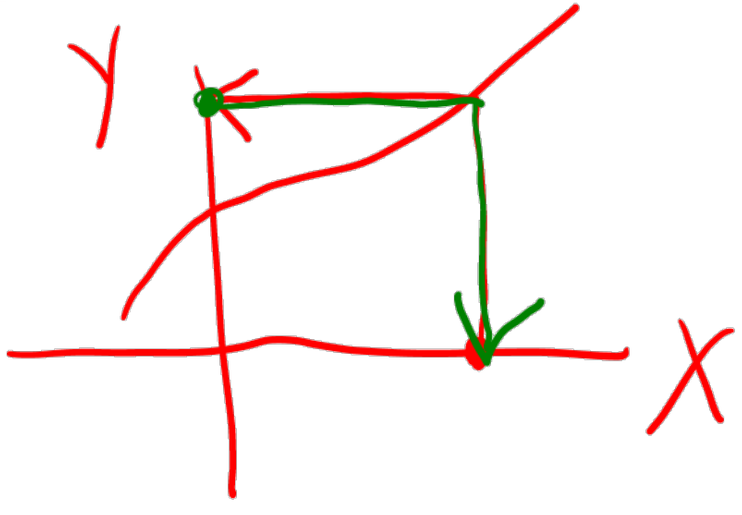

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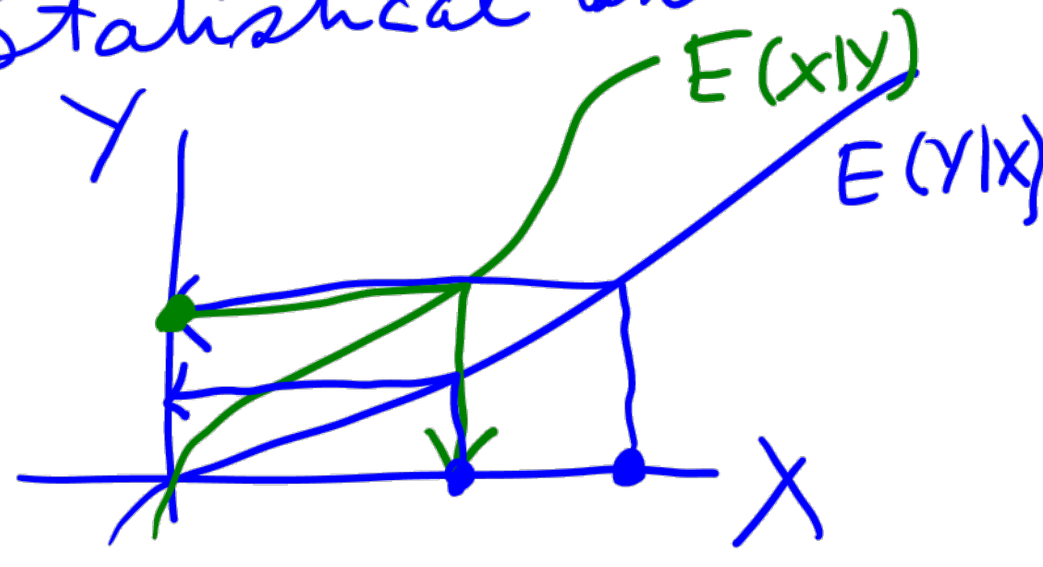

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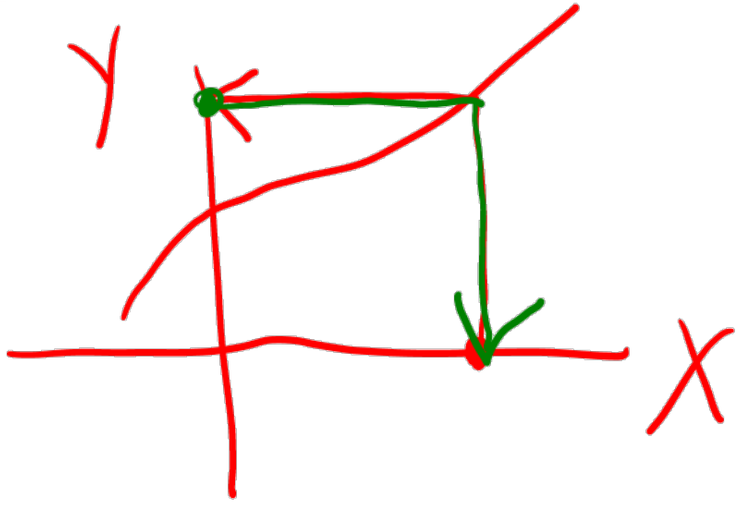
Mathematical inverse



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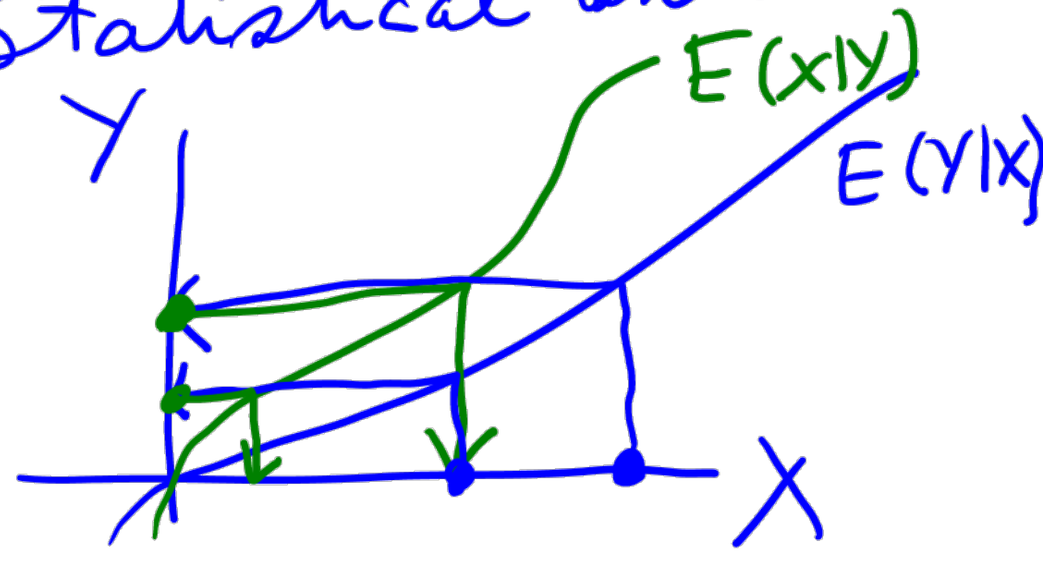

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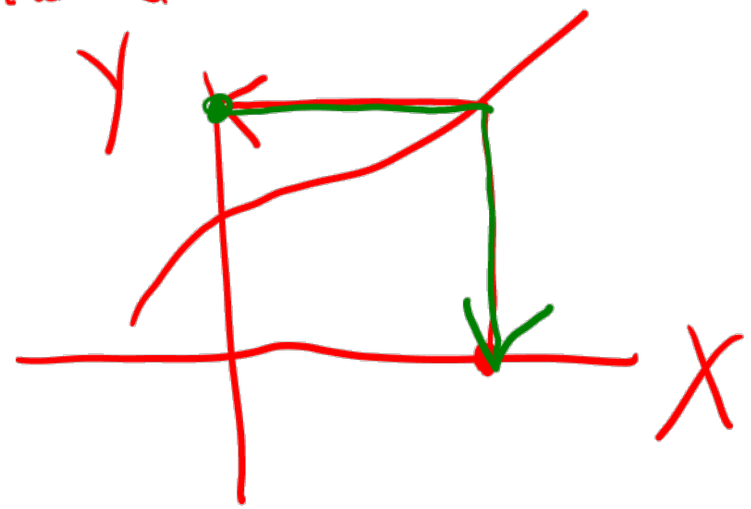
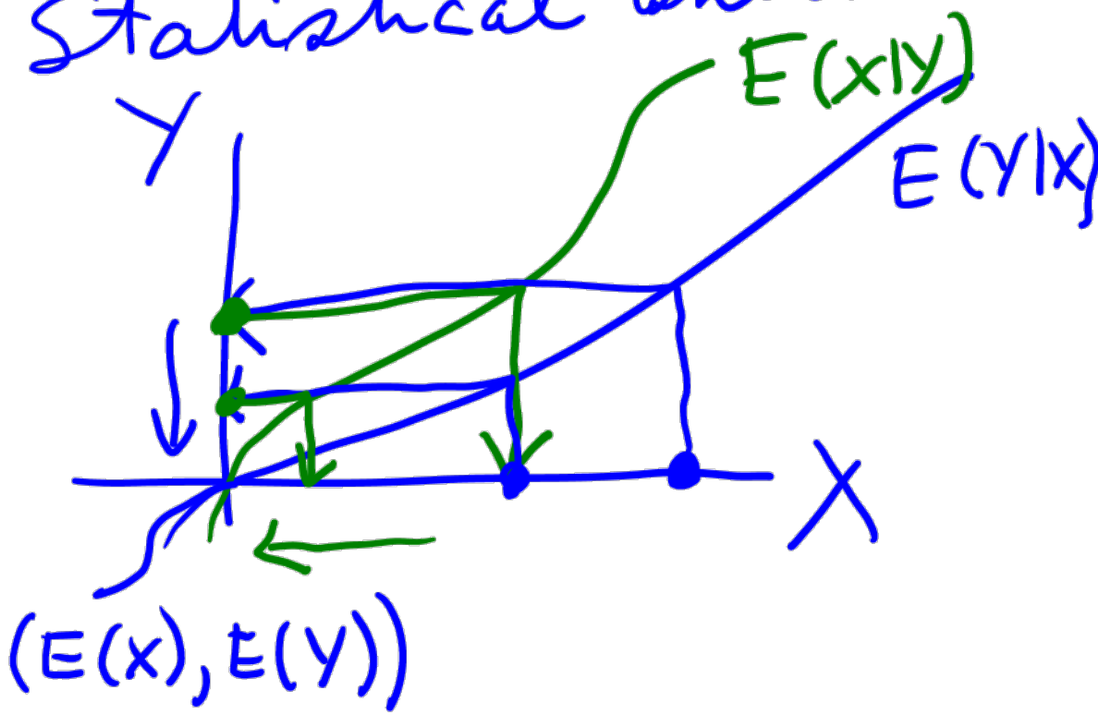


Statistical Inverse



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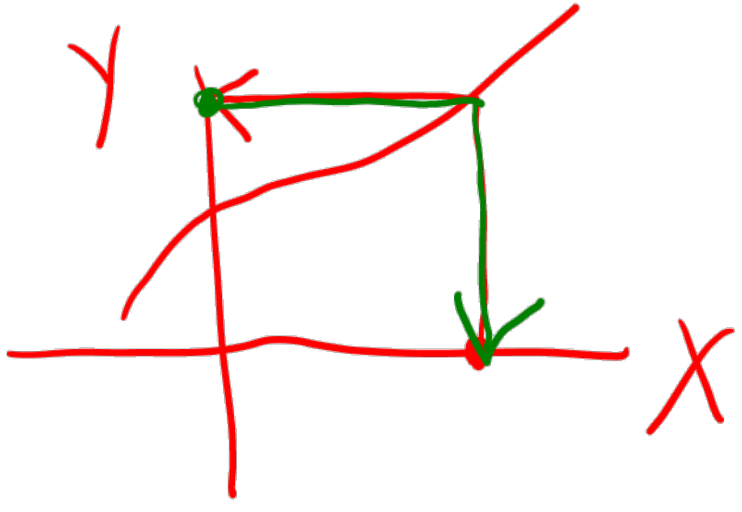
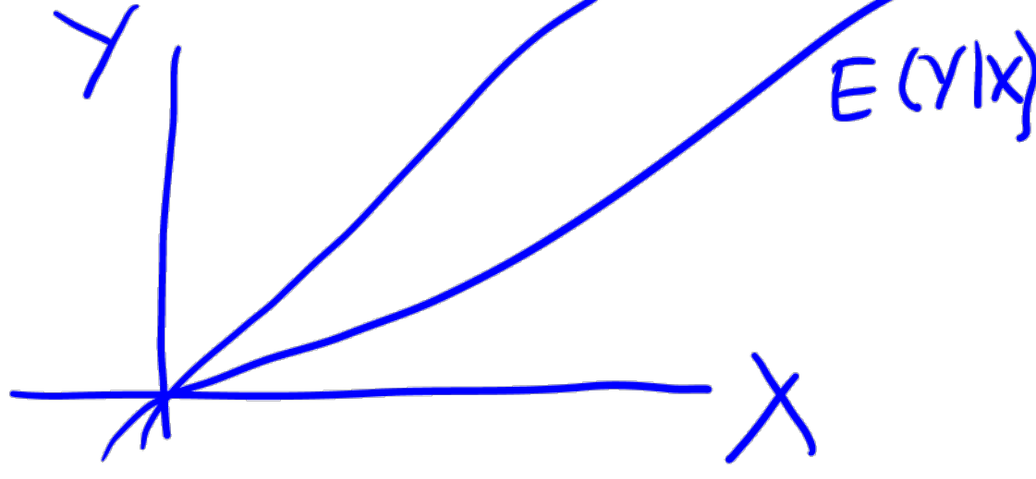
Mathematical inverse



Statistical Inverse



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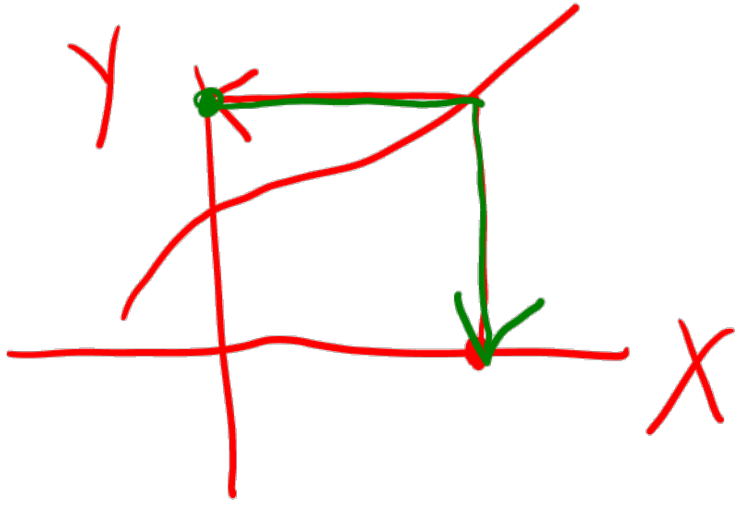
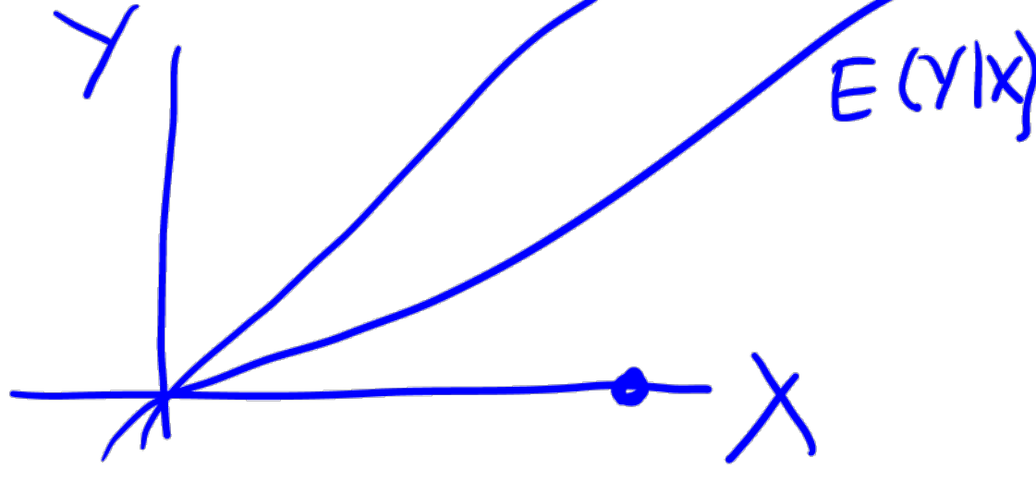


Statistical Inverse



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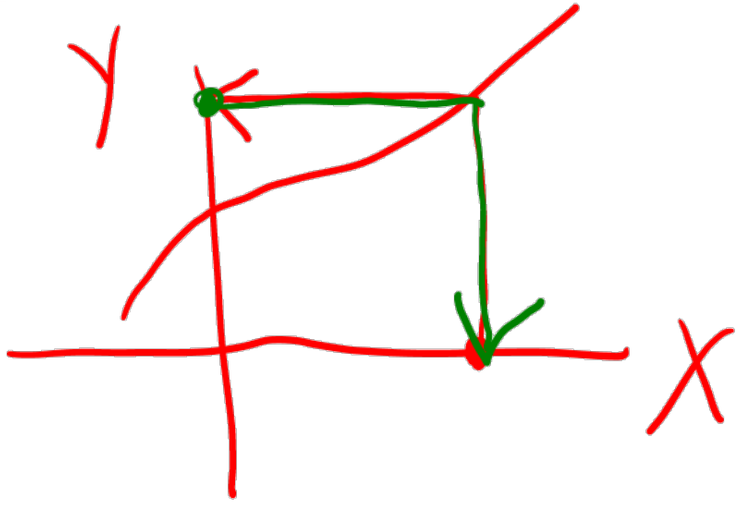
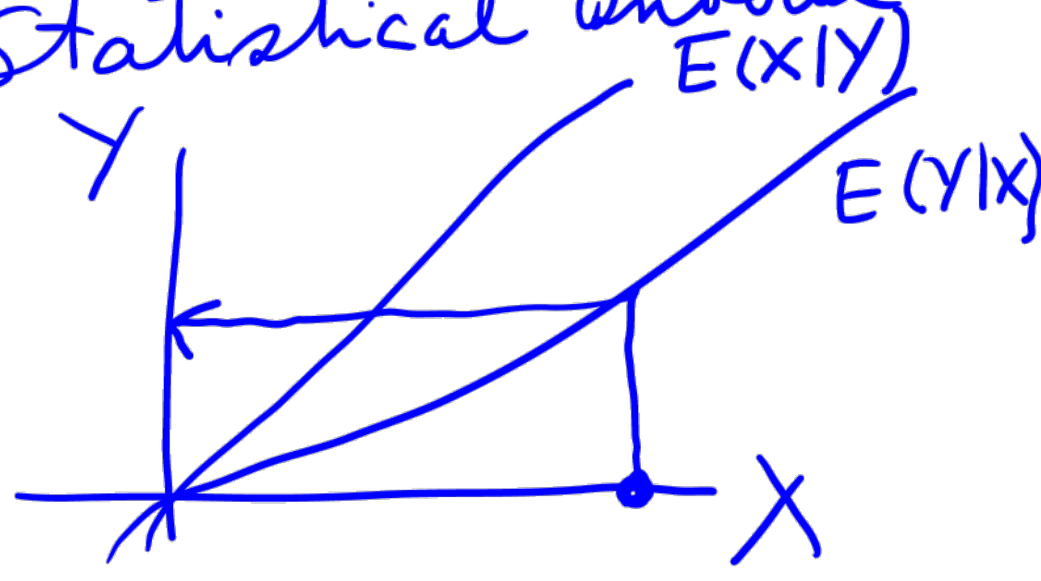
Mathematical inverse



Statistical Inverse



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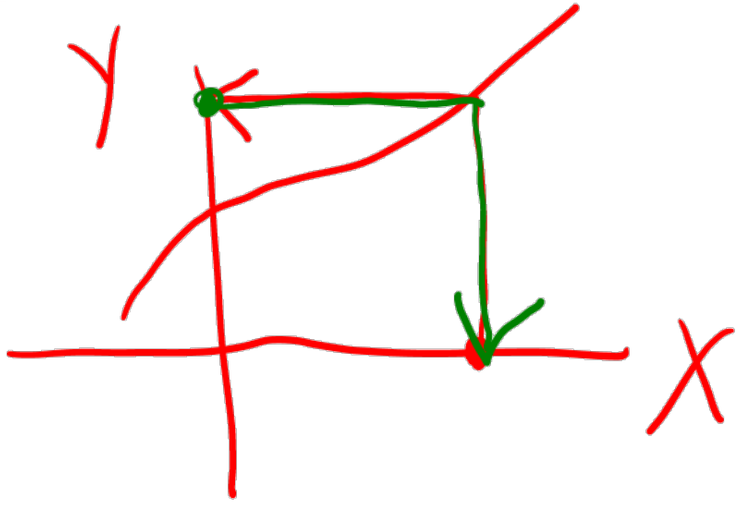
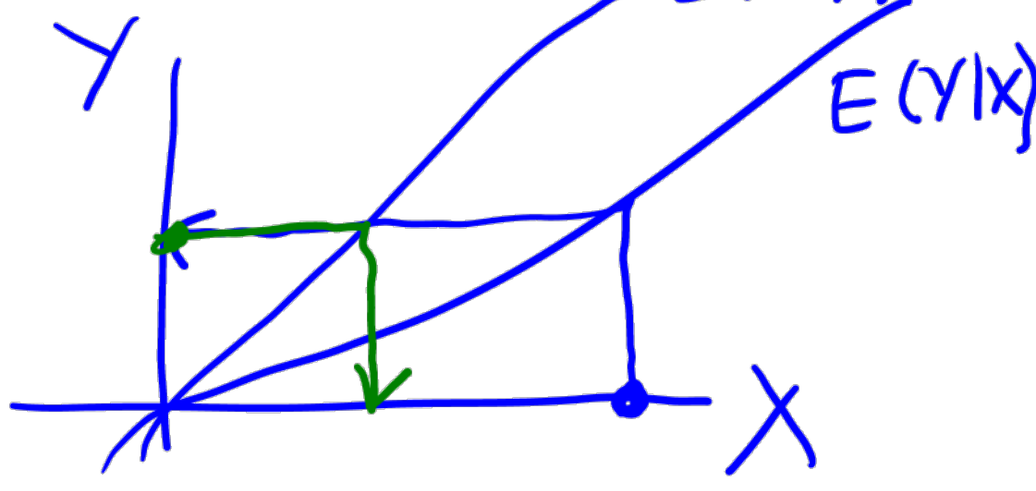


Statistical Inverse



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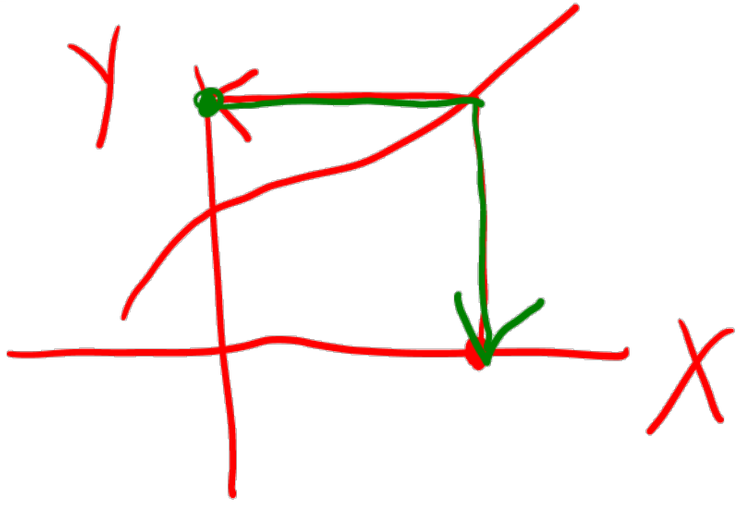
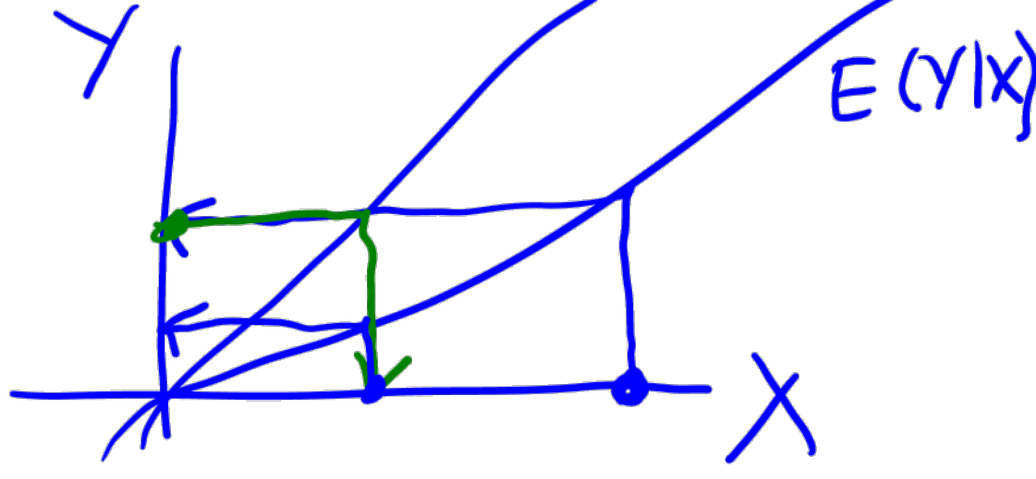


Statistical Inverse



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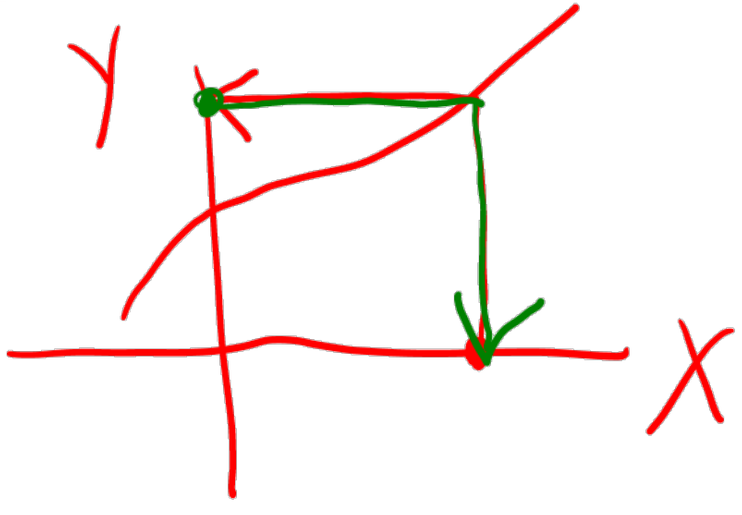
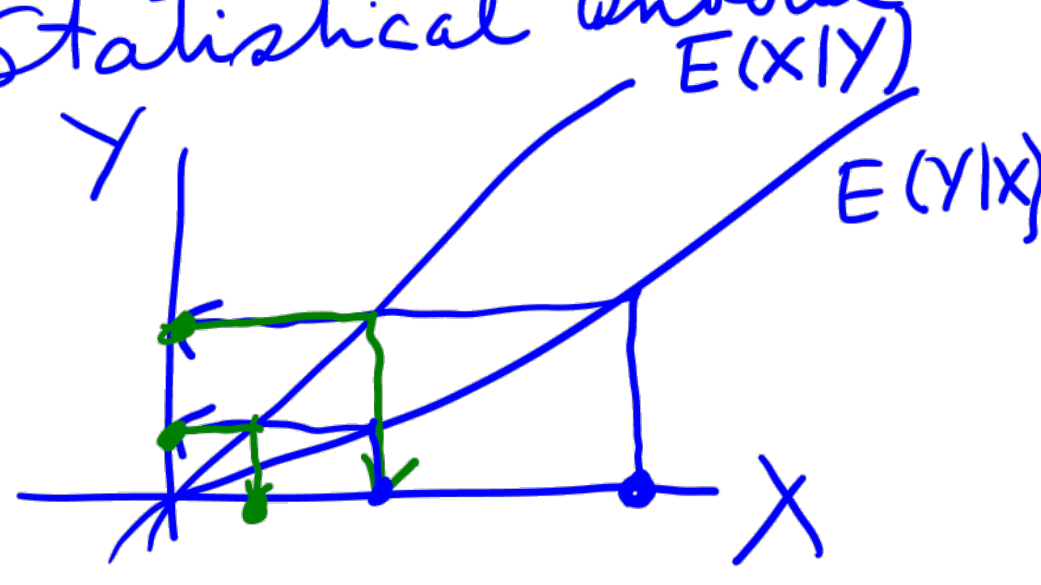
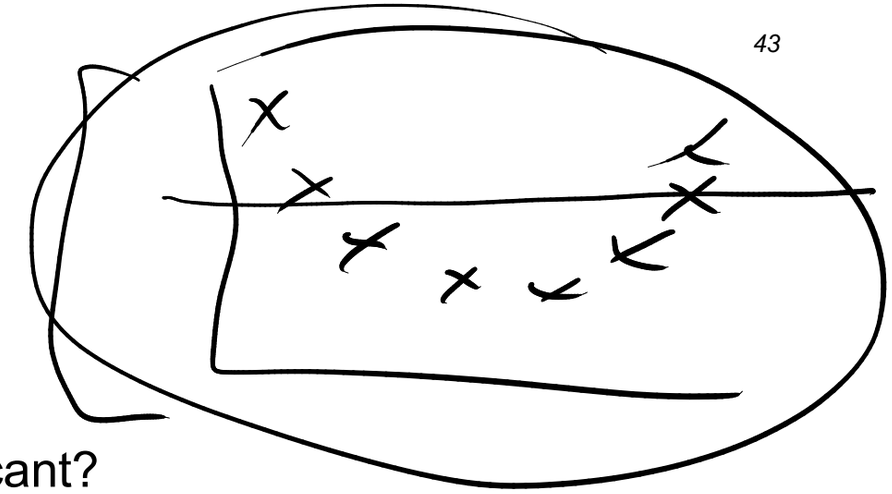


Statistical Inverse



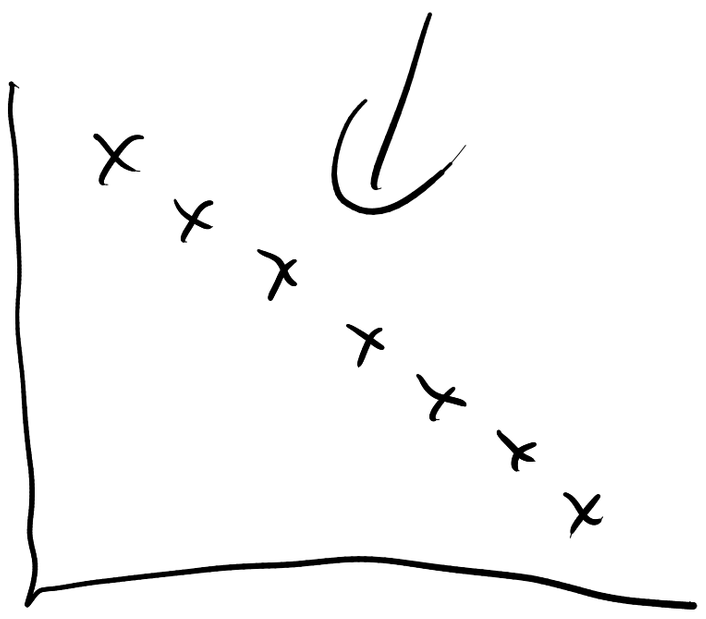
2 Eyeballing a scatterplot

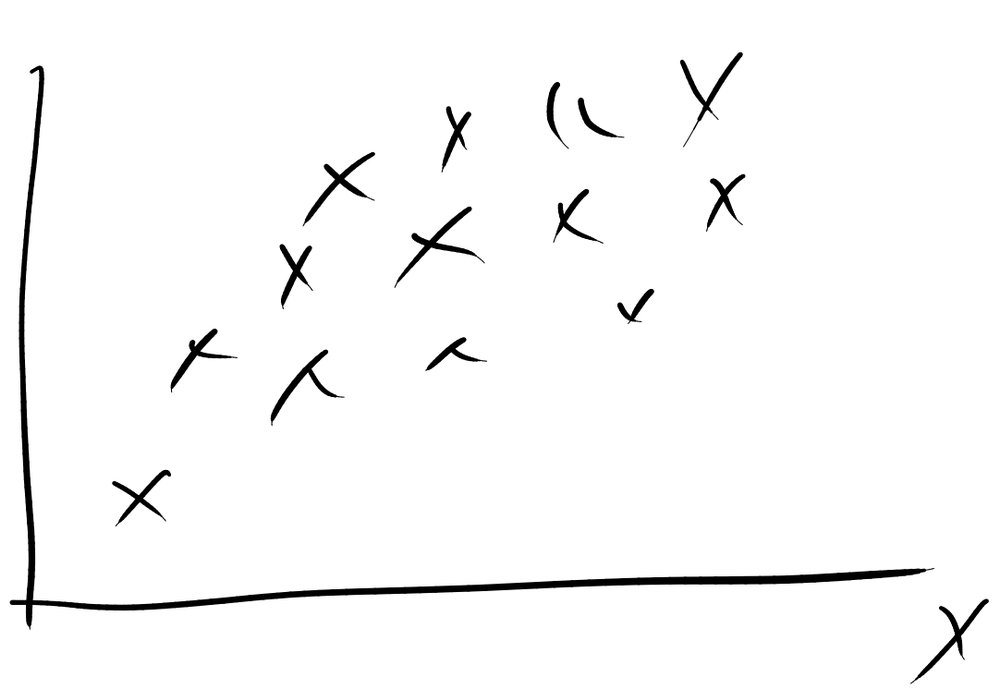
correlation?



At a glance:

- What is the correlation?
- Is the relationship statistically significant?
- Approximate 95% confidence interval?





Some data:

```
1971 Canadian Occupational Prestige Data
[1] Occupational title
[2] Average education of incumbents, years
[3] Average income of incumbents, dollars
[4] Percent of incumbents who are women
[5] Pineo-Porter prestige score for occupation
[6] Canadian Census occupational code
[7] Type of occupation
        prof =  professional and technical
        wc   =  white collar
        bc   =  blue collar
        ?    =  missing (not classified)
Source: Census of Canada, 1971, in Fox (1997)
```

	Education	Income	PercFem	Prestige	Code	Type
GOV_ADMINISTRATORS	13.11	12351	11.16	68.8	1113	prof
GENERAL_MANAGERS	12.26	25879	4.02	69.1	1130	prof
ACCOUNTANTS	12.77	9271	15.70	63.4	1171	prof
PURCHASING_OFFICERS	11.42	8865	9.11	56.8	1175	prof
CHEMISTS	14.62	8403	11.68	73.5	2111	prof
PHYSICISTS	15.64	11030	5.13	77.6	2113	prof
BIOLOGISTS	15.09	8258	25.65	72.6	2133	prof
ARCHITECTS	15.44	14163	2.69	78.1	2141	prof
CIVIL_ENGINEERS	14.52	11377	1.03	73.1	2143	prof
MINING_ENGINEERS	14.64	11023	0.94	68.8	2153	prof

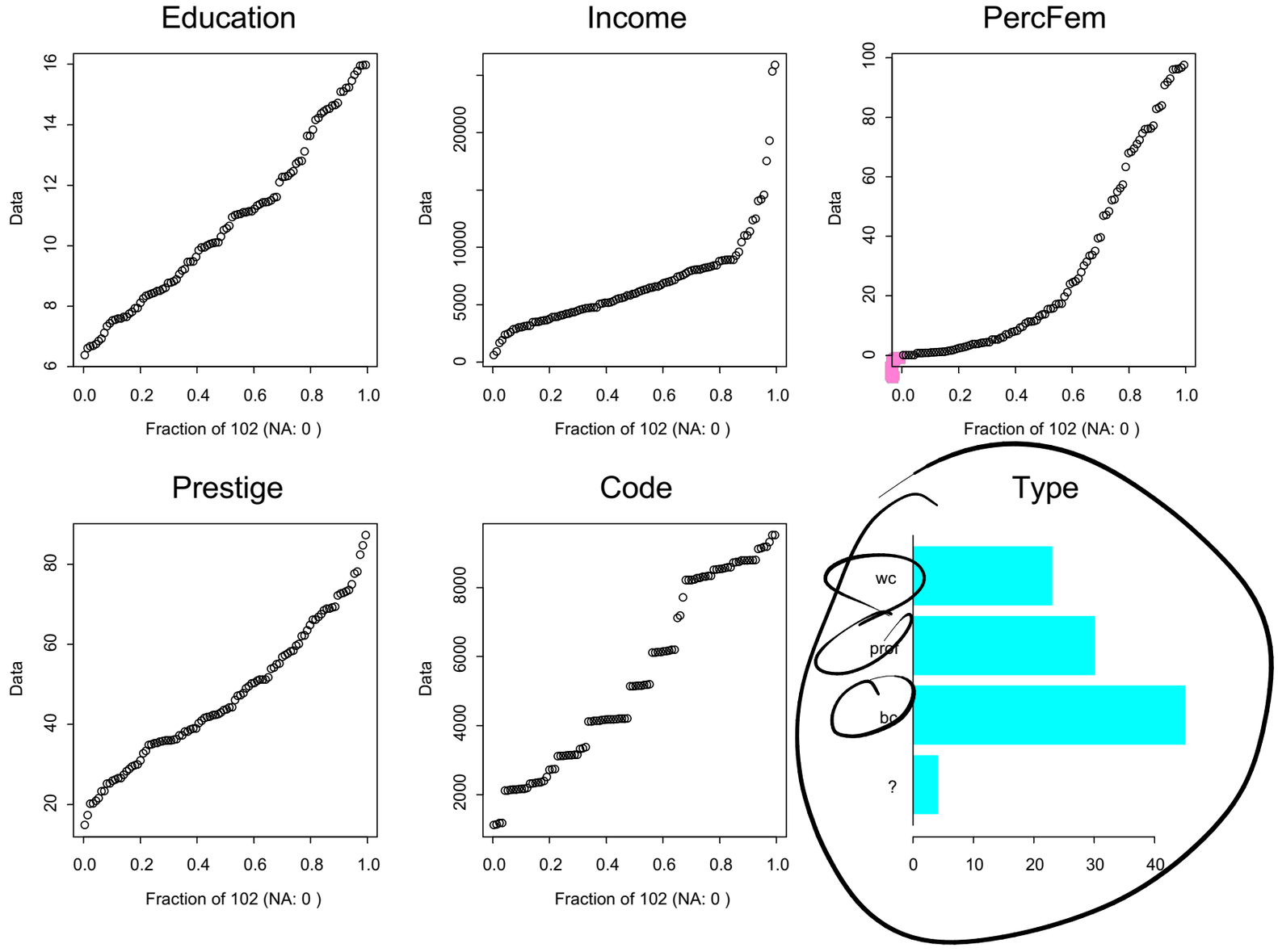
...

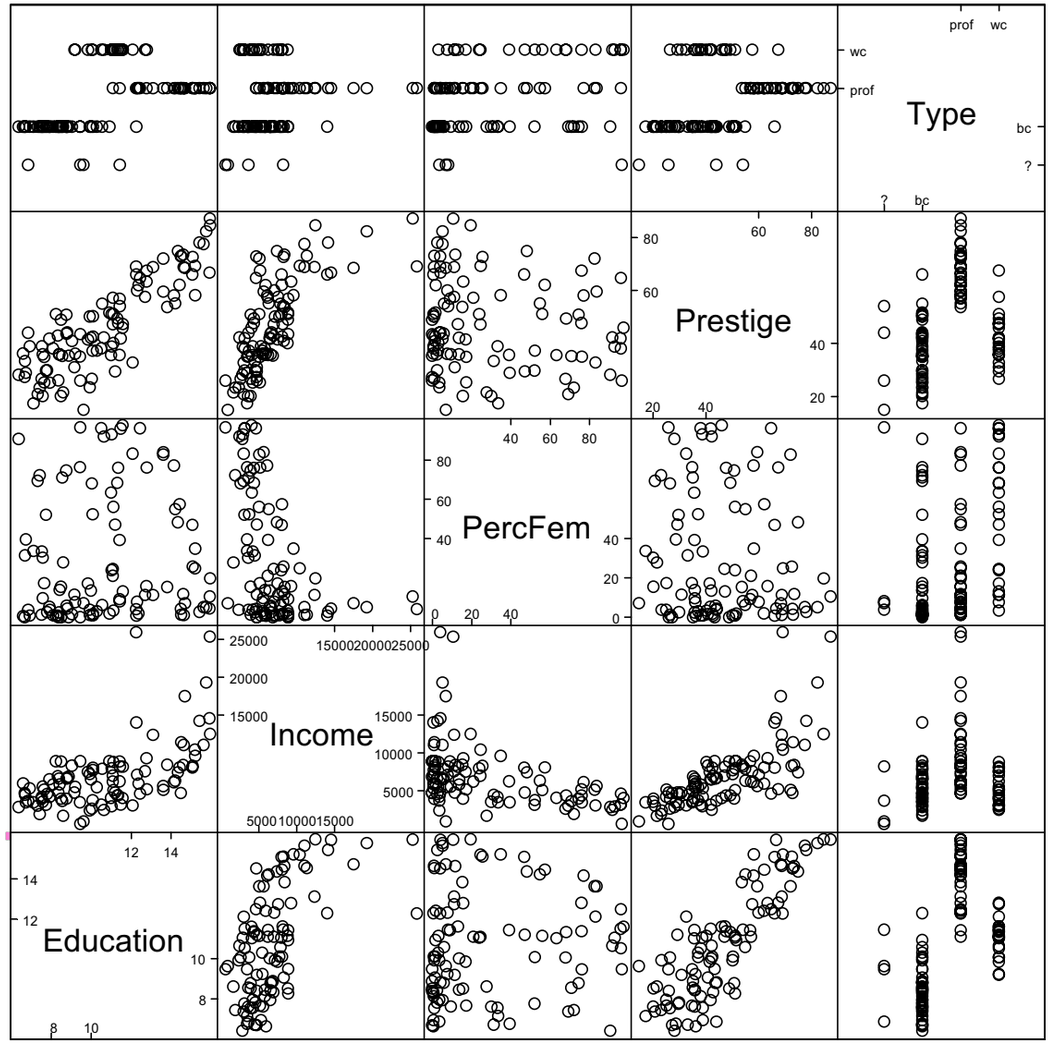
102 occupations

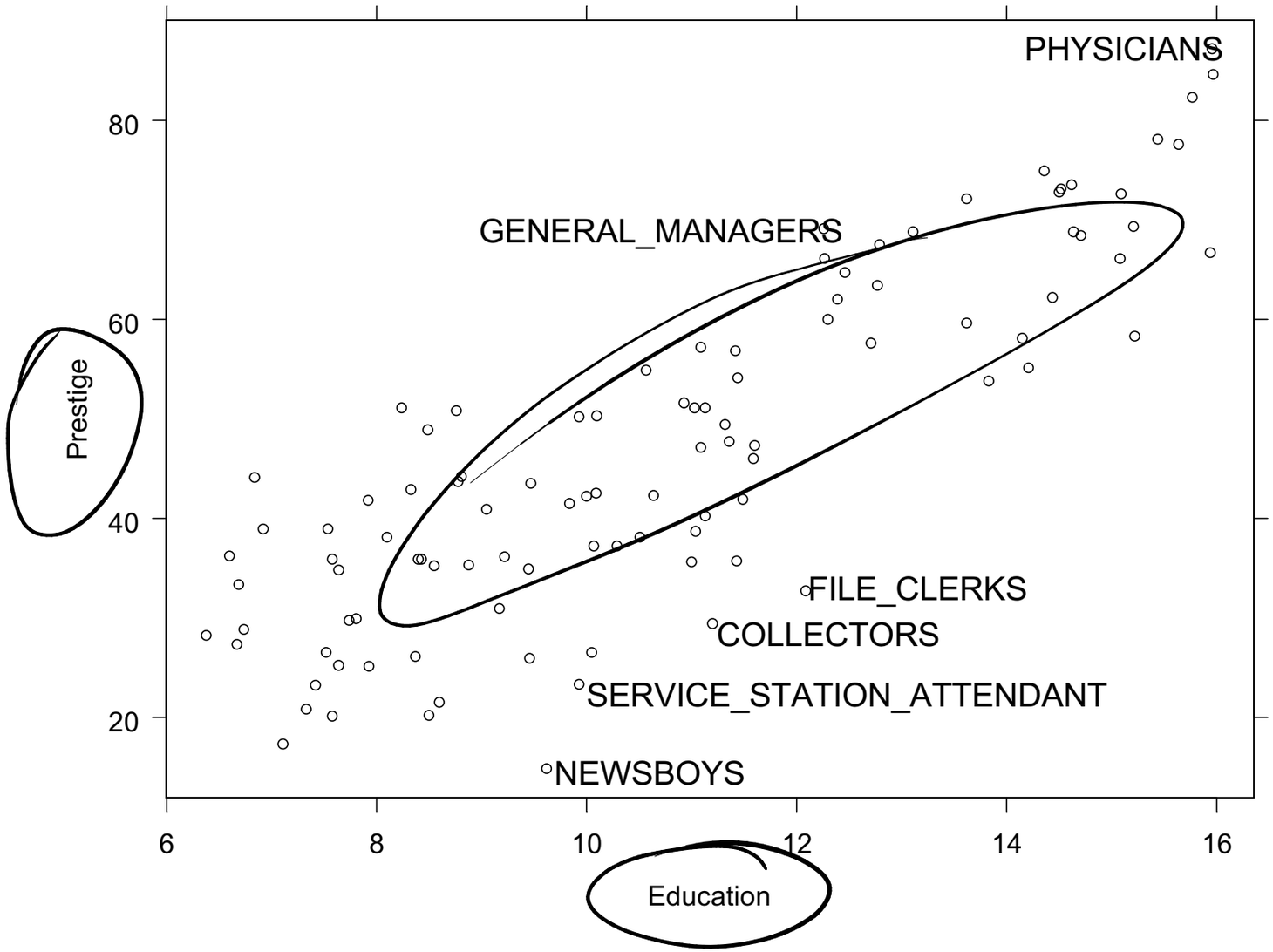
Summary:

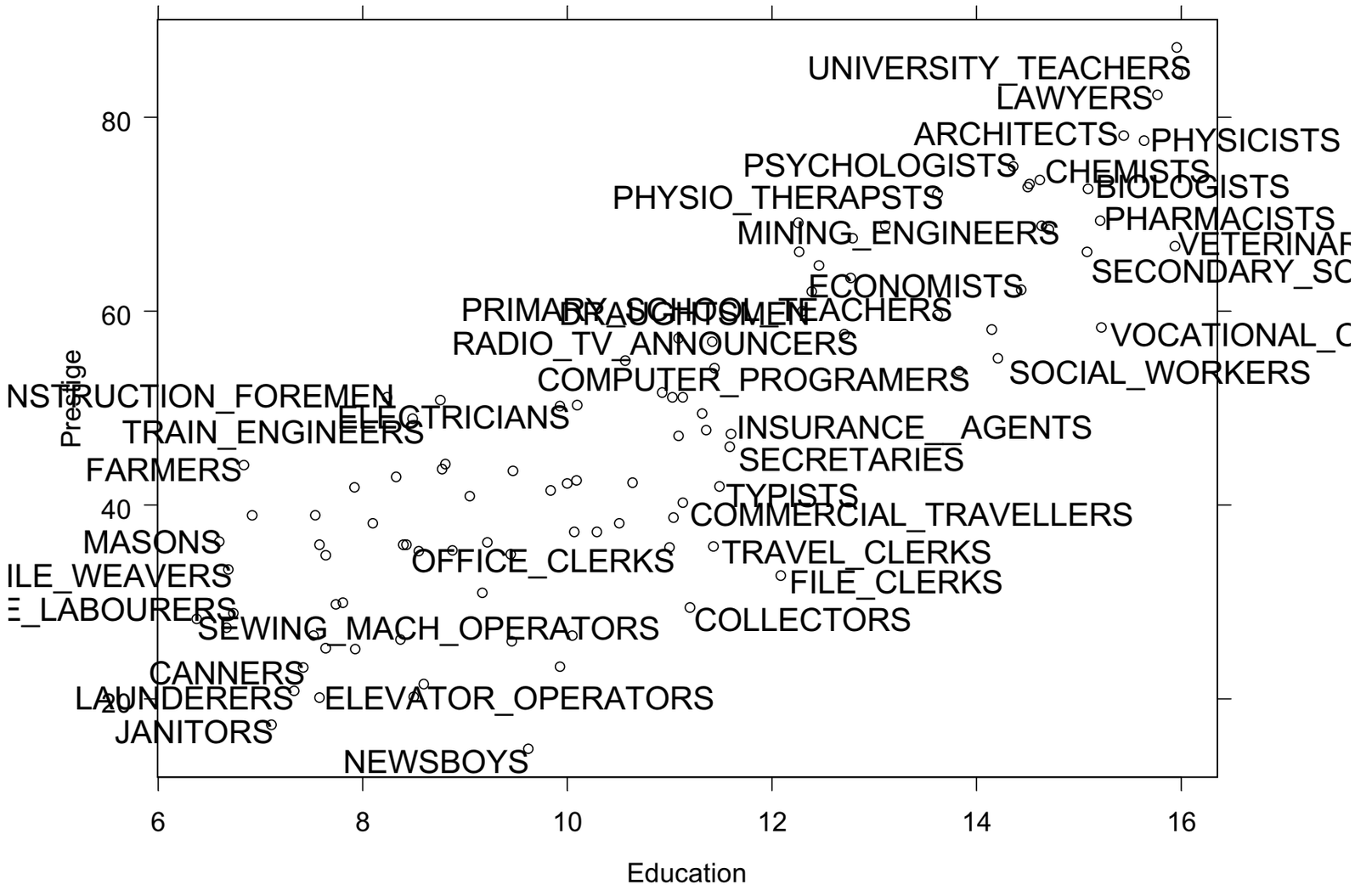
```
   Education          Income          PercFem
   Min.: 6.380      Min.:  611      Min.: 0.000
1st Qu.: 8.445   1st Qu.: 4106   1st Qu.: 3.592
 Median:10.540    Median: 5930    Median:13.600
   Mean:10.740      Mean: 6798      Mean:28.980
3rd Qu.:12.650   3rd Qu.: 8187   3rd Qu.:52.200
   Max.:15.970      Max.:25880      Max.:97.510
```

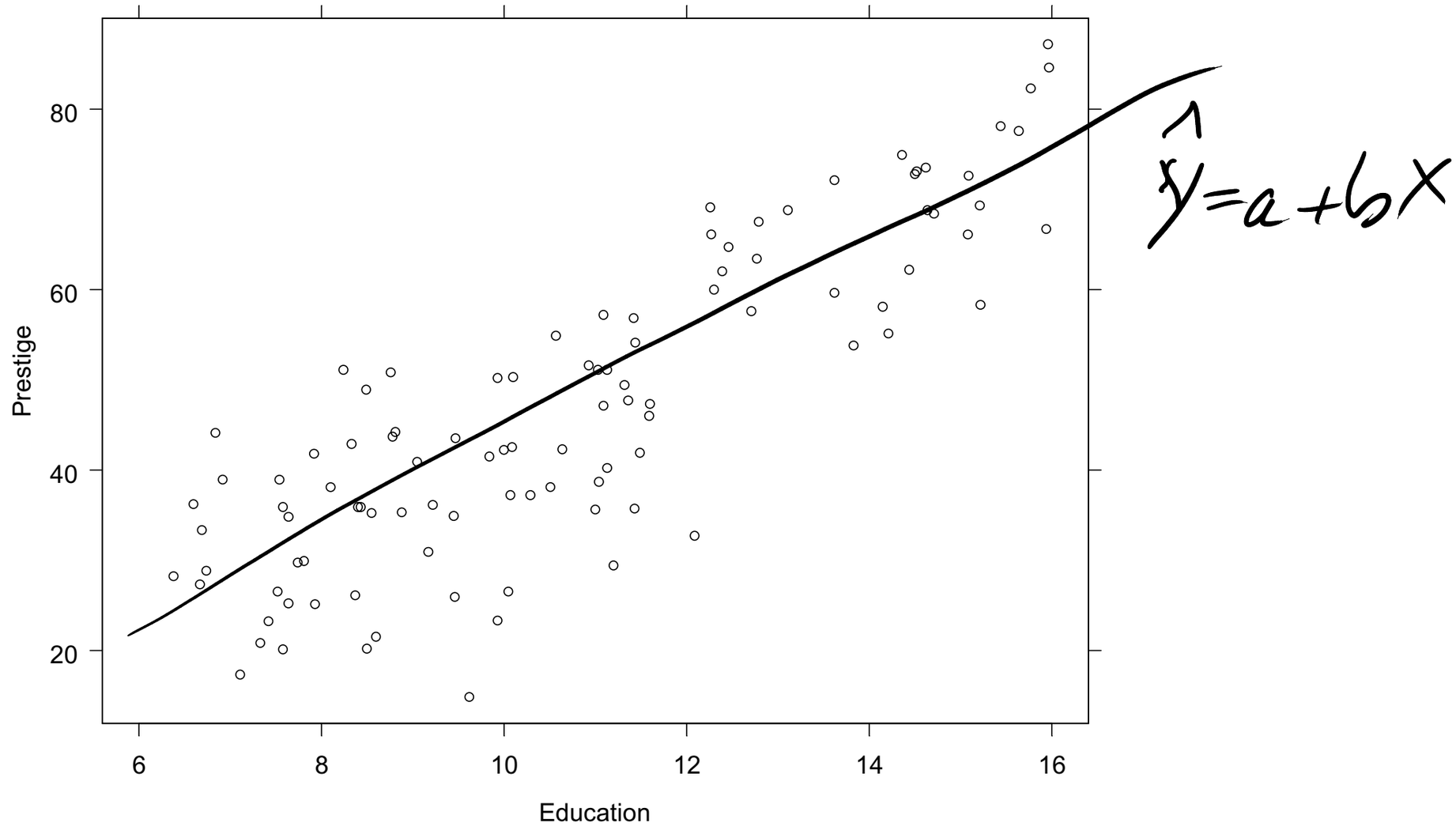
```
    Prestige          Code          Type
   Min.:14.80      Min.:1113      ?: 4
1st Qu.:35.22   1st Qu.:3120     bc:45
 Median:43.60    Median:5135   prof:30
   Mean:46.83      Mean:5402     wc:23
3rd Qu.:59.28   3rd Qu.:8312
   Max.:87.20      Max.:9517
```


Education
Data
Fraction of 102 (NA: 0)
Income
Data
Fraction of 102 (NA: 0)
PercFem
Data
Fraction of 102 (NA: 0)
Prestige
Data
Fraction of 102 (NA: 0)
Code
Data
Fraction of 102 (NA: 0)
Type
wc
prof
bc
?
0
10
20
30
40


Education
Income
PercFem
Prestige
Type
wc
prof
bc
?


PHYSICIANS
GENERAL_MANAGERS
FILE_CLERKS
COLLECTORS
SERVICE_STATION_ATTENDANT
NEWSBOYS
Prestige
Education
80
60
40
20
6
8
10
12
14
16


UNIVERSITY_TEACHERS
LAWYERS
ARCHITECTS
PHYSICISTS
PSYCHOLOGISTS
CHEMISTS
BIOLOGISTS
PHYSIO_THERAPSTS
PHARMACISTS
MINING_ENGINEERS
VETERINAR
SECONDARY_SC
ECONOMISTS
PRIMARY_SCHOOL_TEACHERS
BRAUGHTSMEN
VOCATIONAL_C
RADIO_TV_ANNOUNCERS
SOCIAL_WORKERS
COMPUTER_PROGRAMERS
NSTRUCTION_FOREMEN
ELECTRICIANS
INSURANCE__AGENTS
TRAIN_ENGINEERS
SECRETARIES
FARMERS
TYPISTS
COMMERCIAL_TRAVELLERS
MASONS
TRAVEL_CLERKS
OFFICE_CLERKS
ILE_WEAVERS
FILE_CLERKS
E_LABOURERS
COLLECTORS
SEWING_MACH_OPERATORS
CANNERS
LAUNDERERS
ELEVATOR_OPERATORS
JANITORS
NEWSBOYS
Prestige
Education
80
60
40
20
6
8
10
12
14
16


Prestige
Education
20
40
60
80
6
8
10
12
14
16
$\hat{y} = a + bx$

Typical regression output:

```
> summary(fit <- lm(Prestige ~ Education, pdat))

Call: lm(formula = Prestige ~ Education, data = pdat)
Residuals:
    Min      1Q Median    3Q   Max
 -26.04 -6.523 0.6611 6.743 18.16

Coefficients:
             Value Std. Error  t value Pr(>|t|)
(Intercept) -10.7320   3.6771   -2.9186   0.0043
  Education   5.3609   0.3320   16.1478   0.0000

Residual standard error: 9.103 on 100 degrees of freedom
Multiple R-Squared: 0.7228
F-statistic: 260.8 on 1 and 100 degrees of freedom,
          the p-value is 0
```

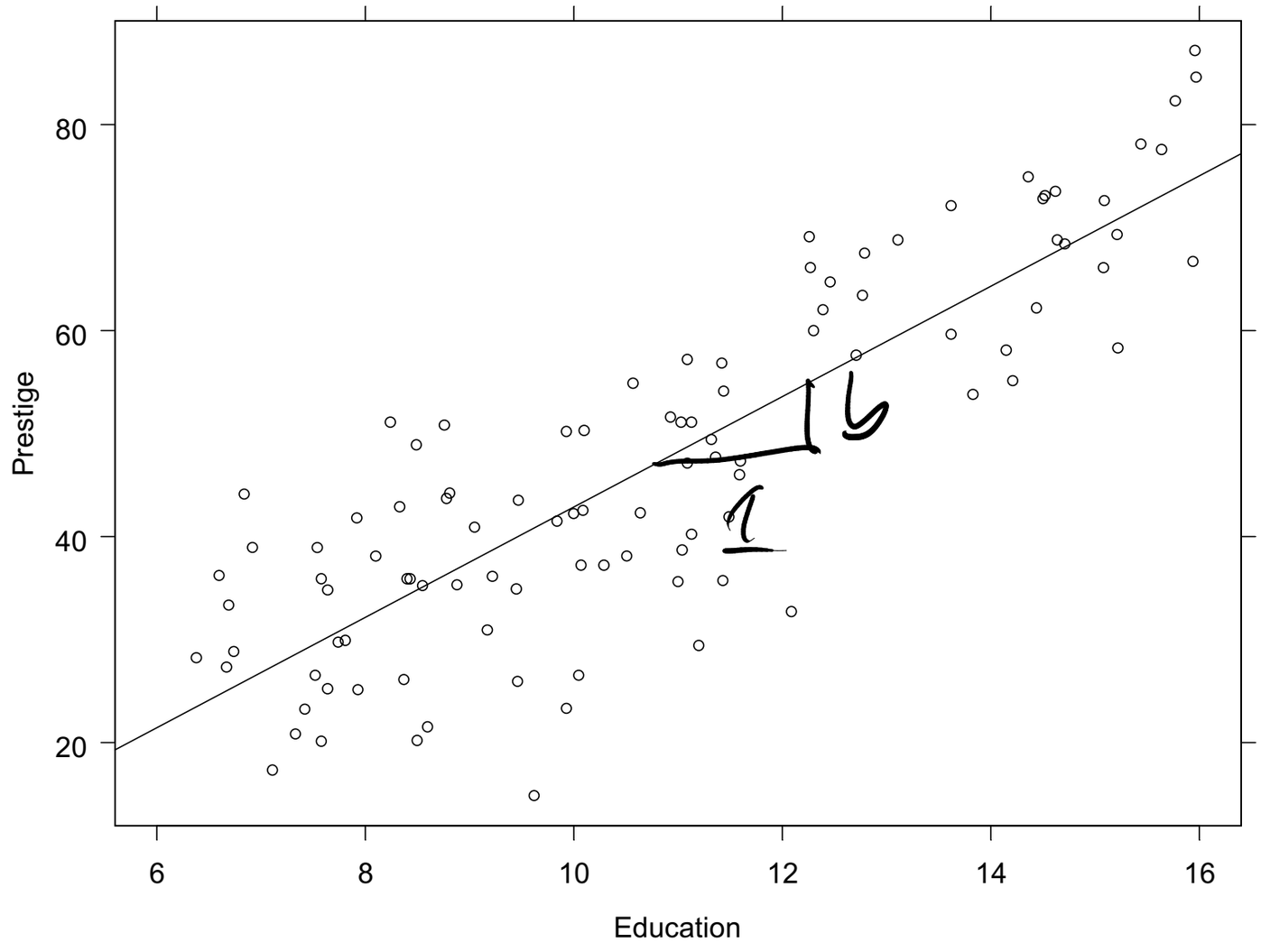
S_e

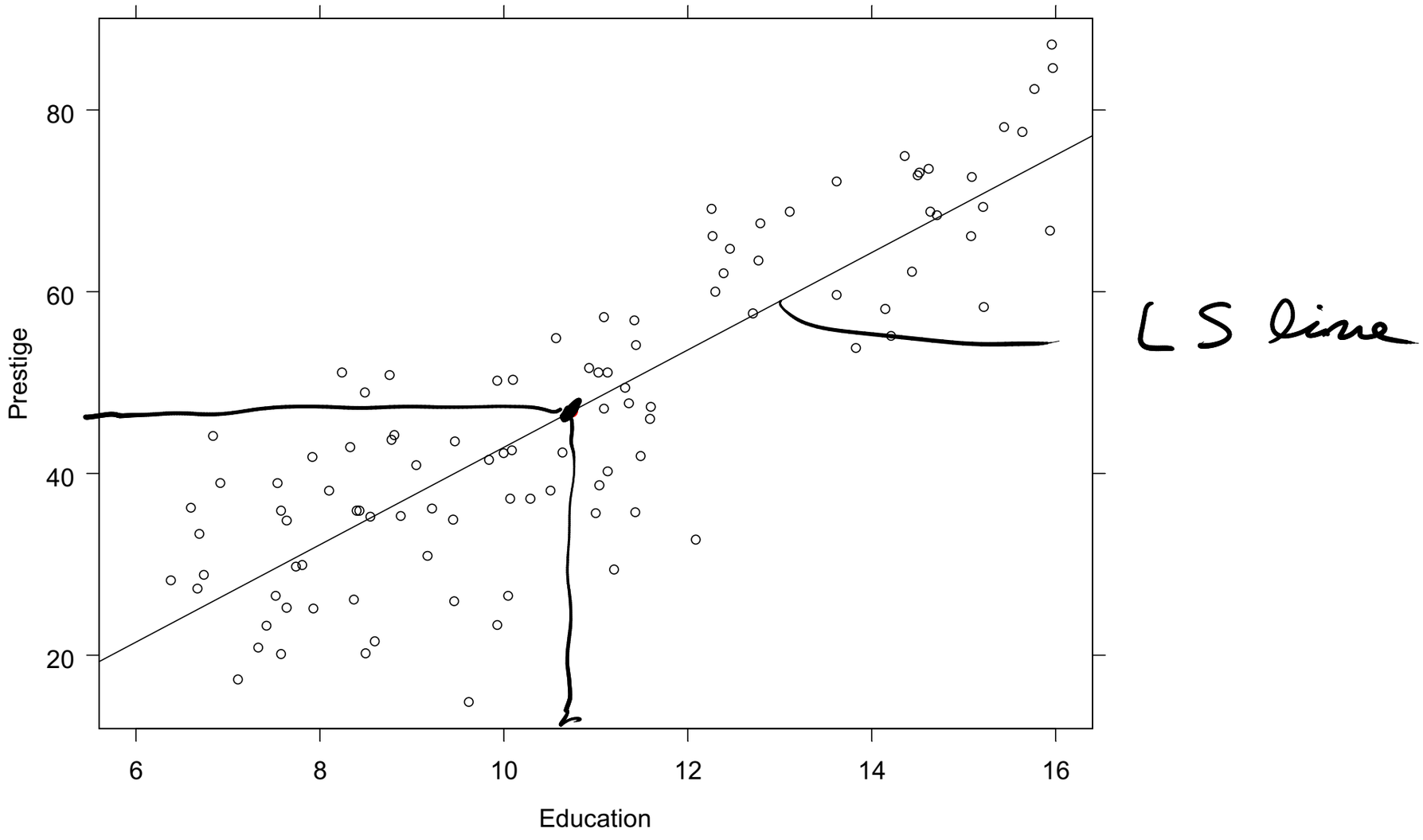
Slope of the least-squares line:

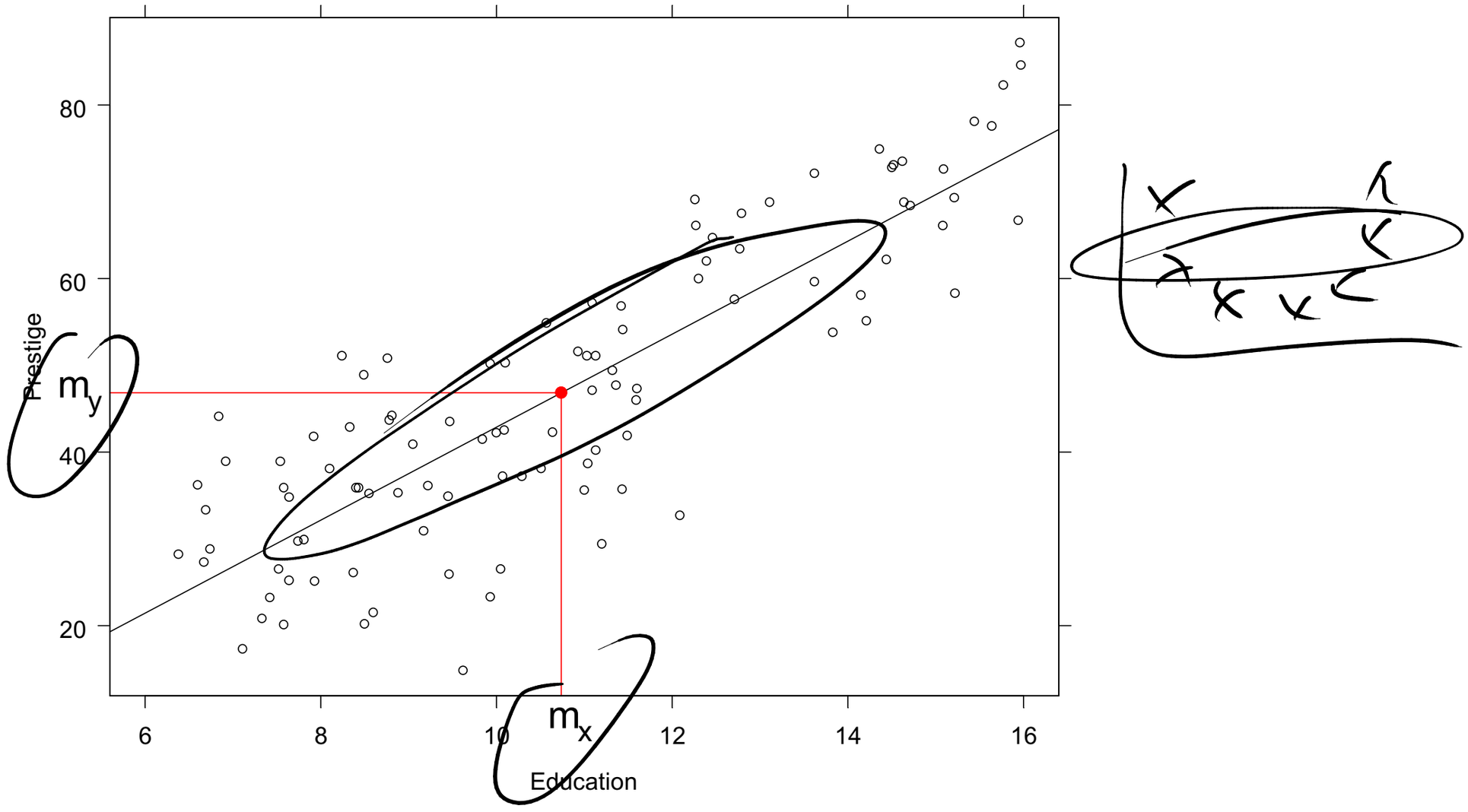
$$\begin{aligned}
\widehat{b} &= \frac{\sum(X_i - \overline{X})(Y_i - \overline{Y})}{\sum(X_i - \overline{X})^2} \\
&= \frac{\left[\sum(X_i - \overline{X})(Y_i - \overline{Y})\right]/[n-1]}{\left[\sum(X_i - \overline{X})^2\right]/[n-1]} \\
&= \frac{\widehat{\mathrm{Cov}}(X,Y)}{\widehat{\mathrm{Var}}(X)} \\
&= \frac{s_{XY}}{s_X^2}
\end{aligned}$$

Intercept:

$$\overline{Y} = a + b\overline{X}$$


Prestige
80
60
40
20
6
8
10
12
14
16
Education


Prestige
80
60
40
20
6
8
10
12
14
16
Education
LS line


80
60
40
20
Prestige
m_y
6
8
10
12
14
16
m_x
Education

Data ellipse:

Data ellipse of radius r :

$$\mathcal{E}_r = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} : \left[\begin{pmatrix} X \\ Y \end{pmatrix} - \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix} \right]' \begin{bmatrix} s_X^2 & s_{XY} \\ s_{XY} & s_Y^2 \end{bmatrix}^{-1} \left[\begin{pmatrix} X \\ Y \end{pmatrix} - \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix} \right] = r^2 \right\}$$

As a transformation of the unit circle:

Let

$$\mathbf{S} = \begin{bmatrix} s_X^2 & s_{XY} \\ s_{XY} & s_Y^2 \end{bmatrix}$$

Let $\mathbf{S}^{1/2}$ be any 'square root' of $\mathbf{S}$ in the sense that:

$$\mathbf{S} = \mathbf{S}^{1/2} \left(\mathbf{S}^{1/2} \right)'$$

A good one is the Choleski square root:

$$\mathbf{S}^{1/2} = \begin{bmatrix} s_X & 0 \\ s_{XY}/s_X & s_{Y \cdot X} \end{bmatrix}$$

where $s_{Y \cdot X}$ denotes the 'partial standard deviation' of Y adjusted for X :

$$s_{Y \cdot X} = \sqrt{s_Y^2 - s_{XY}^2 / s_X^2}$$

Then

$$\mathcal{E}_r = \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix} + r\mathbf{S}^{1/2}\mathcal{U}$$

where $\mathcal{U}$ is the unit circle.

If we let $SSE = \sum (Y_i - \widehat{Y}_i)^2$, then:

$$s_{Y \cdot X} = \sqrt{\frac{SSE}{n-1}}$$

Spectral decomp:

$\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$ = Choleski? Sq. root?

S is a var matrix

$$S = \Gamma \Lambda \Gamma'$$

Γ: orthogonal; Λ: diag with non-neg elements

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

$$\Lambda = \Lambda^{1/2}\Lambda^{1/2}$$

$$S = \underbrace{\Gamma \Lambda^{1/2}}_{A}\underbrace{\Lambda^{1/2}\Gamma'}_{A^T}$$

$$S = AA'$$

The usual standard error of regression is:

$$s_e = \sqrt{\frac{SSE}{n-2}}$$

$$\hat{\sigma}_e = \sqrt{\frac{SSS}{n}}$$

So:

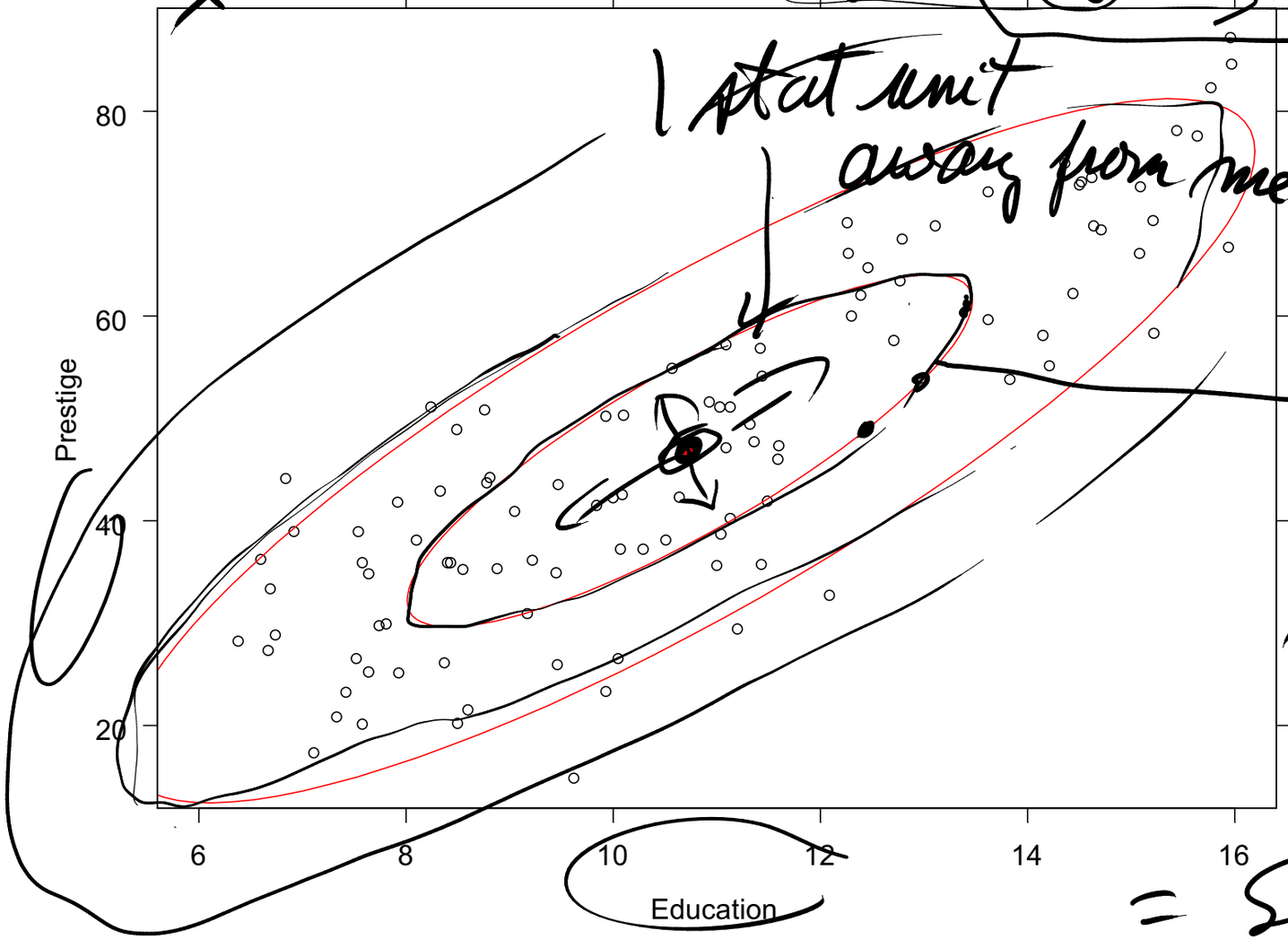
$$s_{Y \cdot X} \approx s_e$$

$$s_x = \sqrt{\frac{\Sigma (x-\bar{x})^2}{n-1}}$$

$$\hat{\sigma}_x = \sqrt{\frac{\Sigma (x-\bar{x})^2}{n}}$$

$$\frac{(x-\bar{x})^2}{s_x^2} = z_x^2$$

$$\mathcal{E}_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right]^T S^{-1} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right) = 1 \right\}$$



1 stat unit away from mean.

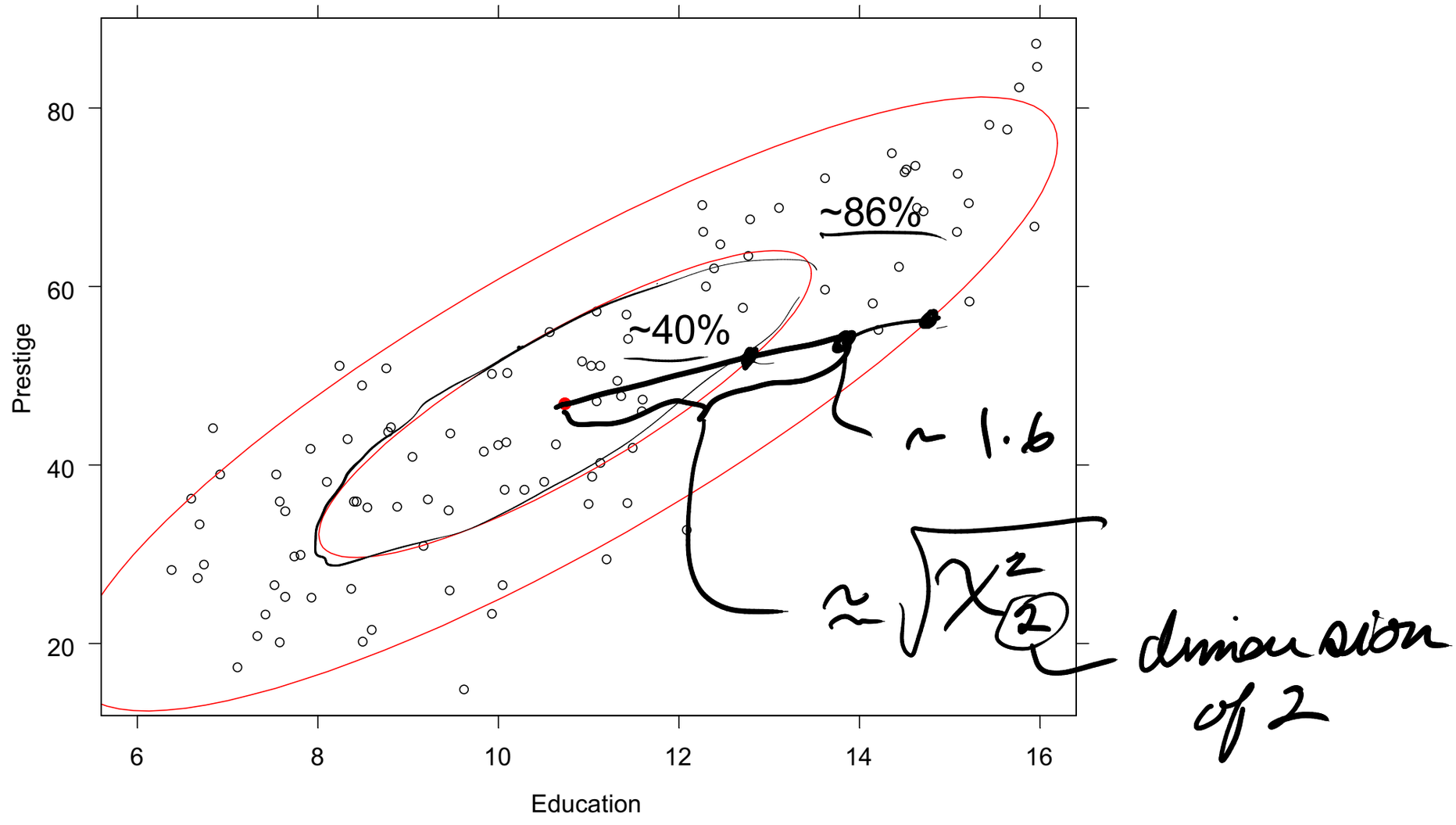
$= 2^2$
3^2

$\mathcal{E}_1$

$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \right.$

"Mahalanobis distance"

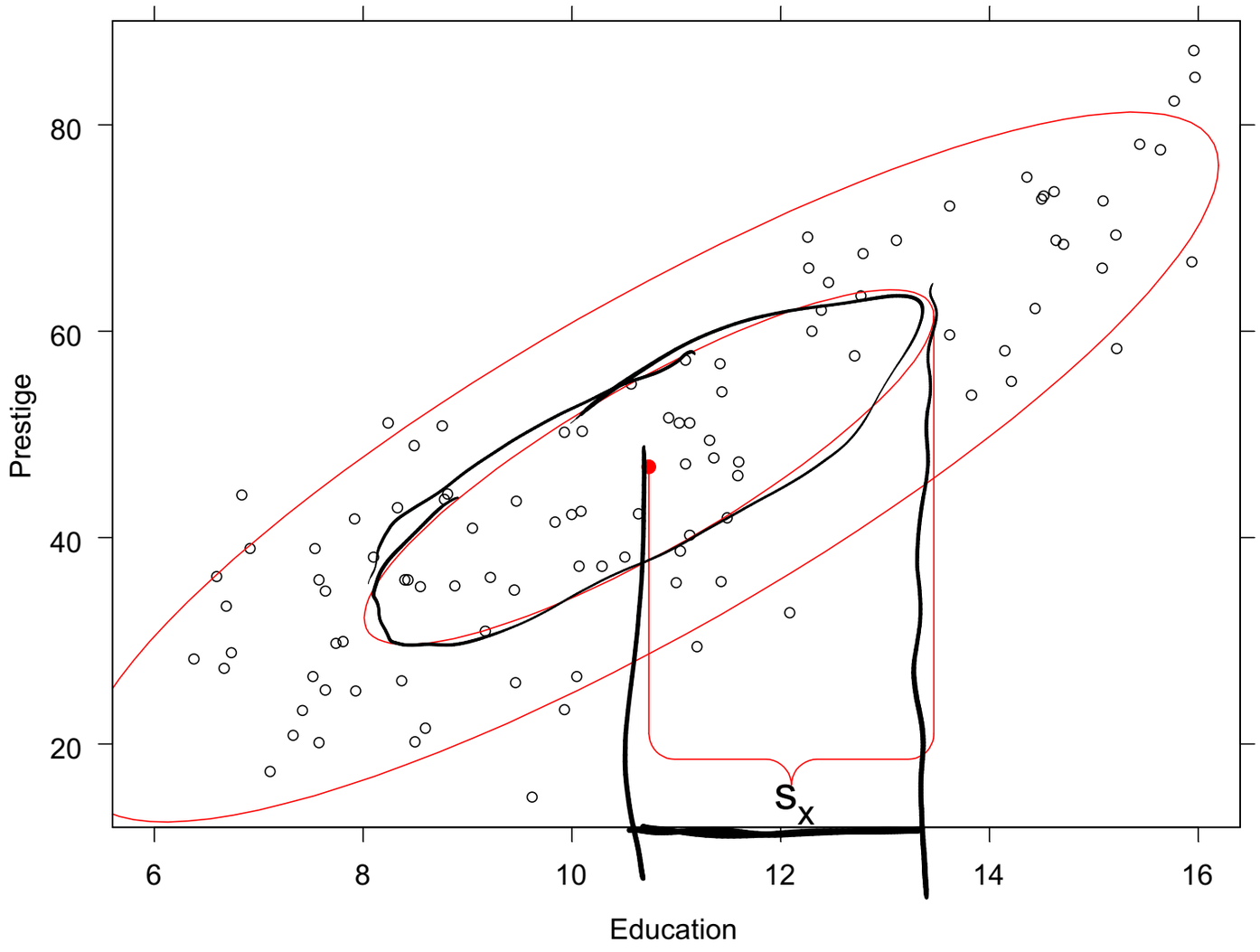
= Statistical distance

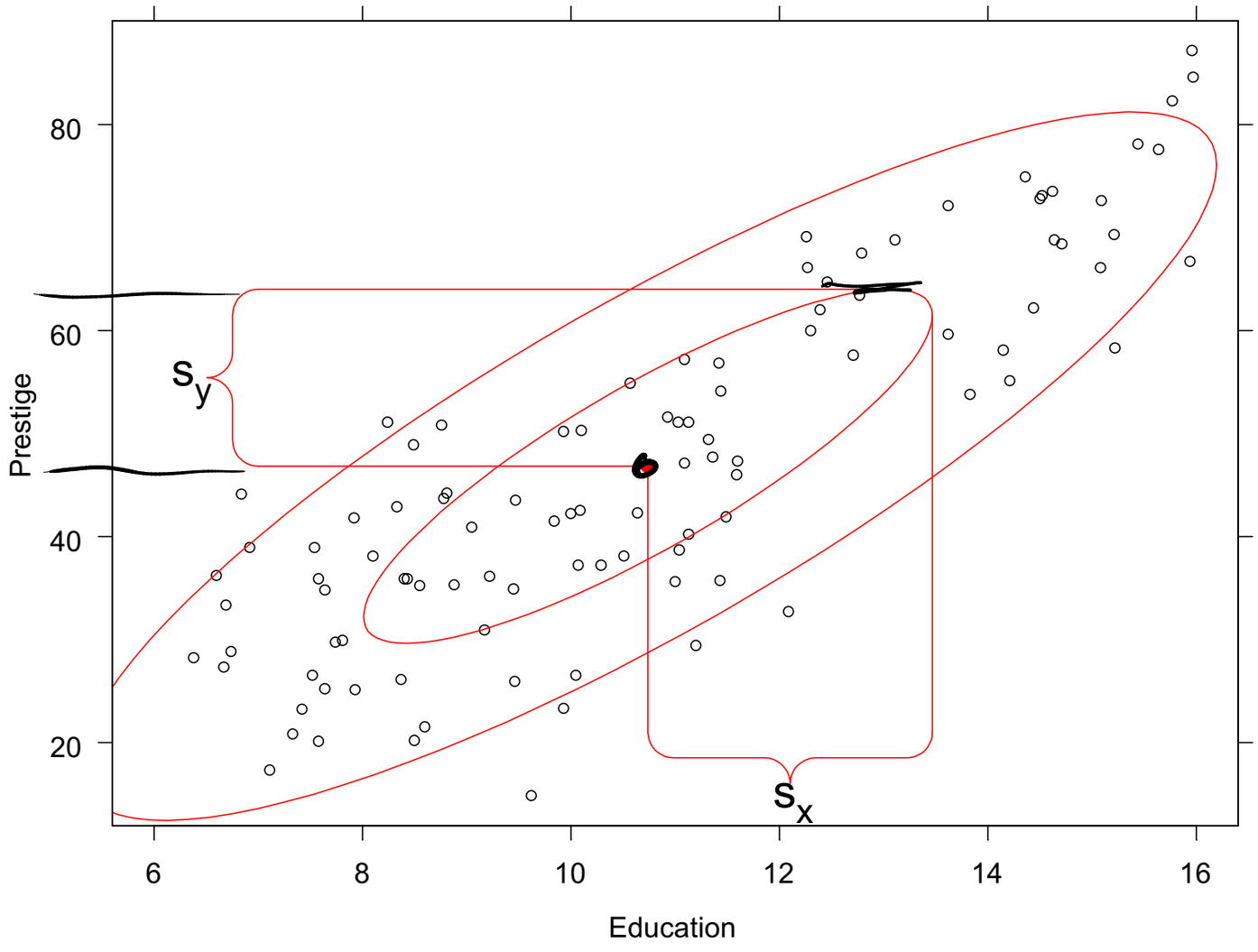

Prestige
80
60
40
20
6
8
10
12
14
16
Education
~86%
~40%
~ 1.6
≈ √χ²(2)
dimension of 2

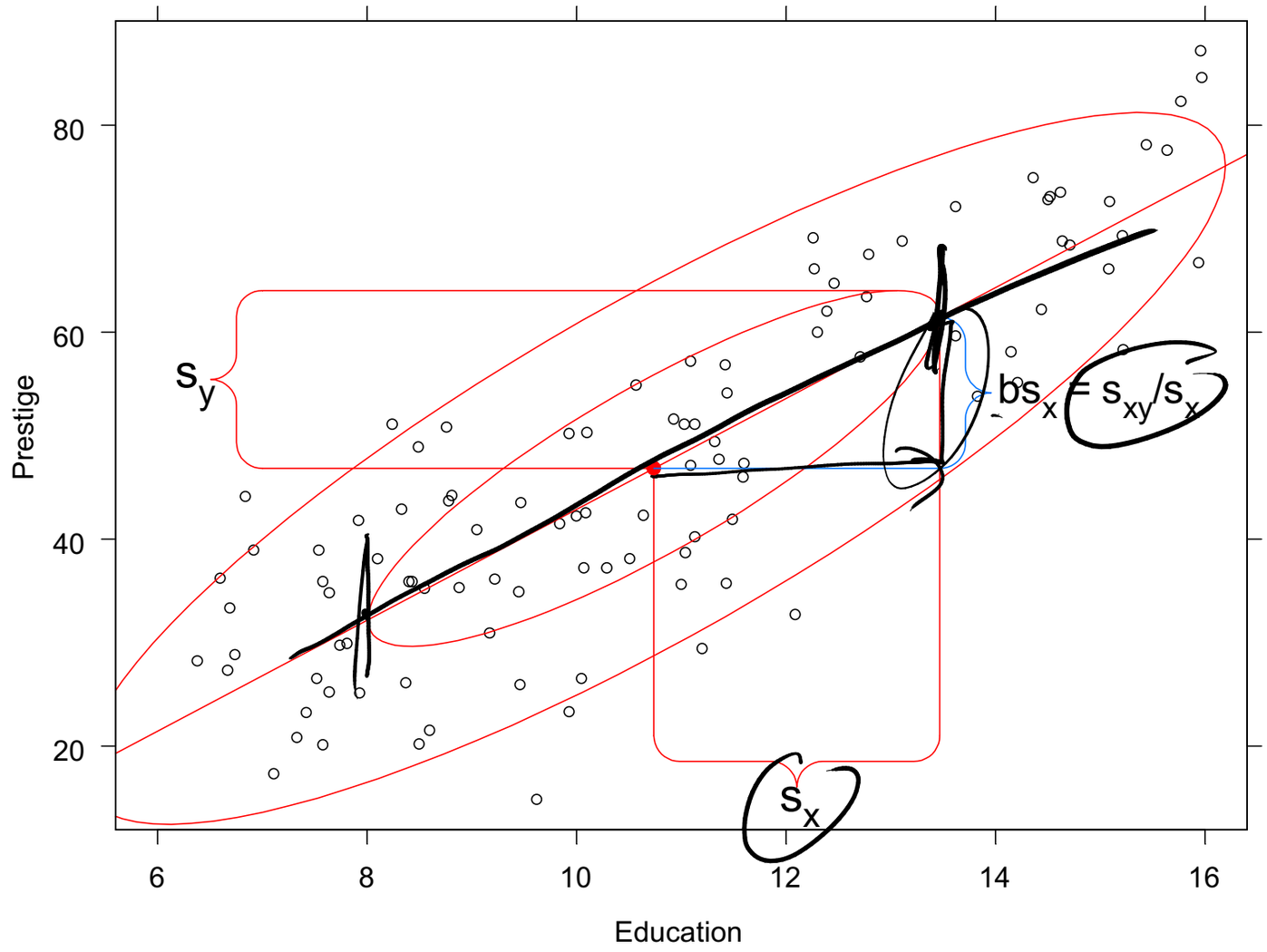
If the data cloud is approx. bivariate normal, then the proportion of points in $\mathcal{E}_r$ is approx:

$$\Pr(\chi_2^2 \leq r^2)$$

where χ_2^2 has a χ^2 distribution with 2 degrees of freedom.

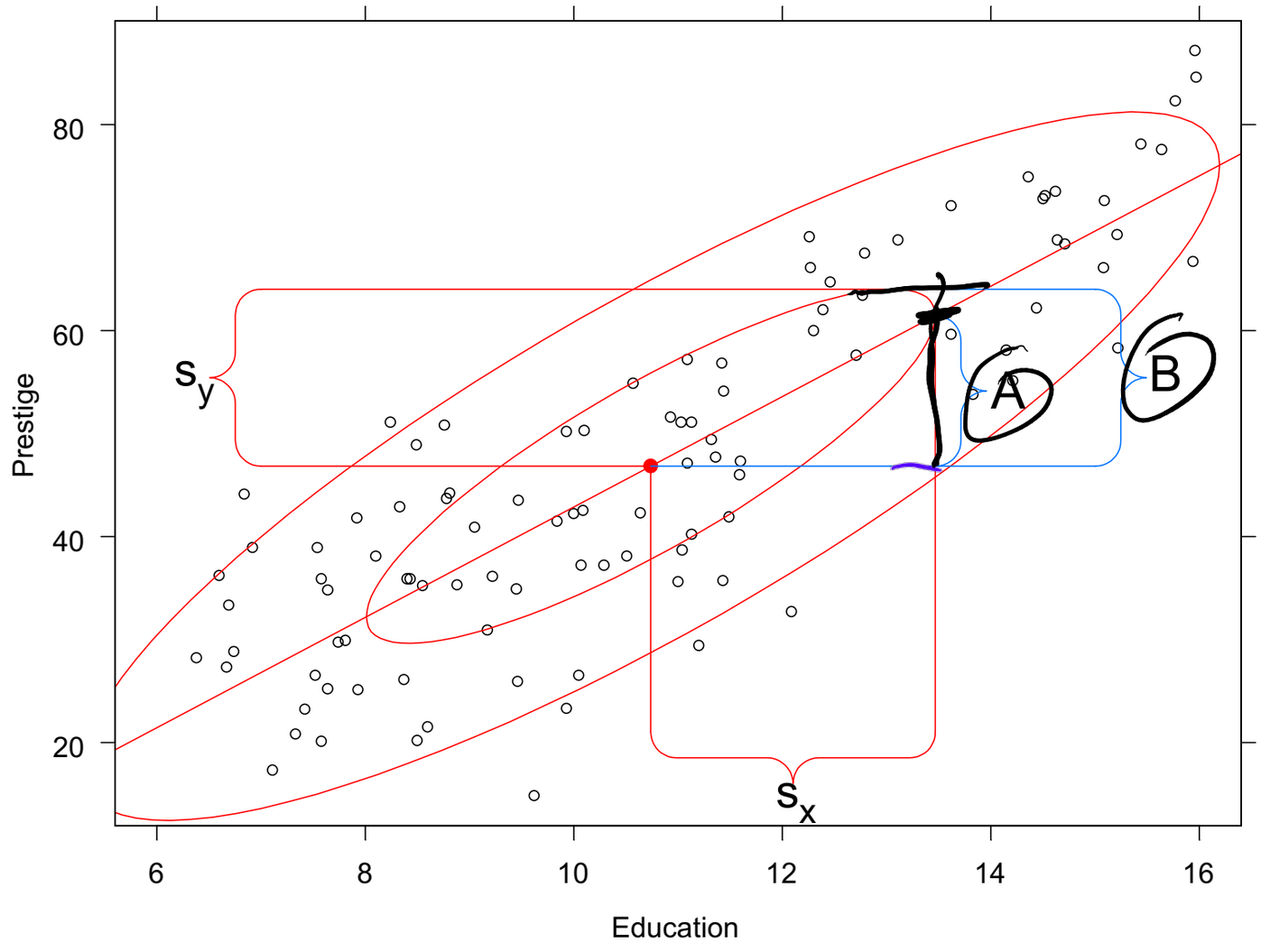

80
60
40
20
Prestige
6
8
10
12
14
16
Education
S_X

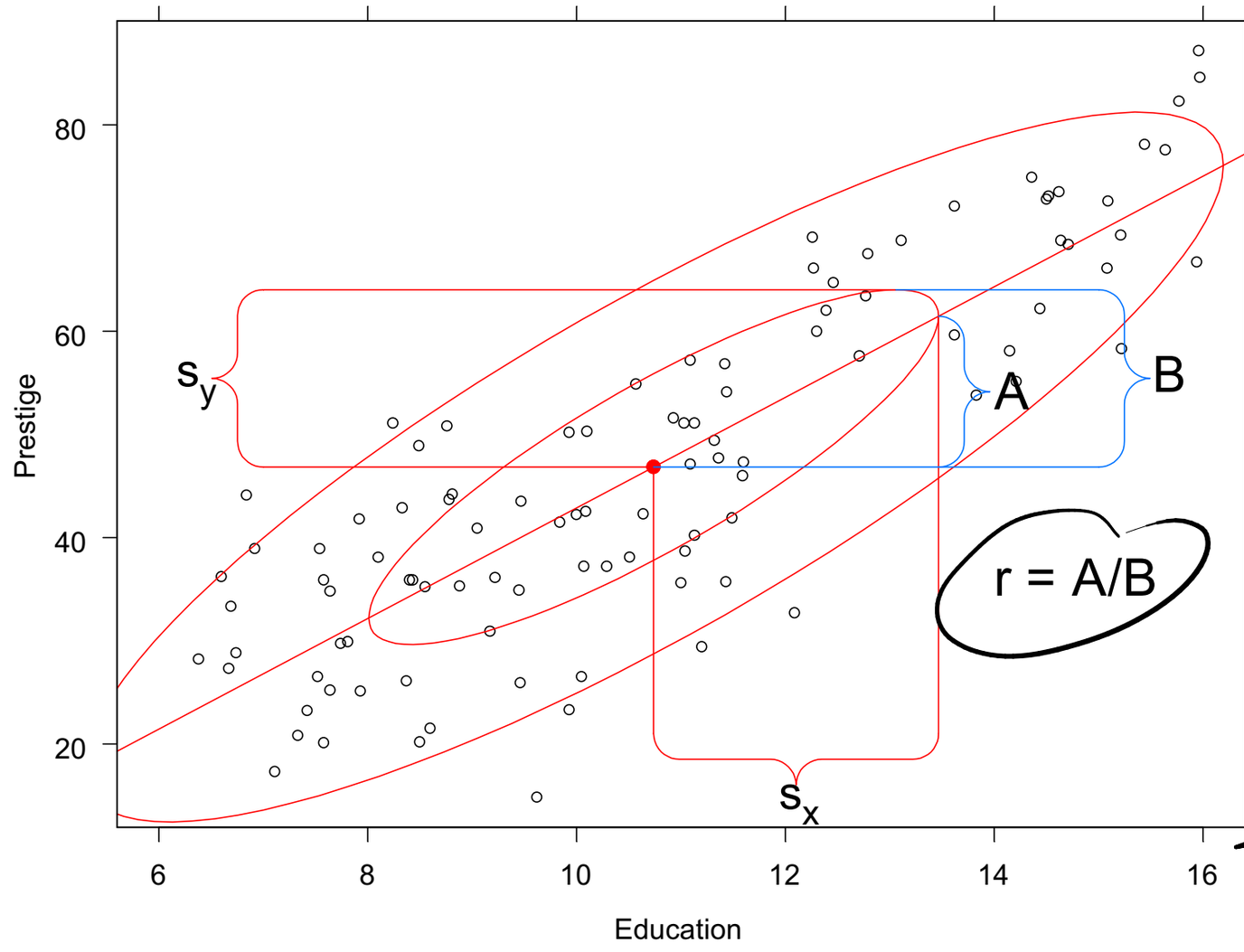

80
60
40
20
Prestige
6
8
10
12
14
16
Education
s_y
s_x

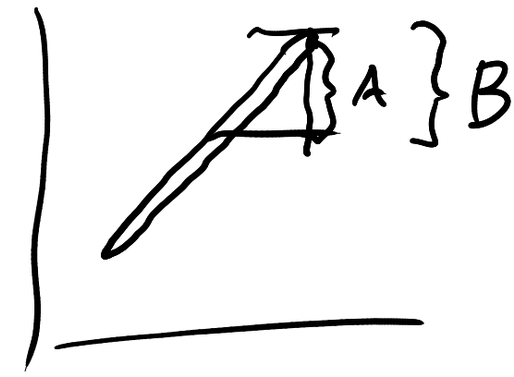


$$b = \frac{s_{xy}}{s_x^2}$$

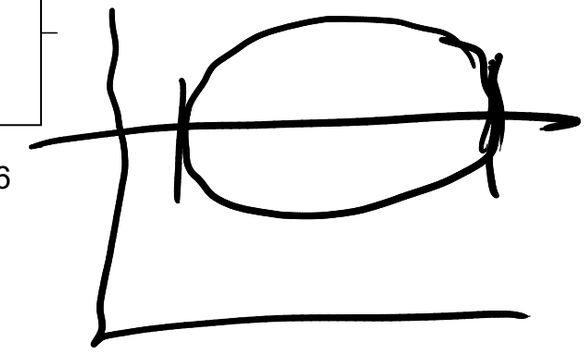
$$b s_x = \frac{s_{xy}}{s_x}$$

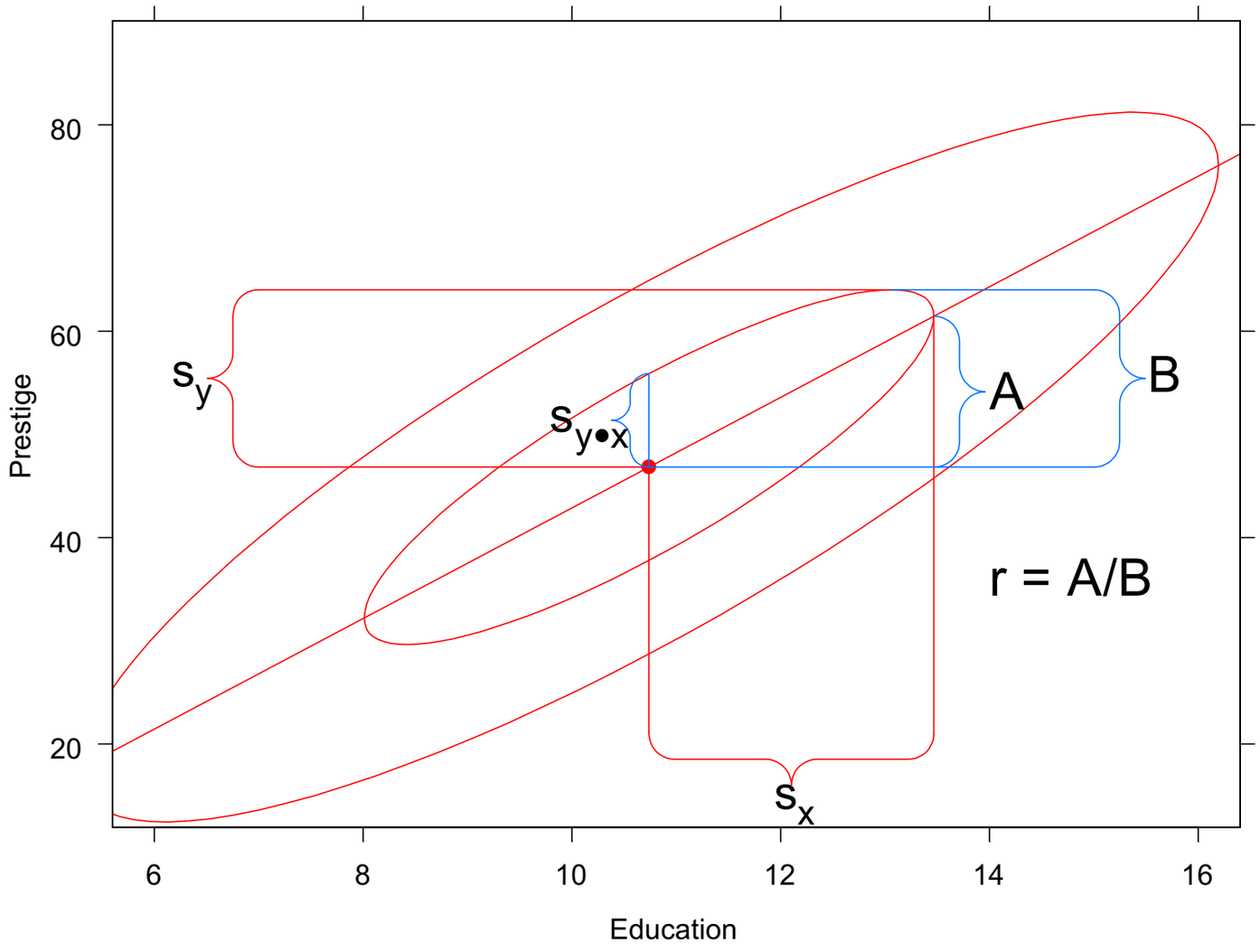

Prestige
80
60
40
20
6
8
10
12
14
16
Education
s_y
s_x
A
B

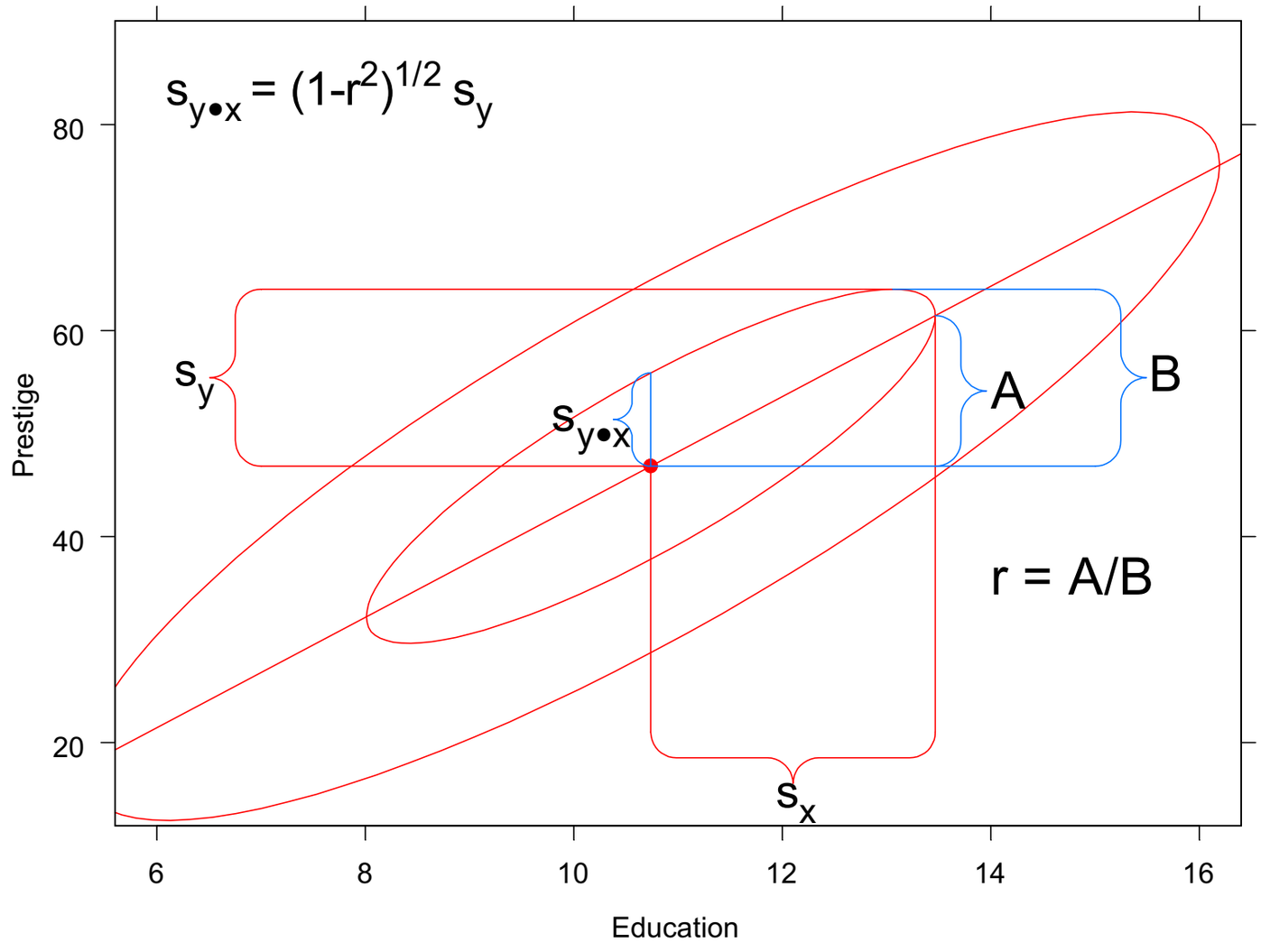


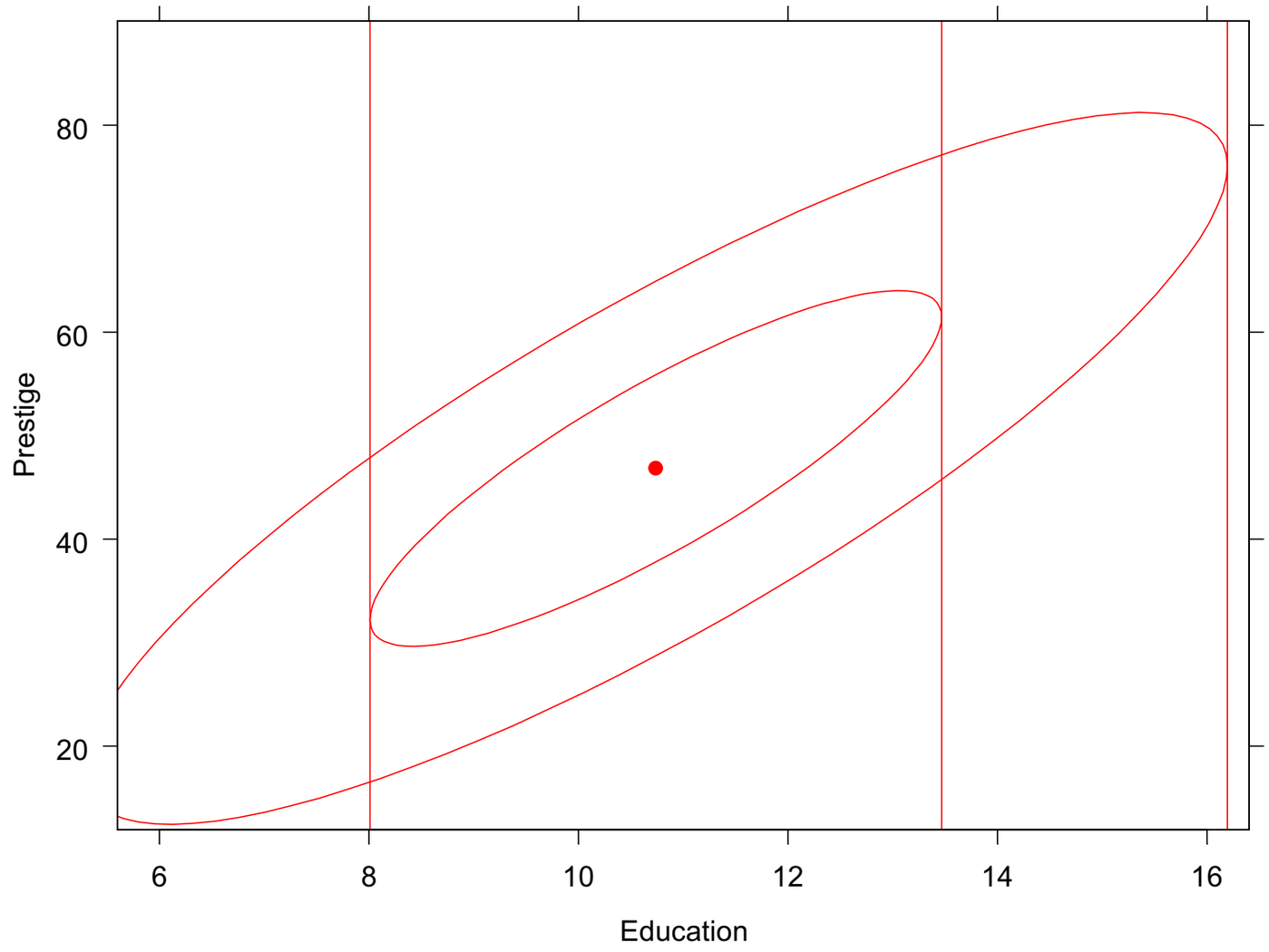


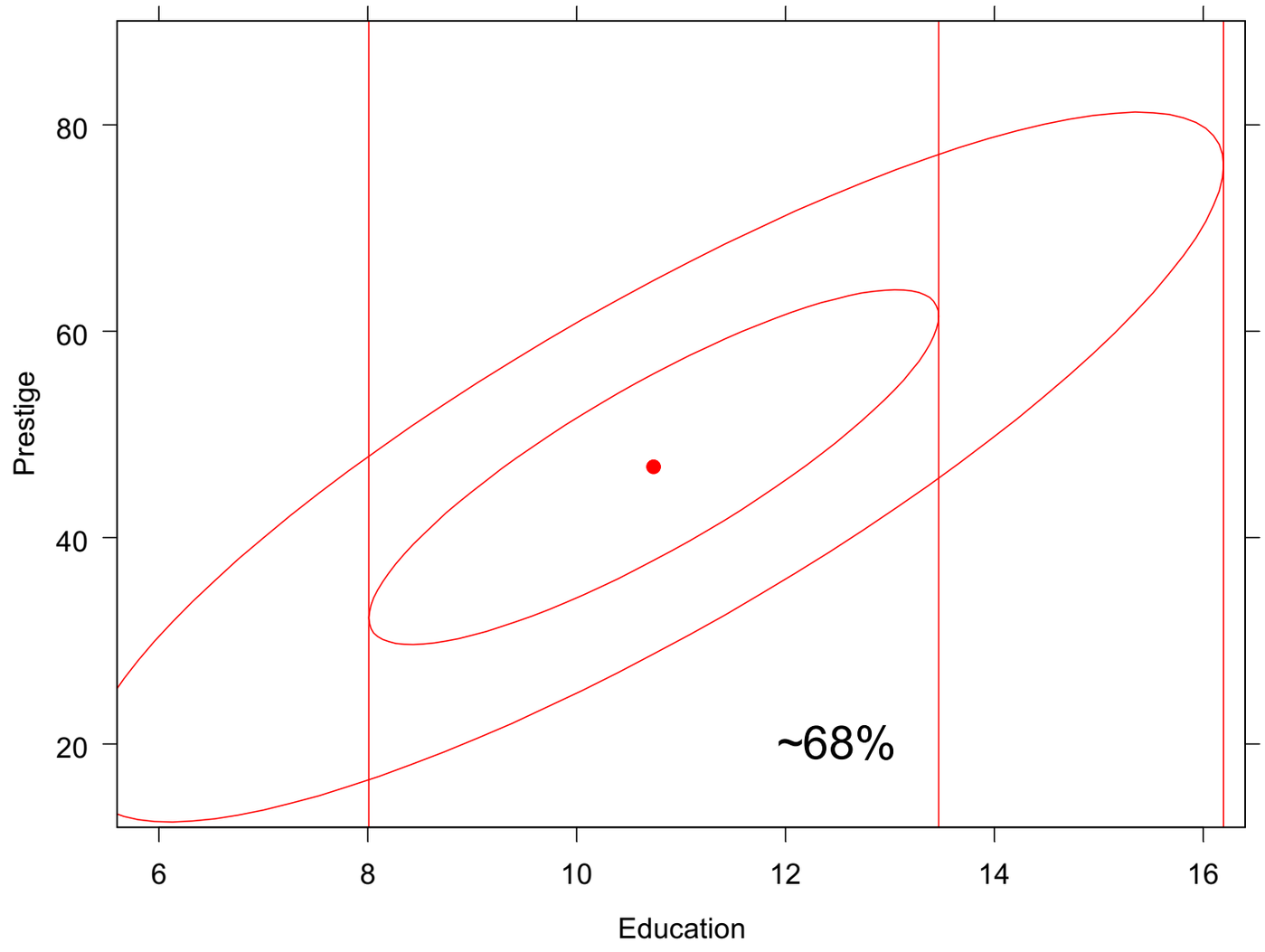
$\frac{A}{B}$ close to 1

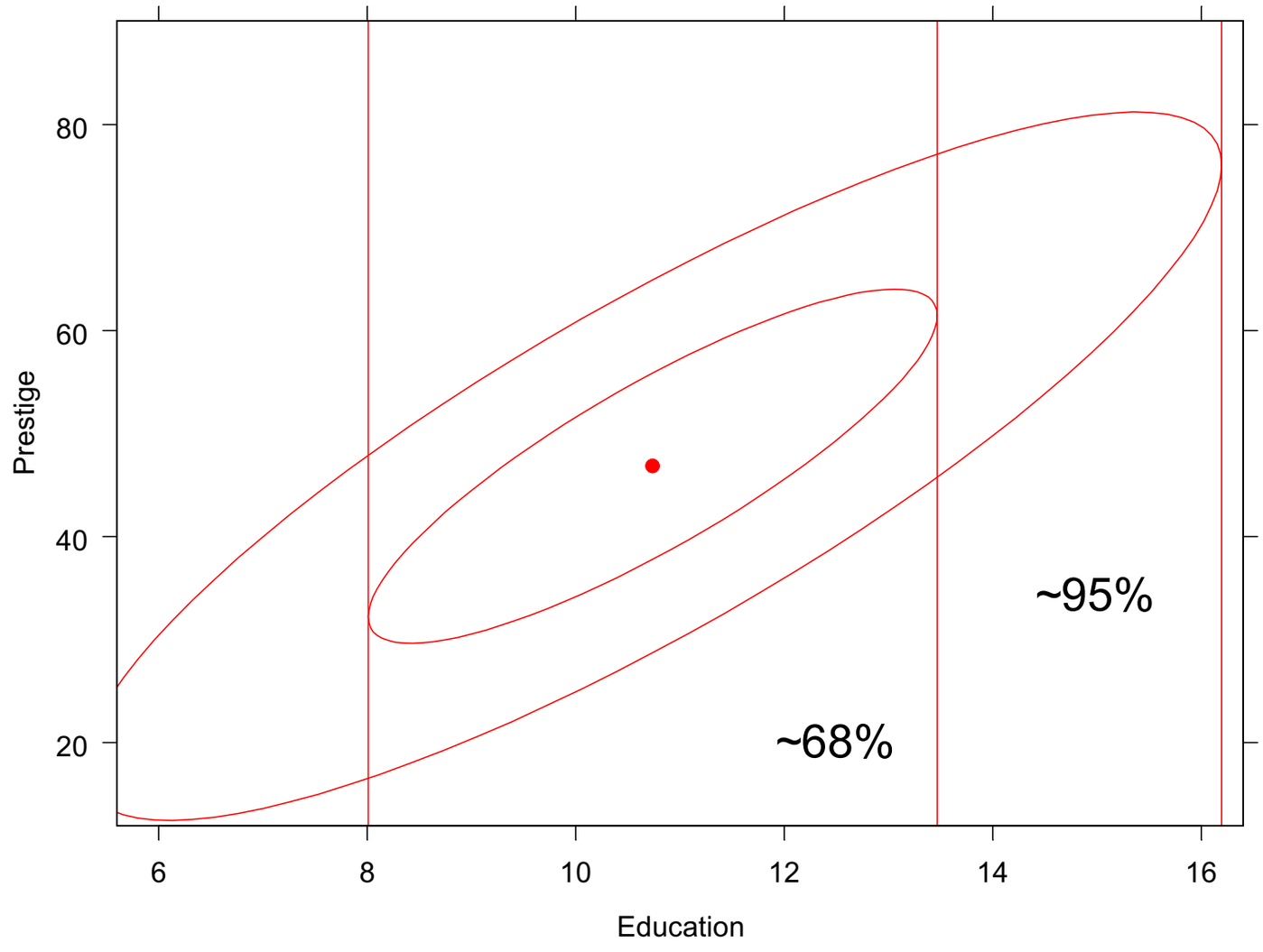


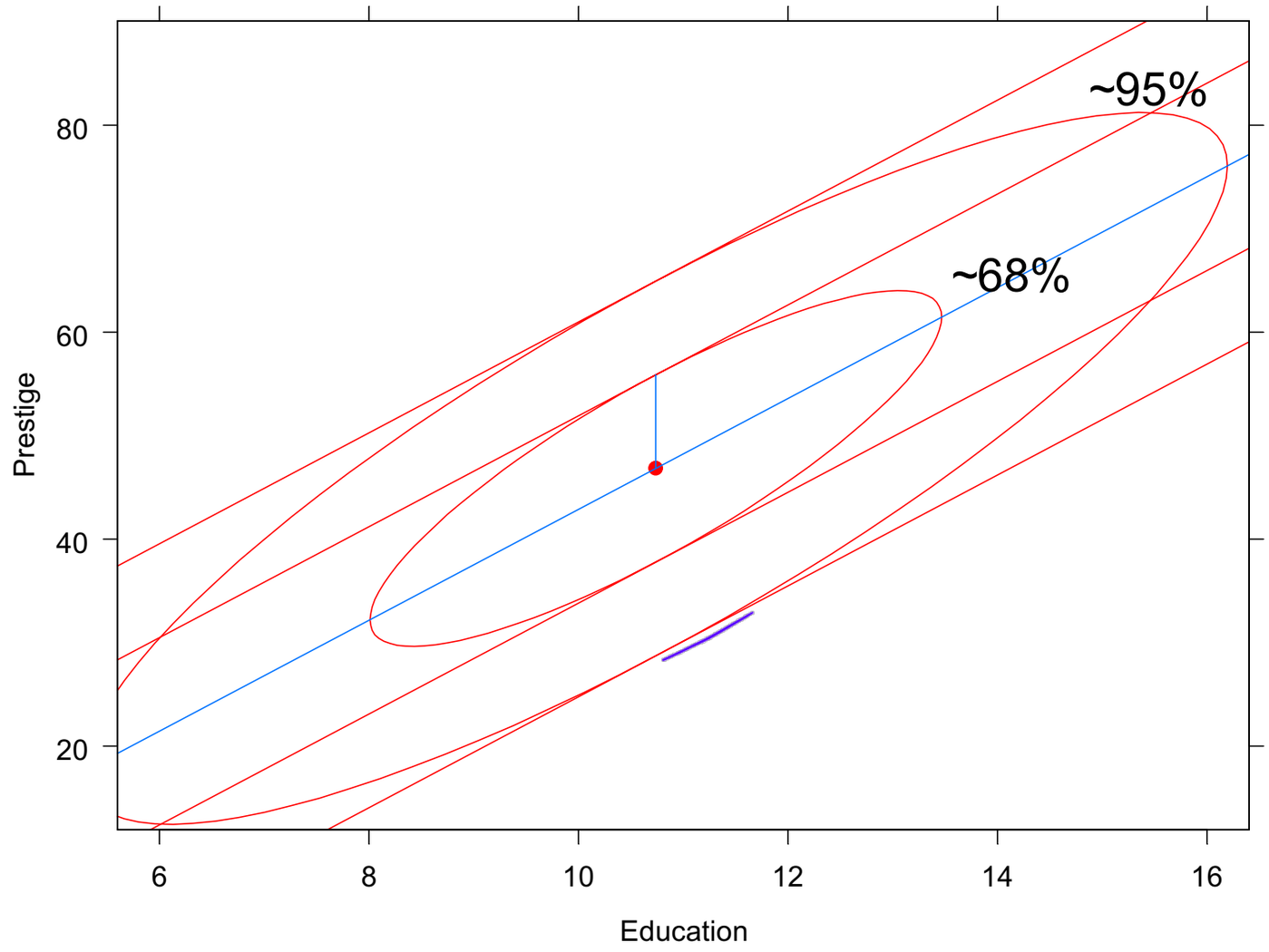

80
60
40
20
Prestige
6
8
10
12
14
16
Education
s_y
$s_{y \bullet x}$
s_x
A
B
r = A/B


$s_{y \bullet x} = (1-r^2)^{1/2} s_y$
s_y
$s_{y \bullet x}$
A
B
r = A/B
s_x
Prestige
Education
80
60
40
20
6
8
10
12
14
16


80
60
40
20
Prestige
6
8
10
12
14
16
Education


80
60
40
20
Prestige
6
8
10
12
14
16
Education
~68%


80
60
40
20
Prestige
6
8
10
12
14
16
Education
~95%
~68%


~95%
~68%
80
60
40
20
Prestige
6
8
10
12
14
16
Education

If the data cloud is approx. bivariate normal, then the proportion of points in a *band* containing $\mathcal{E}_r$ is approx:

$$\Pr(\chi_1^2 \leq r^2)$$

where χ_1^2 has a χ^2 distribution with 1 degrees of freedom.

3 Visual 95% Confidence Interval

95% Confidence interval for slope:

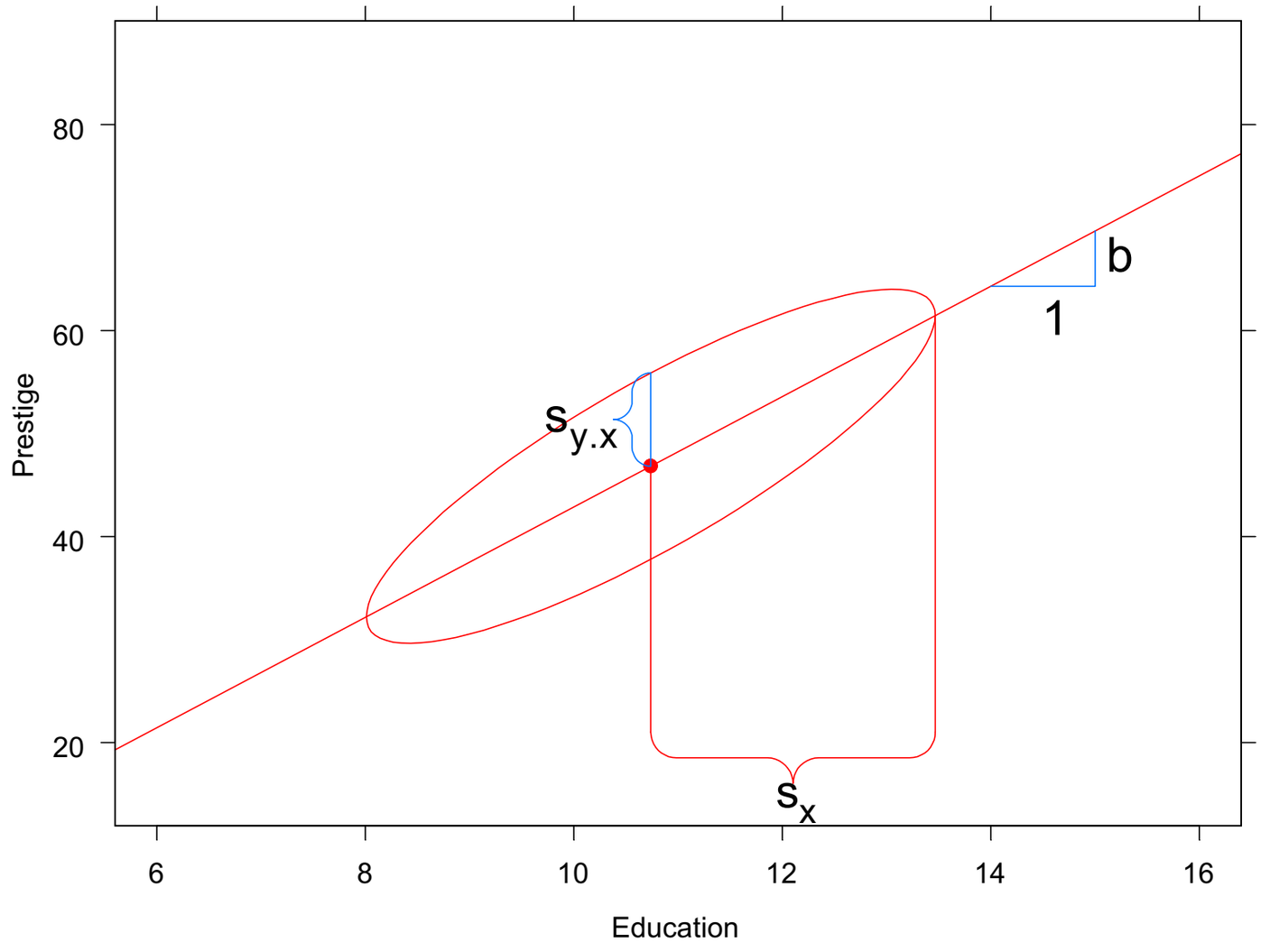
$$b \pm t_{0.975} \times SE(b)$$

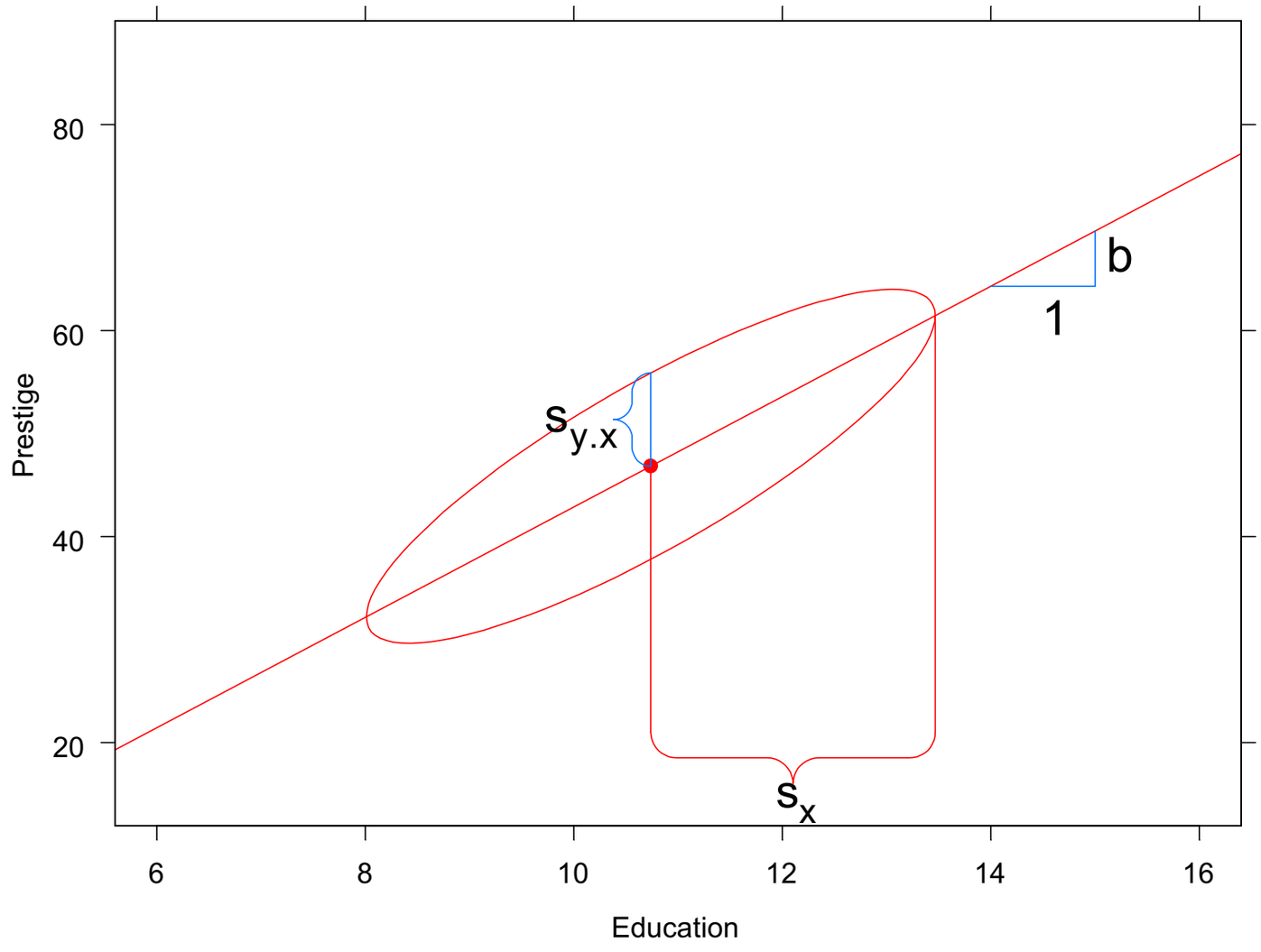
Now,

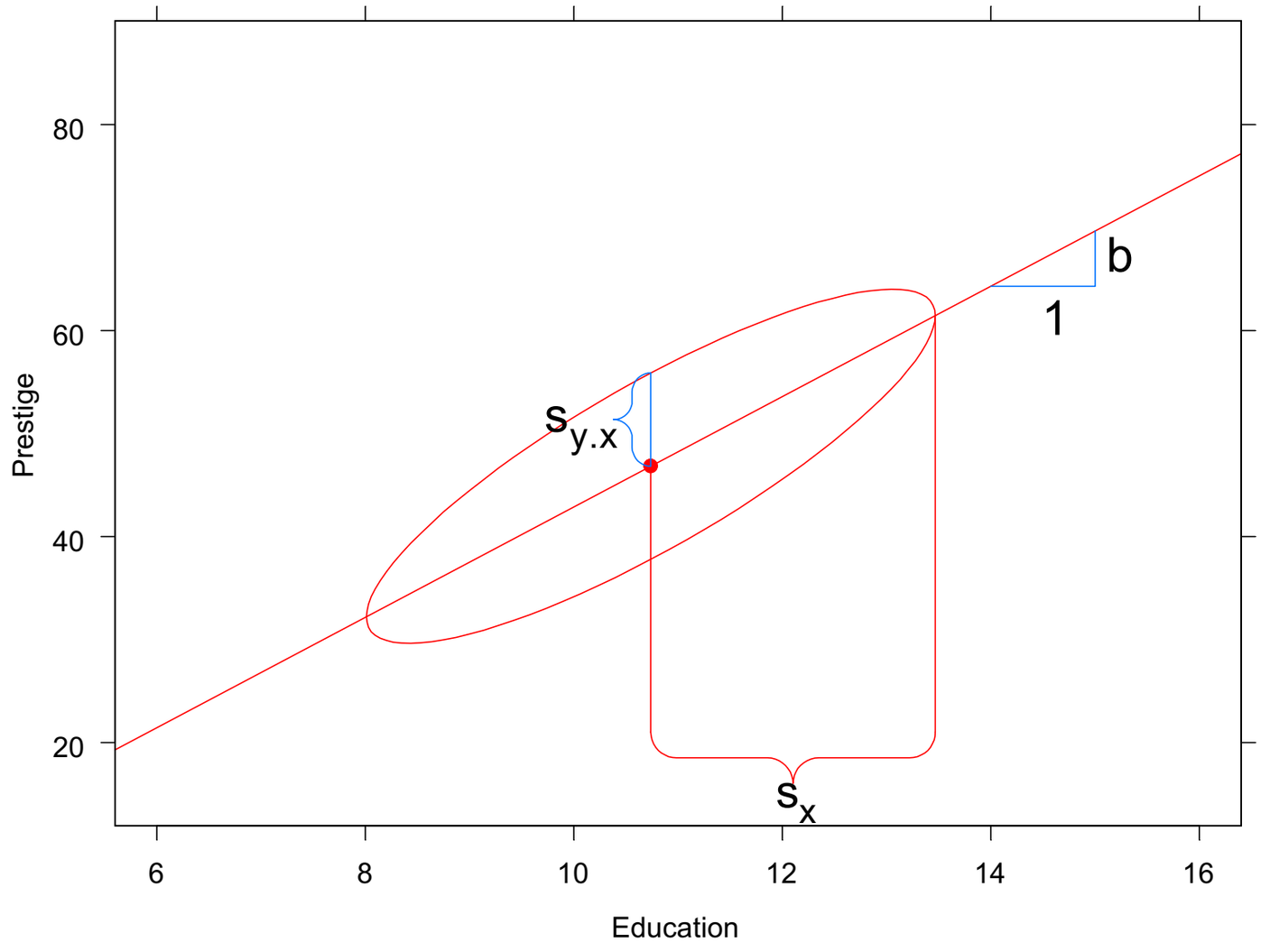
$$\begin{aligned} SE(b) &= \frac{1}{\sqrt{n}} \frac{s_{y.x}}{s_x} \times \frac{\sqrt{n}}{\sqrt{n-2}} \\ &\simeq \frac{1}{\sqrt{n}} \frac{s_{y.x}}{s_x} \end{aligned}$$

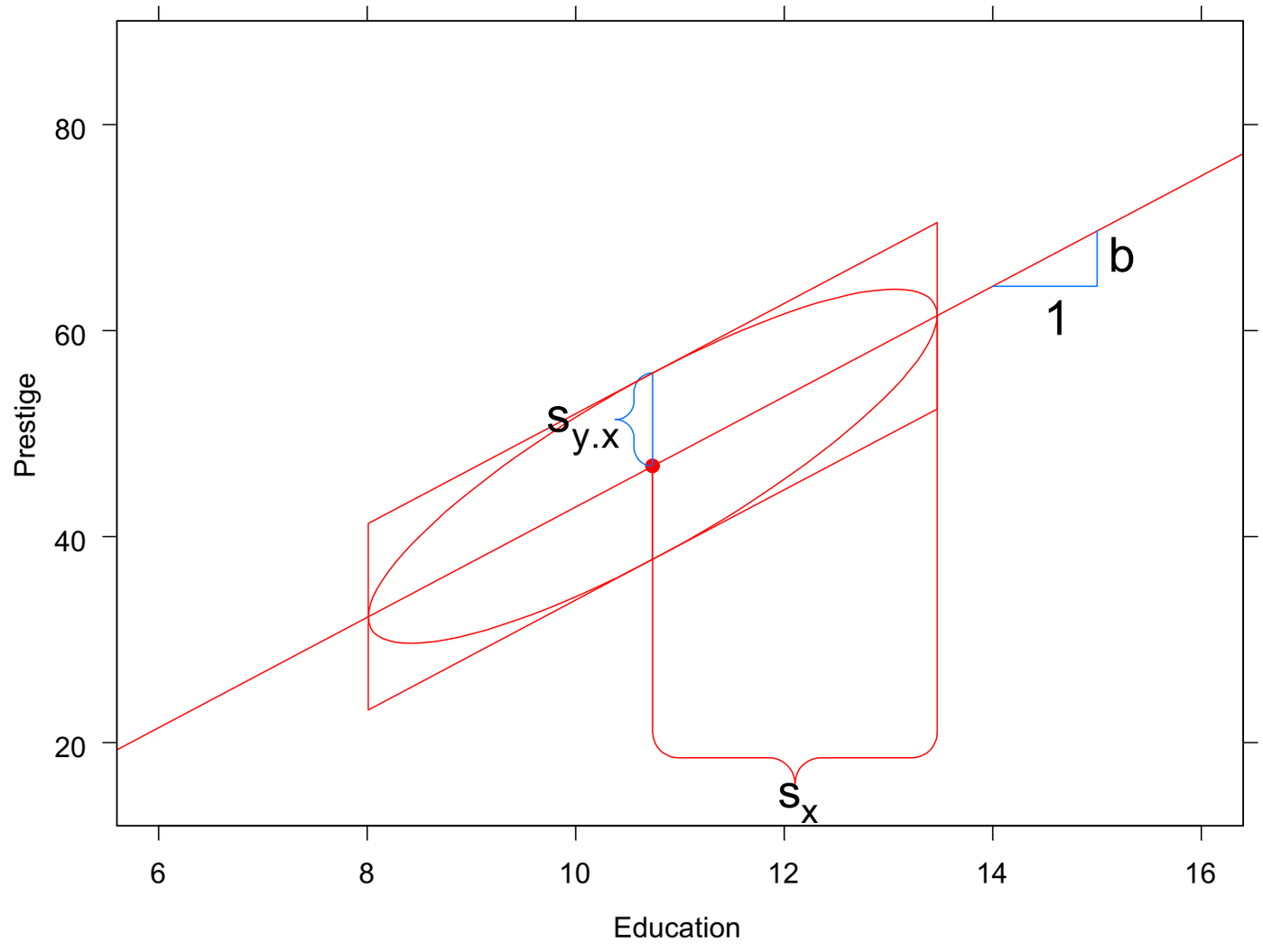
and taking $t_{0.975} \simeq 2$, we have an approximate 95% CI:

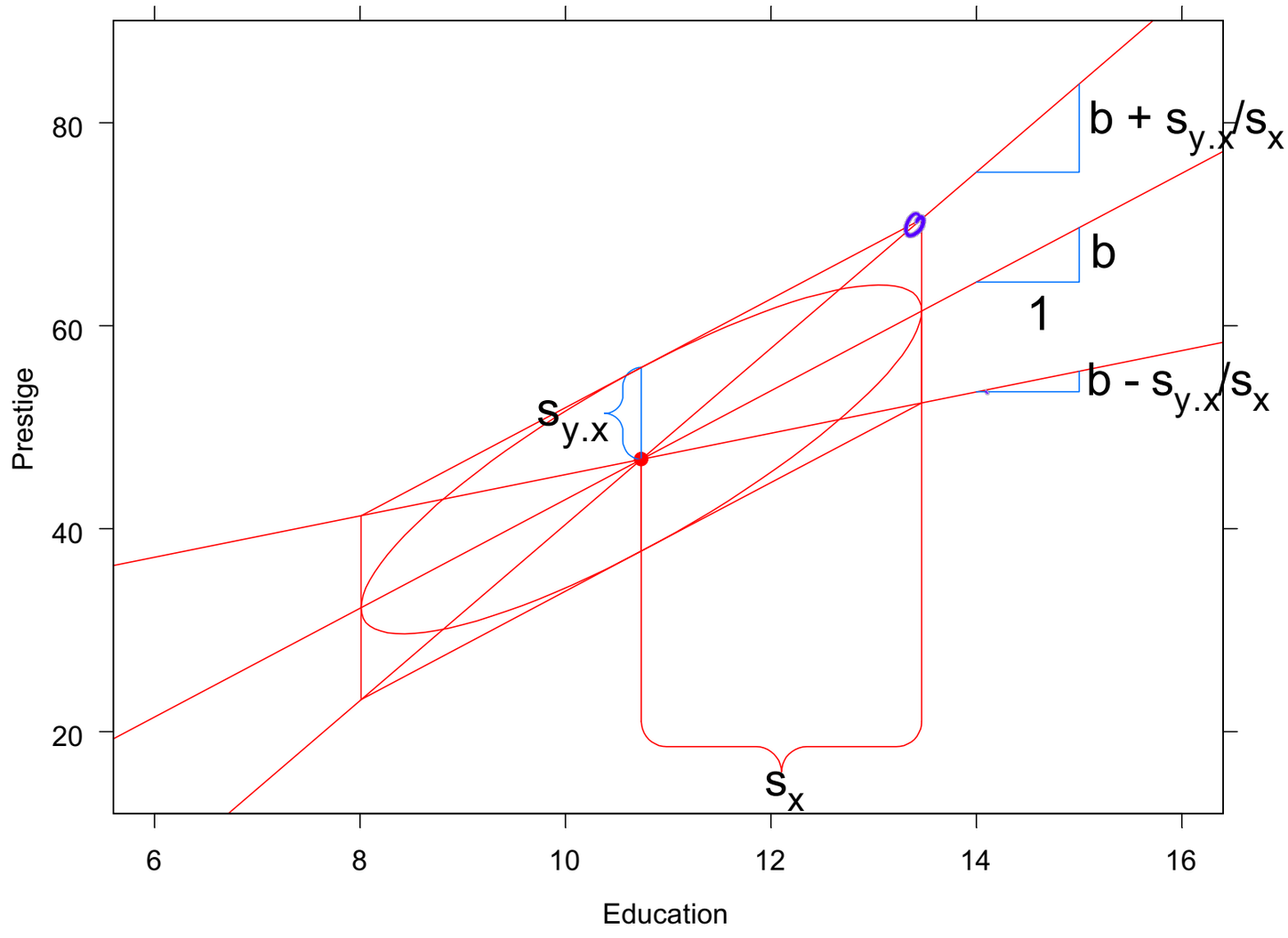
$$b \pm \frac{2}{\sqrt{n}} \times \frac{s_{y.x}}{s_x}$$

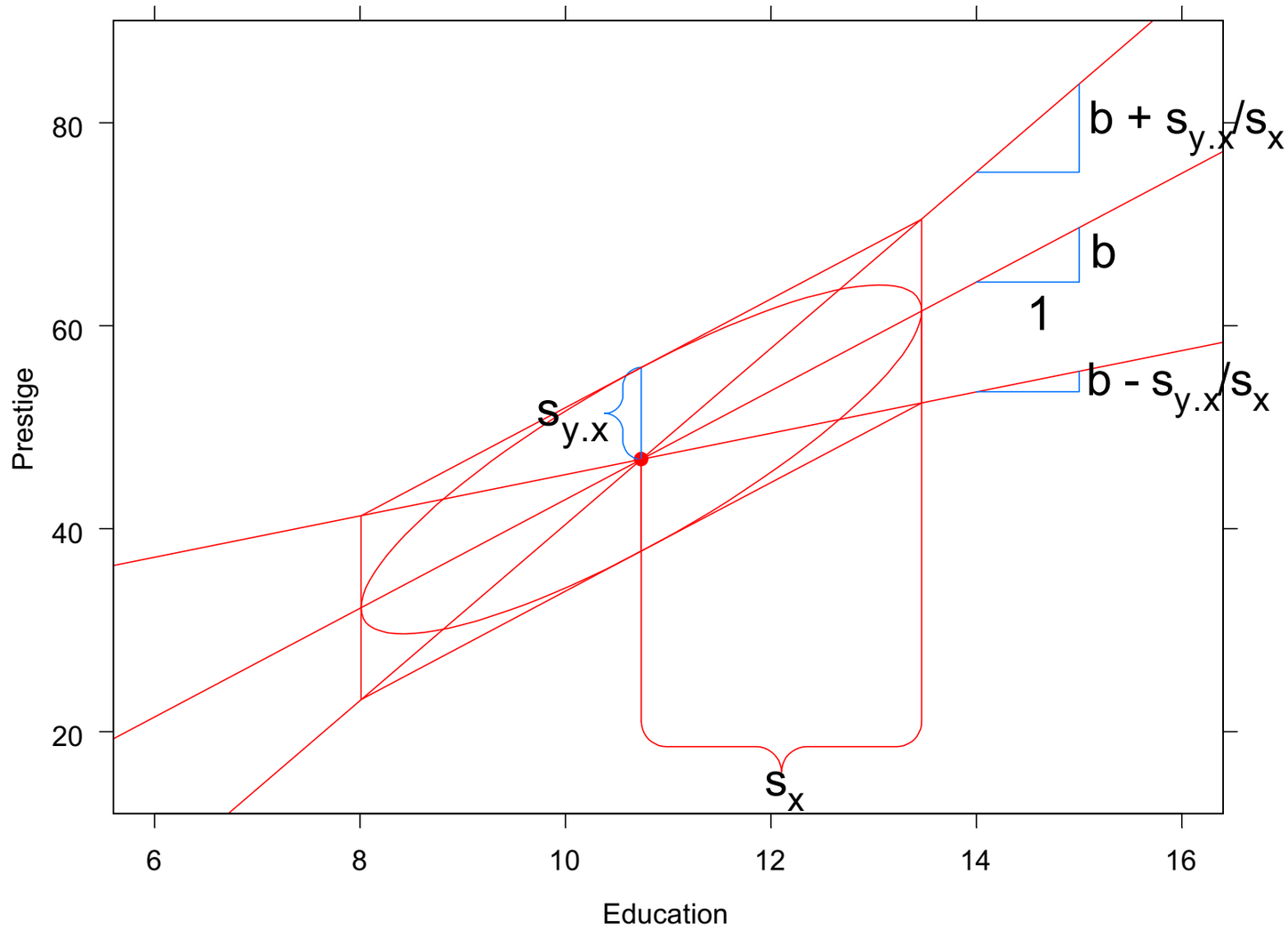

80
60
40
20
Prestige
6
8
10
12
14
16
Education
b
1
$s_{y.x}$
s_x

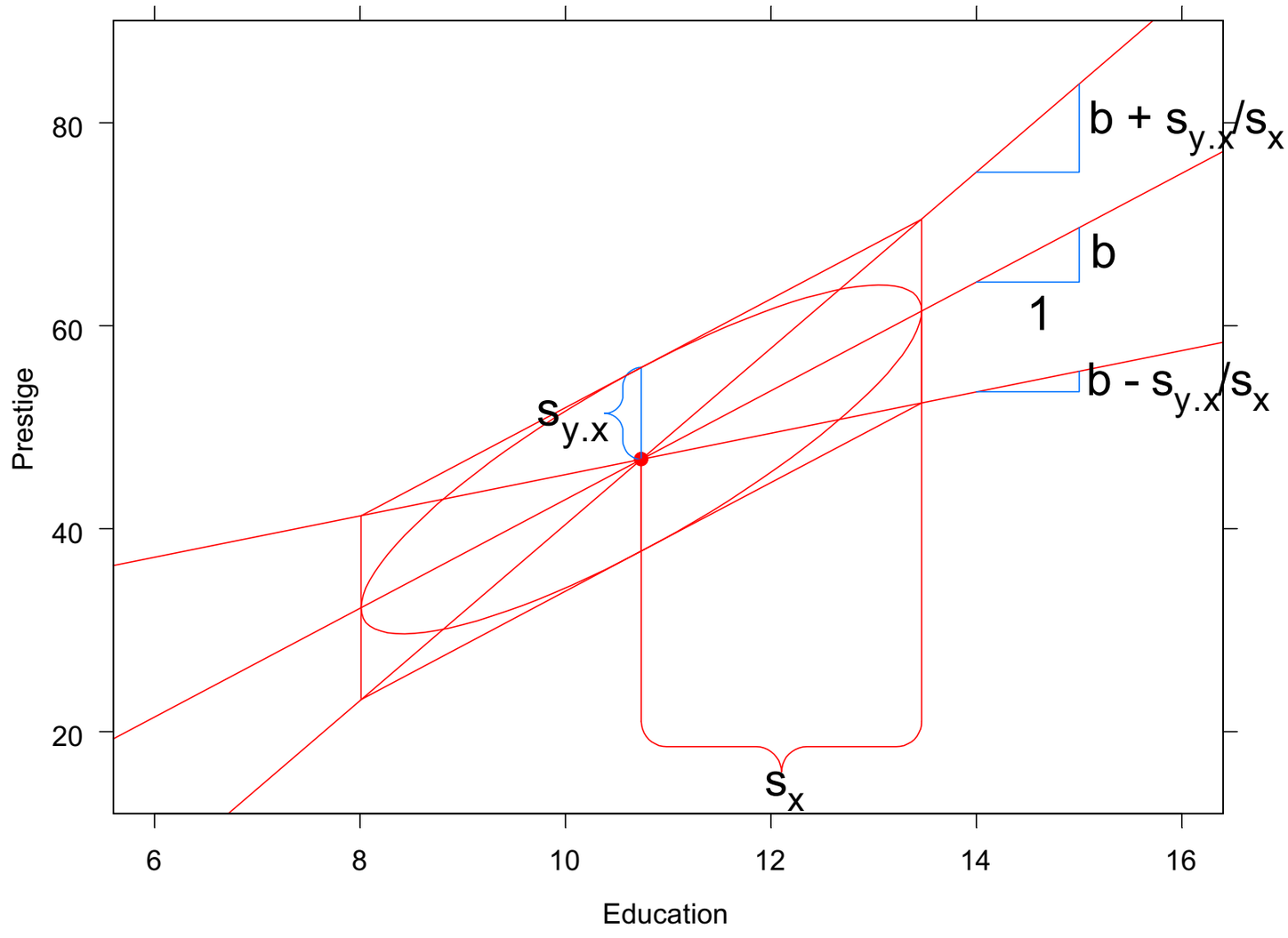

80
60
40
20
Prestige
6
8
10
12
14
16
Education
b
1
$s_{y.x}$
s_x

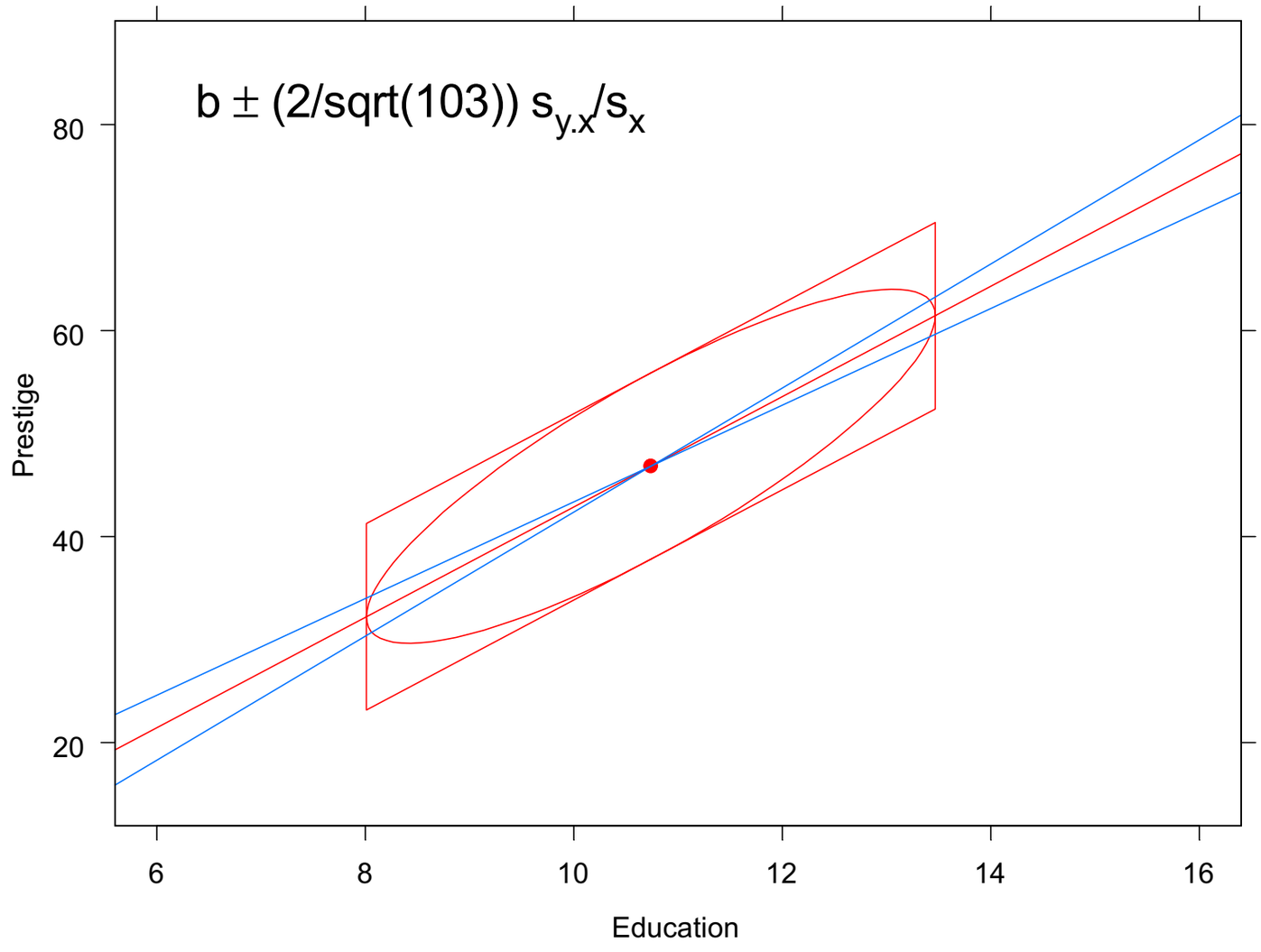

80
60
40
20
Prestige
6
8
10
12
14
16
Education
b
1
$s_{y.x}$
s_x

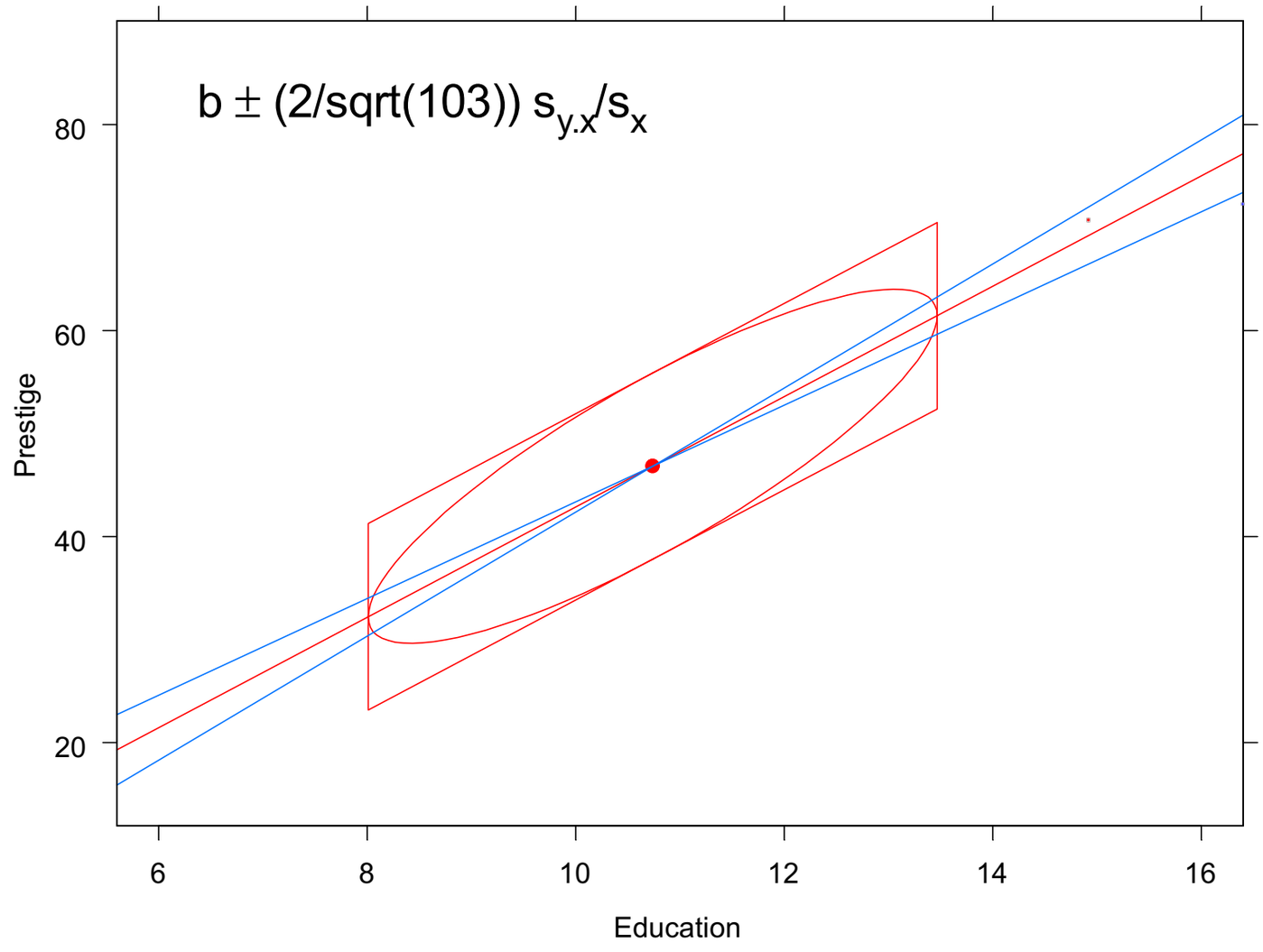

80
60
40
20
Prestige
6
8
10
12
14
16
Education
$s_{y.x}$
s_x
b
1


Prestige
Education
80
60
40
20
6
8
10
12
14
16
b + s_y.x/s_x
b
1
b - s_y.x/s_x
s_y.x
s_x


Prestige
Education
80
60
40
20
6
8
10
12
14
16
b + s_y.x/s_x
b
1
b - s_y.x/s_x
s_y.x
s_x

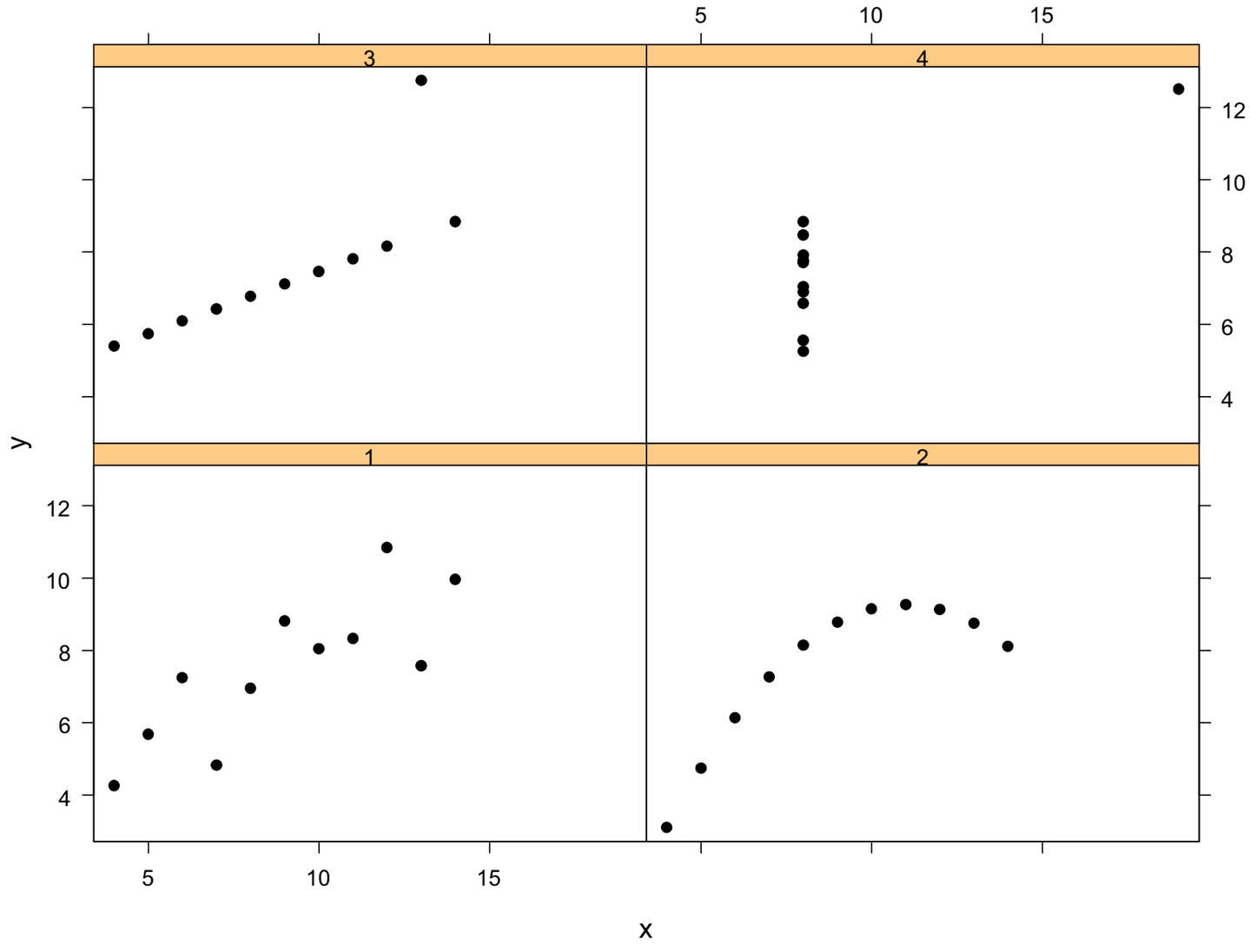

Prestige
Education
80
60
40
20
6
8
10
12
14
16
b + s_y.x/s_x
b
1
b - s_y.x/s_x
s_y.x
s_x


b ± (2/sqrt(103)) s_y.x/s_x
Prestige
Education
80
60
40
20
6
8
10
12
14
16


b ± (2/sqrt(103)) s_y.x/s_x
Prestige
Education
80
60
40
20
6
8
10
12
14
16

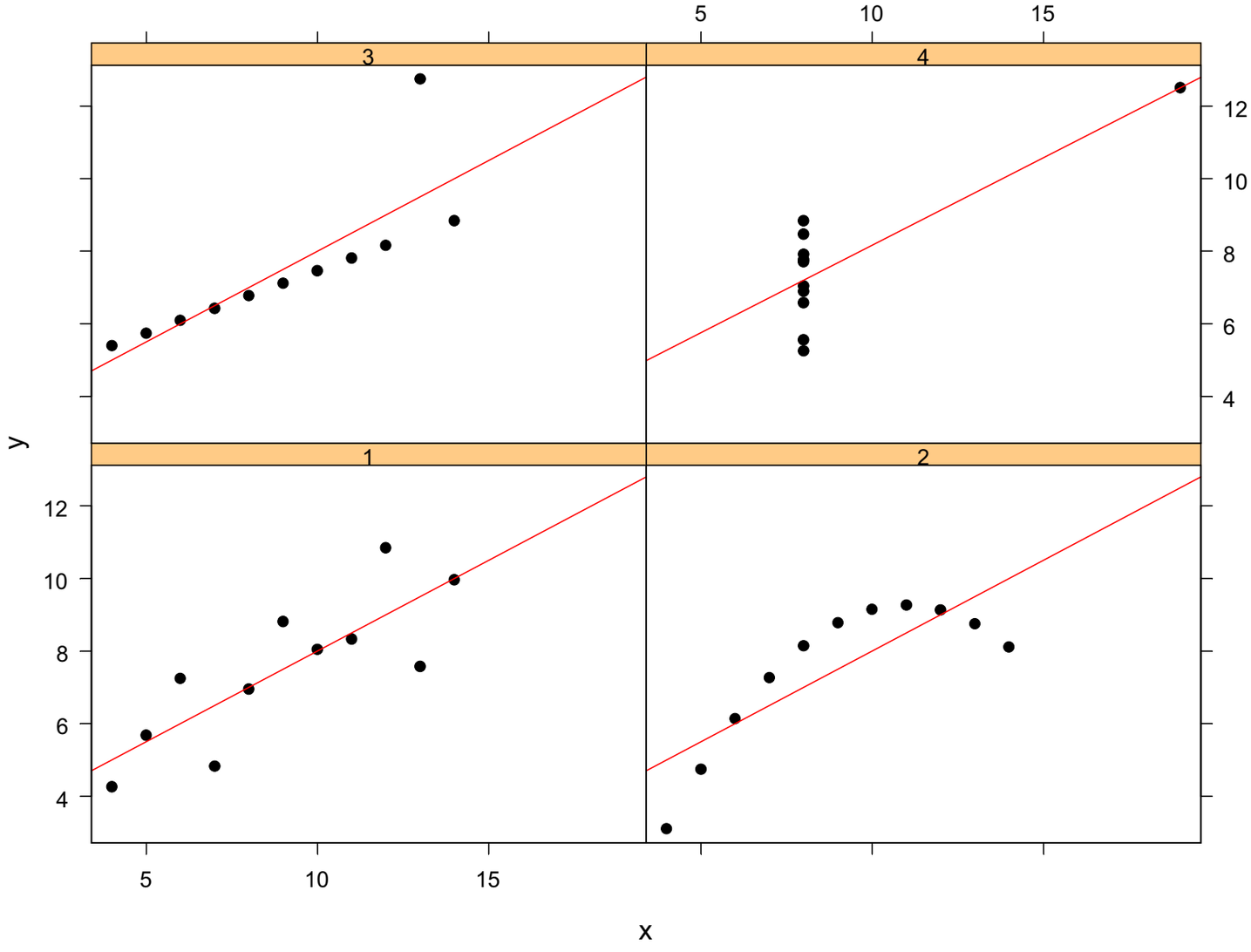
4 Anscombe Examples

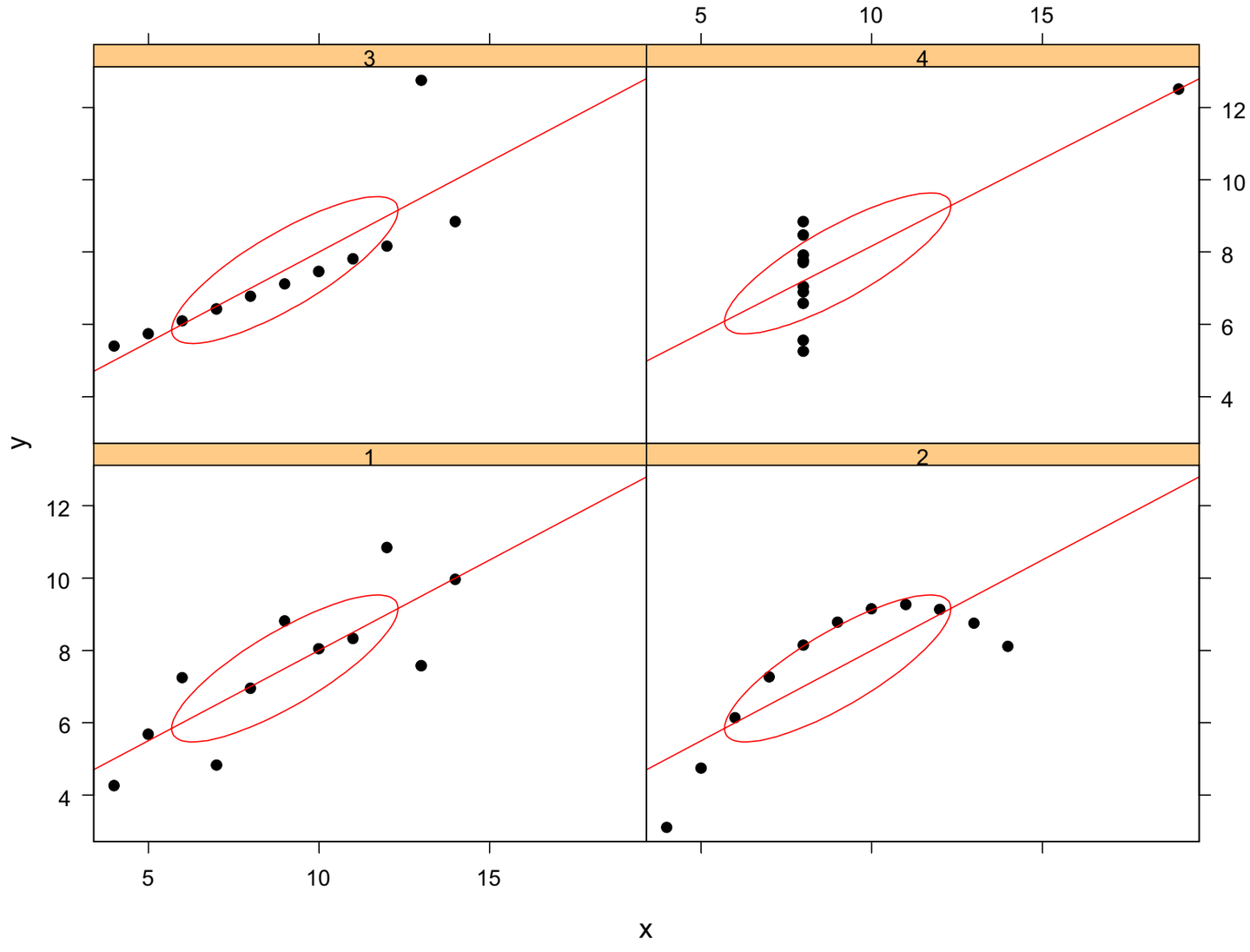
Four datasets:
Same least-squares regression but very different stories


3
4
1
2
y
x
5
10
15
4
6
8
10
12

Same means, variances and covariances... so same least-squares regression results:

```
Call: lm(formula = y ~ x, data = Anscombe, subset = type == 1)
Residuals:
    Min      1Q   Median     3Q   Max
 -1.921 -0.4558 -0.04136 0.7094 1.839

Coefficients:
            Value Std. Error t value Pr(>|t|)
(Intercept) 3.0001 1.1247     2.6673  0.0257
          x 0.5001 0.1179     4.2415  0.0022

Residual standard error: 1.237 on 9 degrees of freedom
Multiple R-Squared: 0.6665
F-statistic: 17.99 on 1 and 9 degrees of freedom,
       the p-value is 0.00217
```


3
4
1
2
y
x
5
10
15
4
6
8
10
12


3
4
1
2
y
x
5
10
15
4
6
8
10
12

5 Influence and Andrews' Configuration

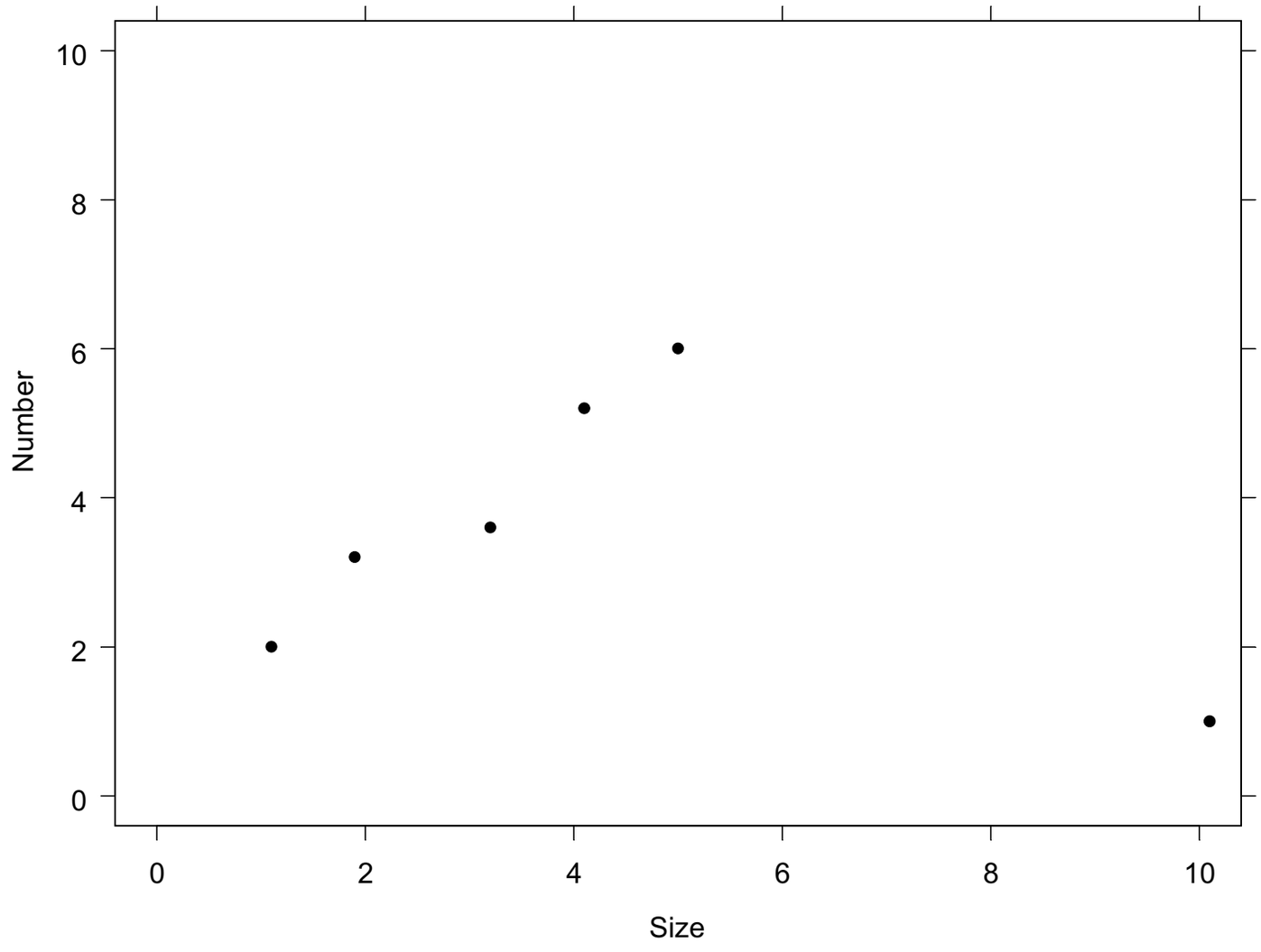
Darwin's data on Galapagos Islands:

X: size of island

Y: number of finch species

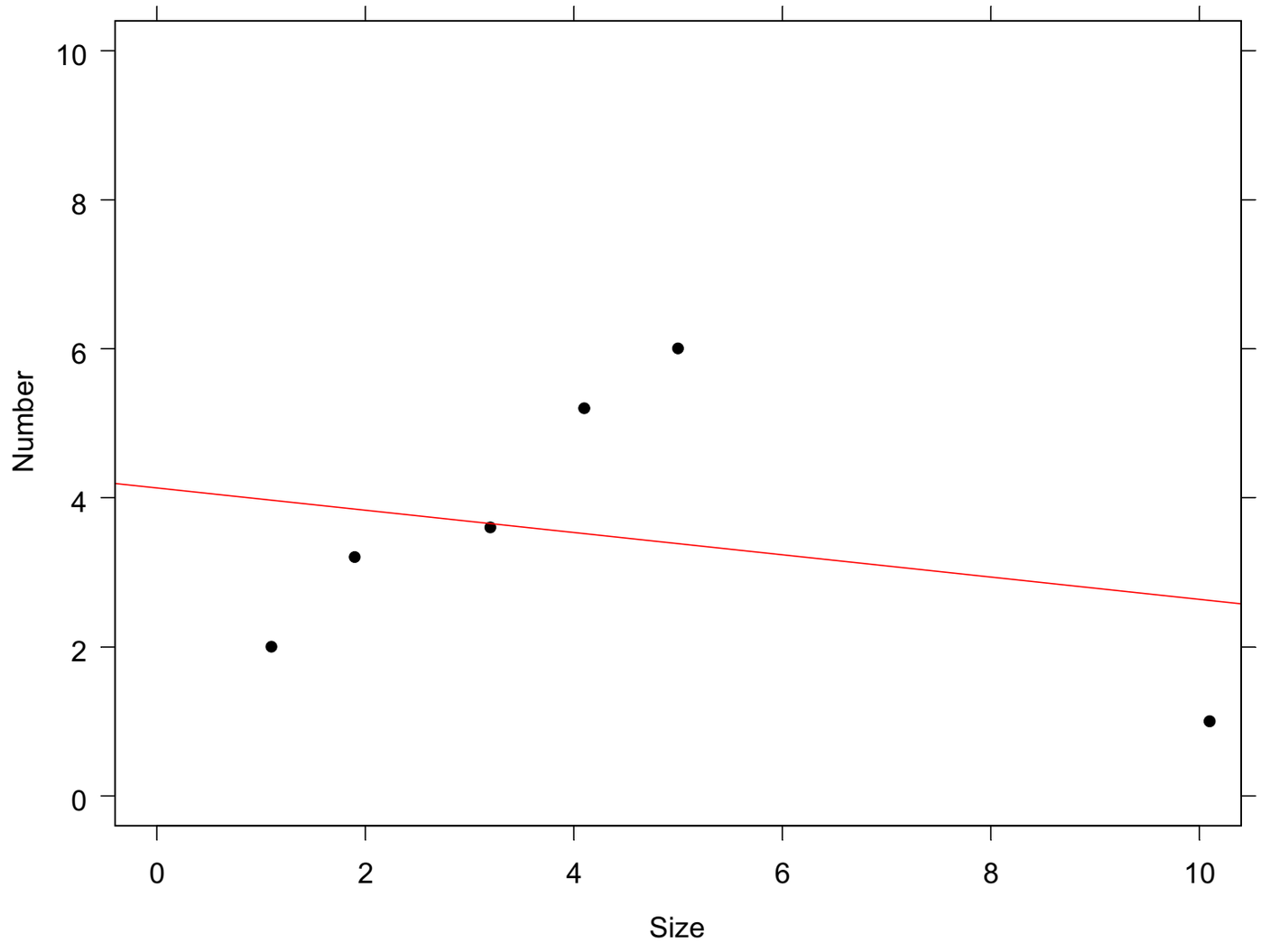
Hypothesis: X and Y should have a positive relationship if a larger area provides more evolutionary niches

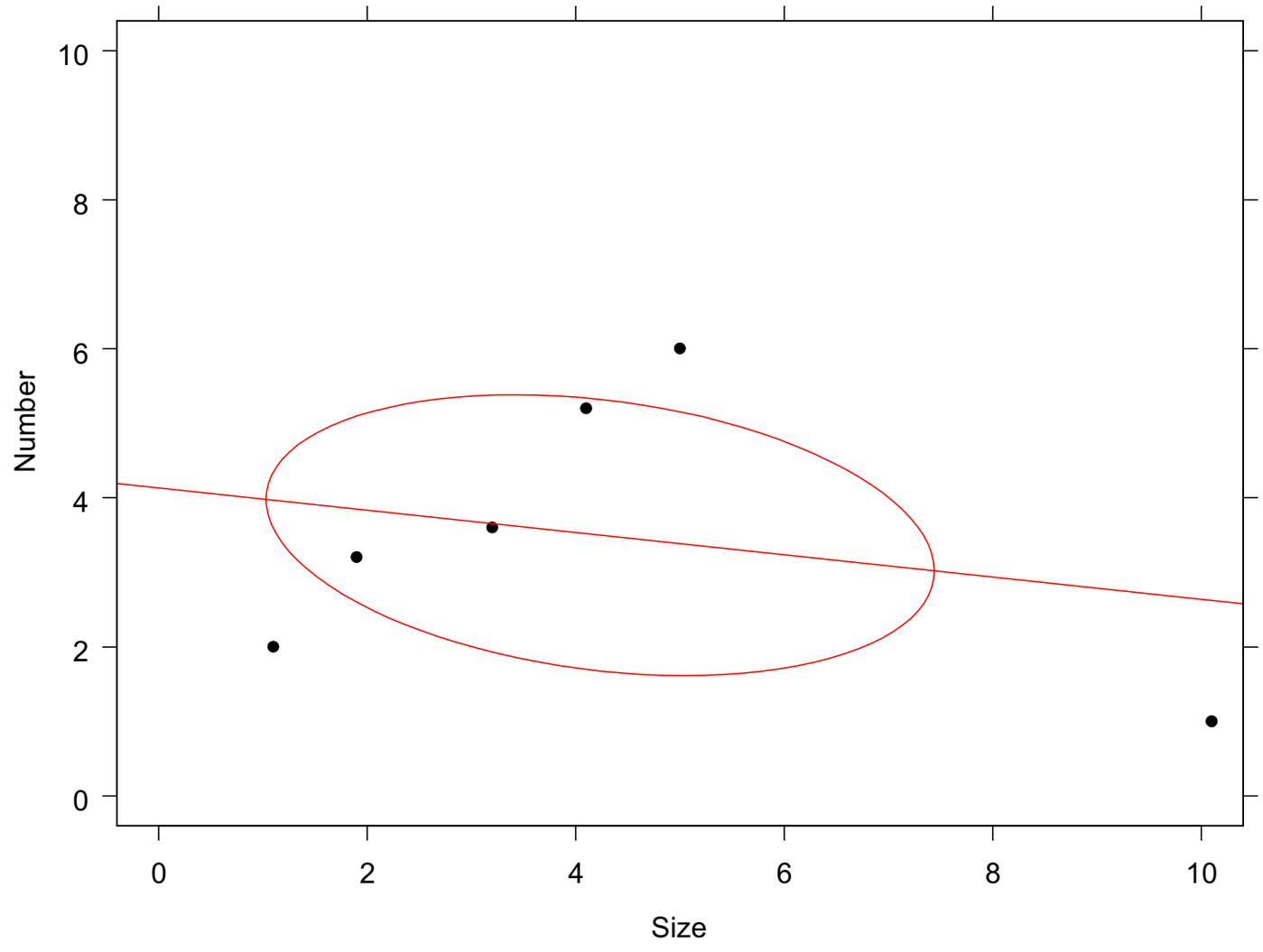
Idealized version of Darwin's data = Andrews Configuration

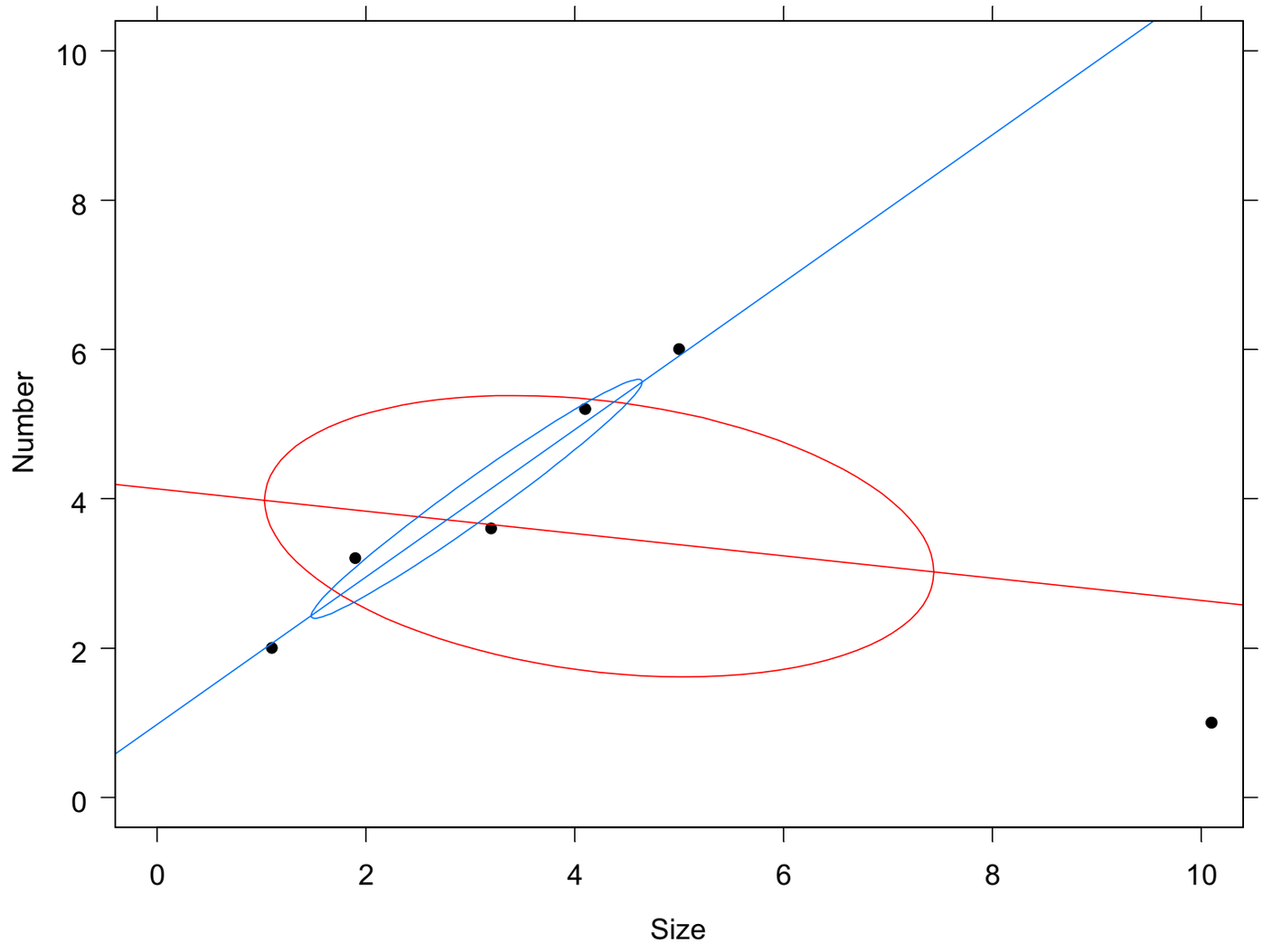

10
8
6
4
2
0
Number
0
2
4
6
8
10
Size

Least-squares regression:

```
> fit <- lm(Number ~Size, Andrews)
> summary(fit)

Call: lm(formula = Number ~Size, data = Andrews)
Residuals:
      1       2         3    4     5       6
 -1.969  -0.649  -0.05454 1.68 2.615  -1.623

Coefficients:
             Value Std. Error t value Pr(>|t|)
(Intercept)  4.1331  1.4625     2.8261  0.0475
       Size -0.1496  0.2842    -0.5262  0.6266

Residual standard error: 2.037 on 4 degrees of freedom
Multiple R-Squared: 0.06474
F-statistic: 0.2769 on 1 and 4 degrees of freedom,
   the p-value is 0.6266
```


Number
Size
0
2
4
6
8
10


Number
Size
0
2
4
6
8
10

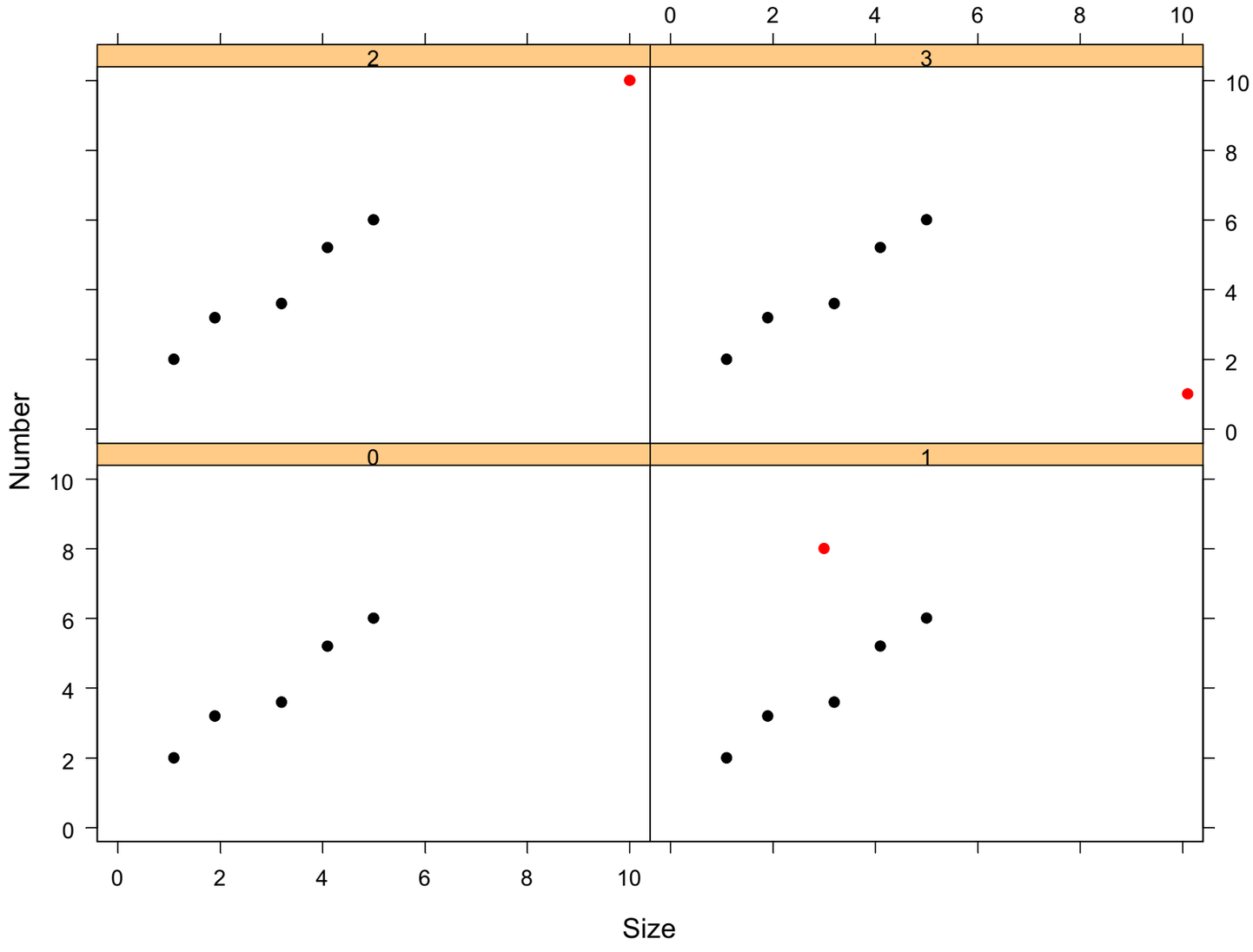

Number
Size
0
2
4
6
8
10

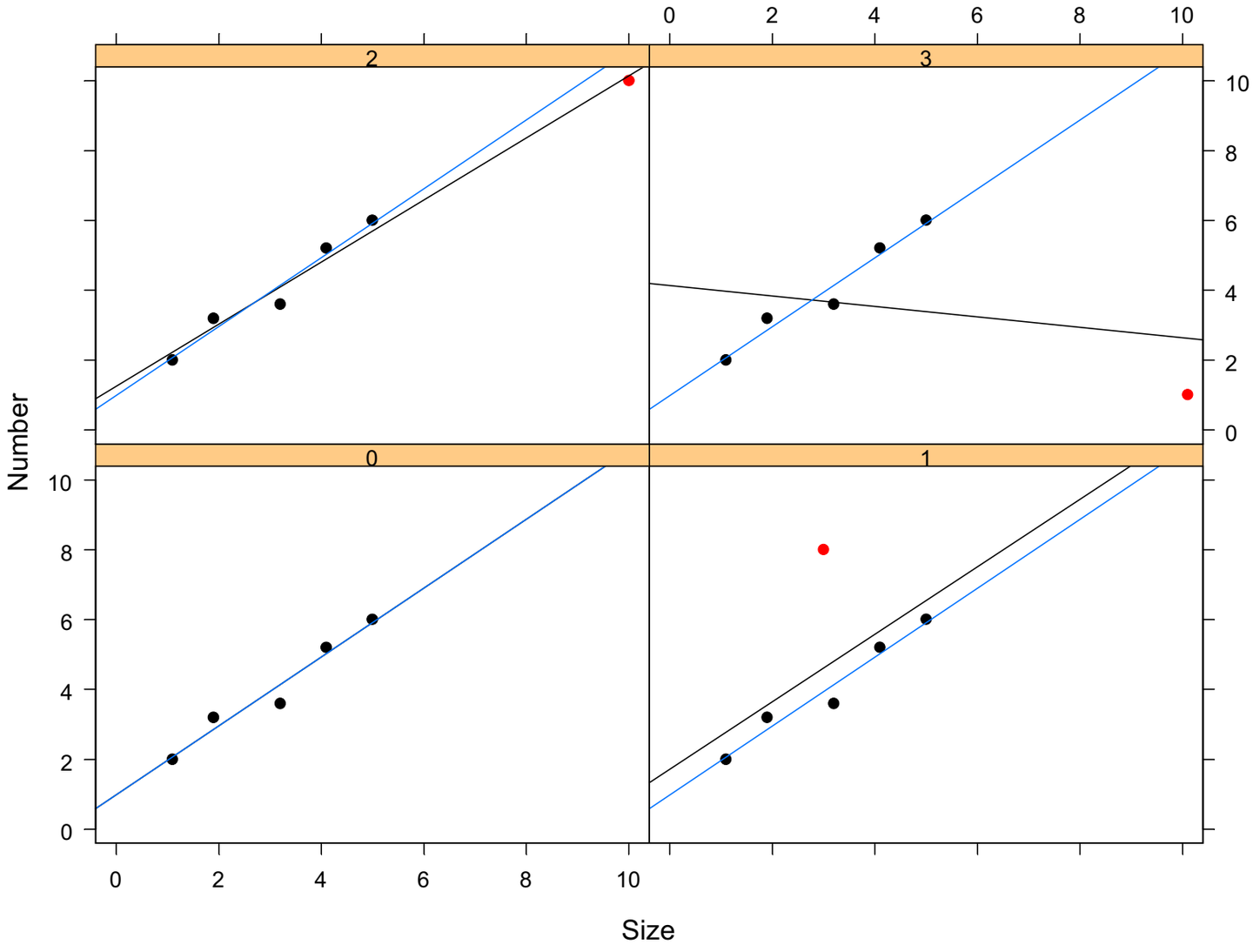
6 Influence and Leverage

What happens when you take good data and add a wild point?

Three archetypal wild points:

- Typical X, bad fit
- Unusual X, good fit
- Unusual X, bad fit


2
3
0
1
Number
Size
0
2
4
6
8
10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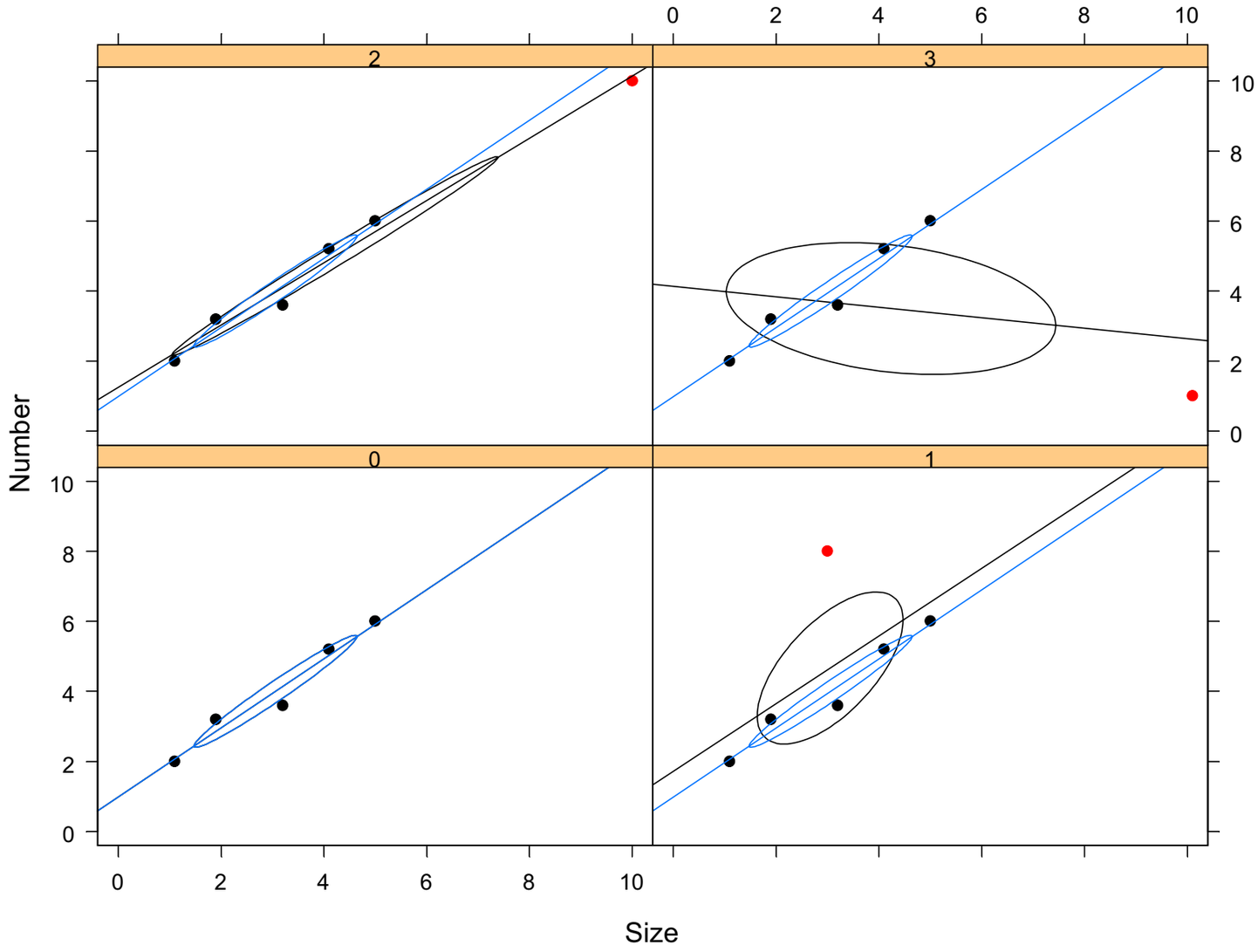
3
0
1
Number
Size
0
2
4
6
8
10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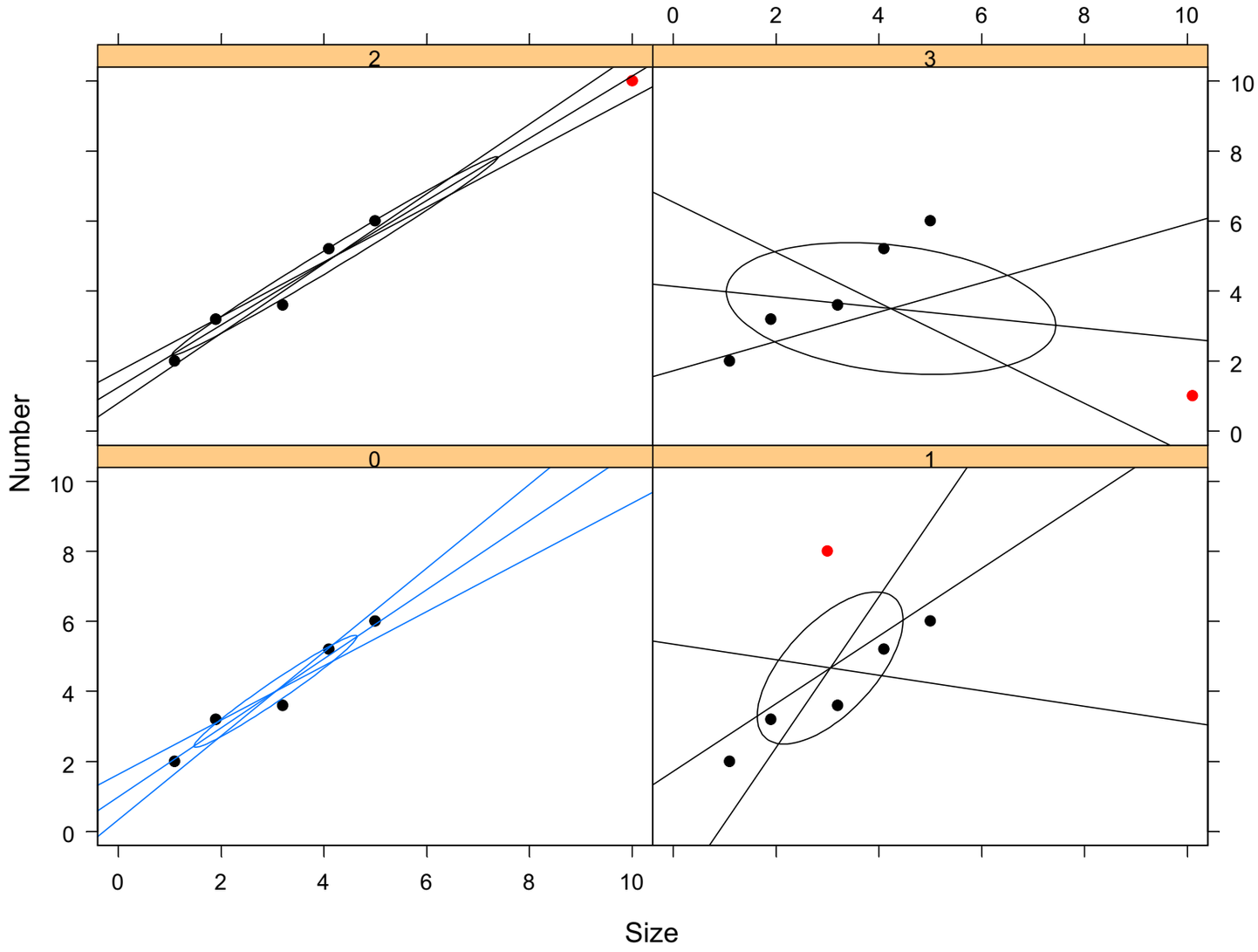
3
0
1
Number
Size
0
2
4
6
8
10

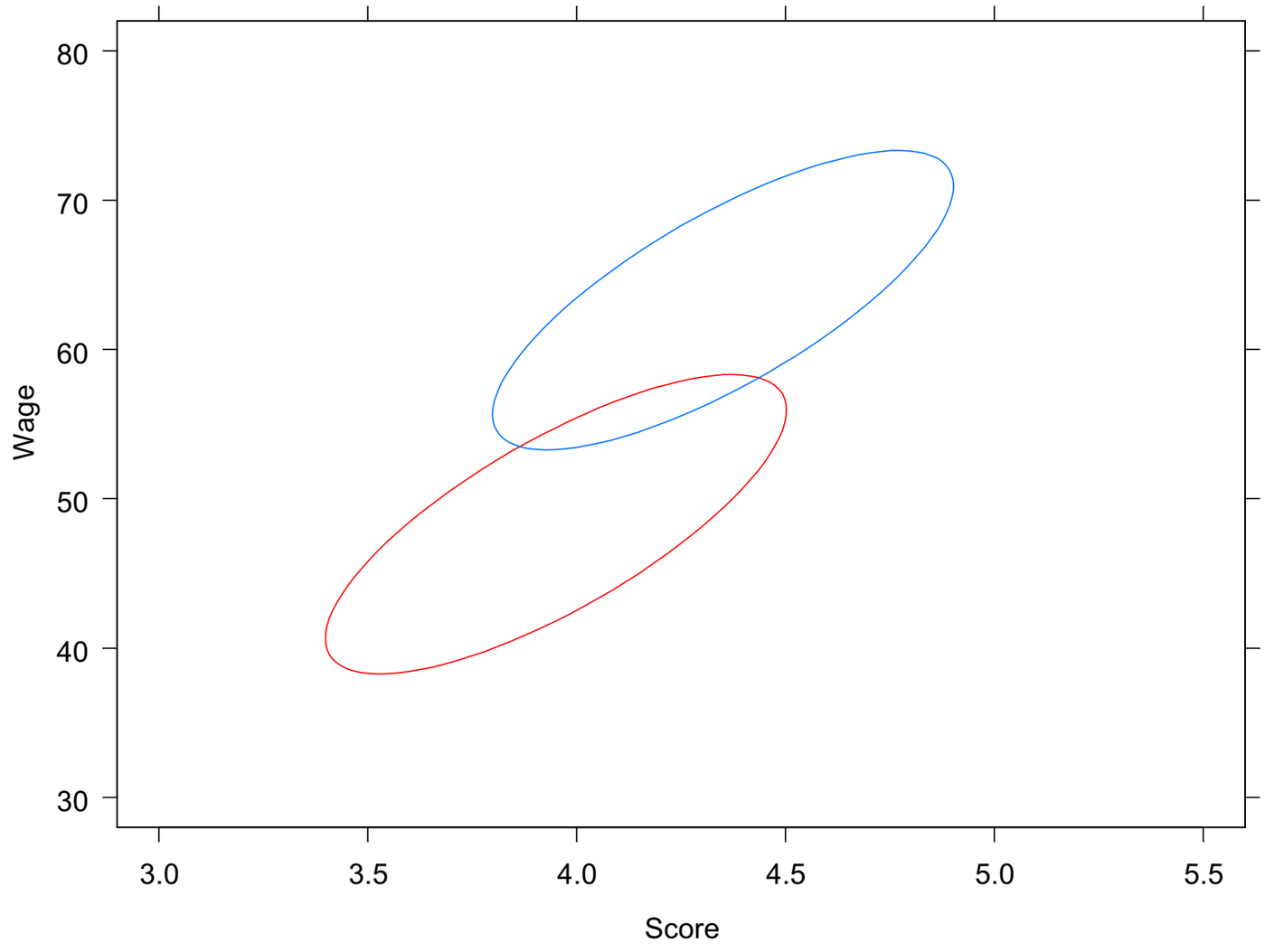

2
3
0
1
Number
Size
0
2
4
6
8
10

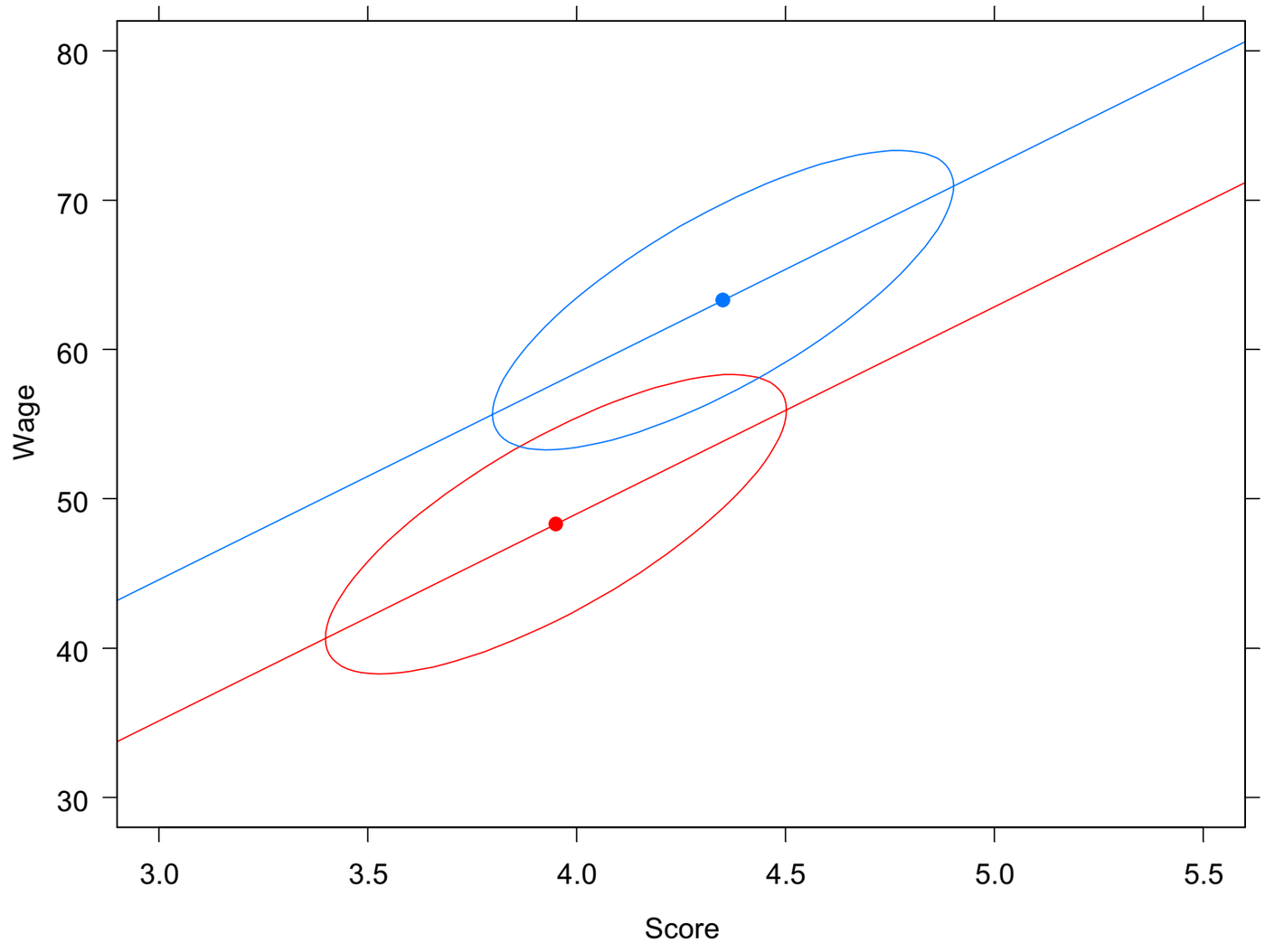
7 Measurement error

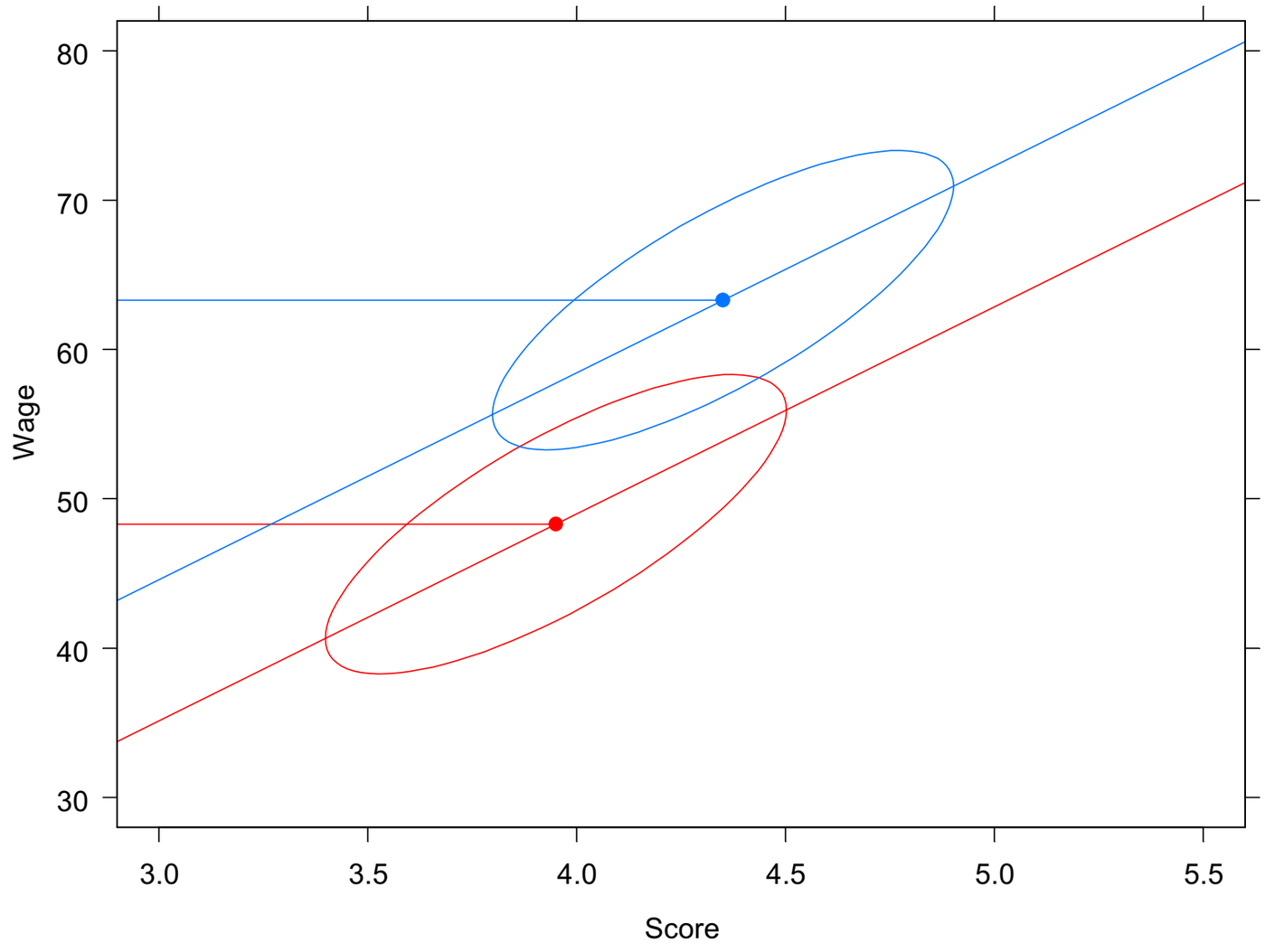
What happens when a covariate is measured with error?

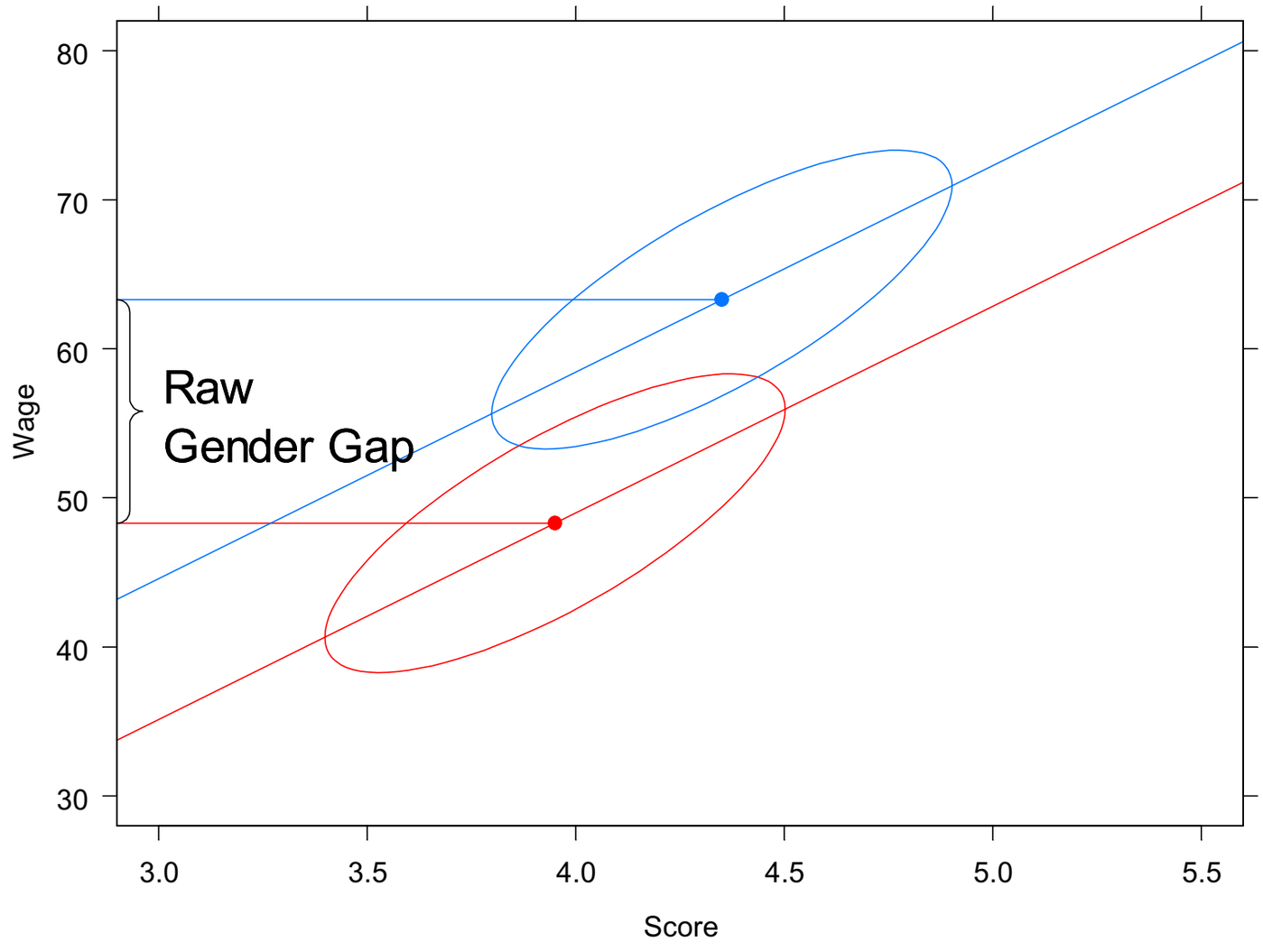
Example: Pay Equity – Estimating the Gender Gap

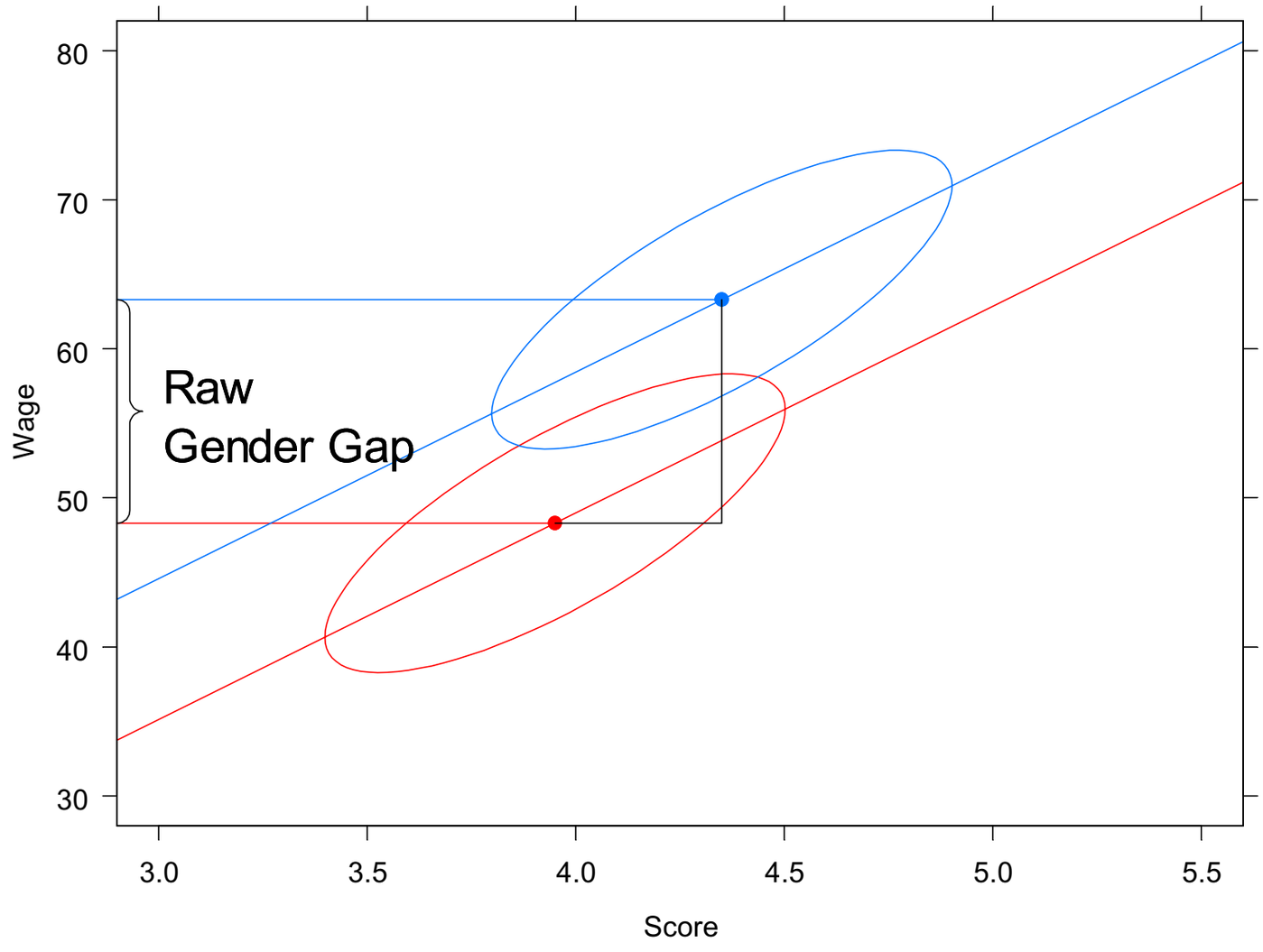
Wages vs Job Value by Gender

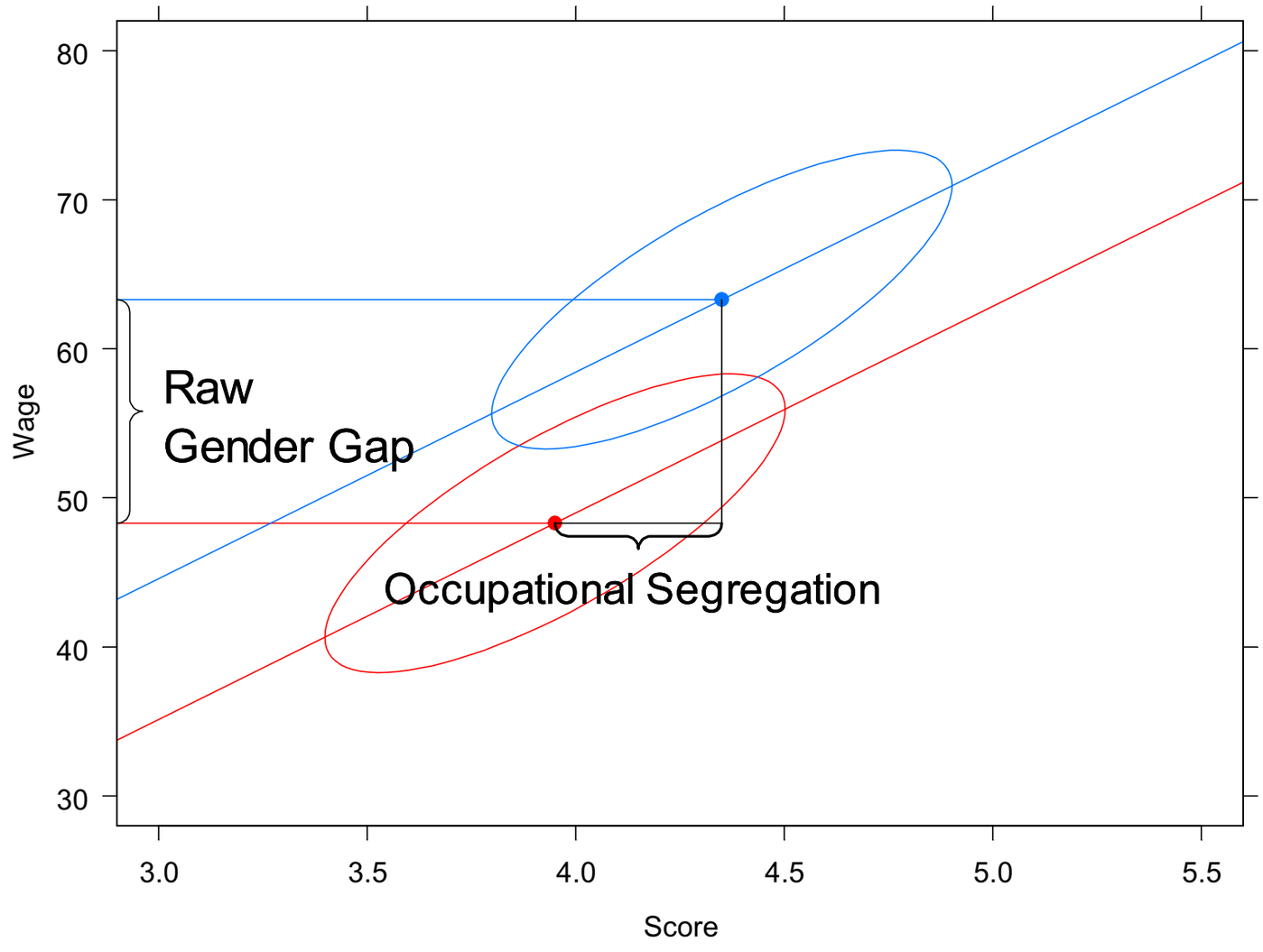

80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score

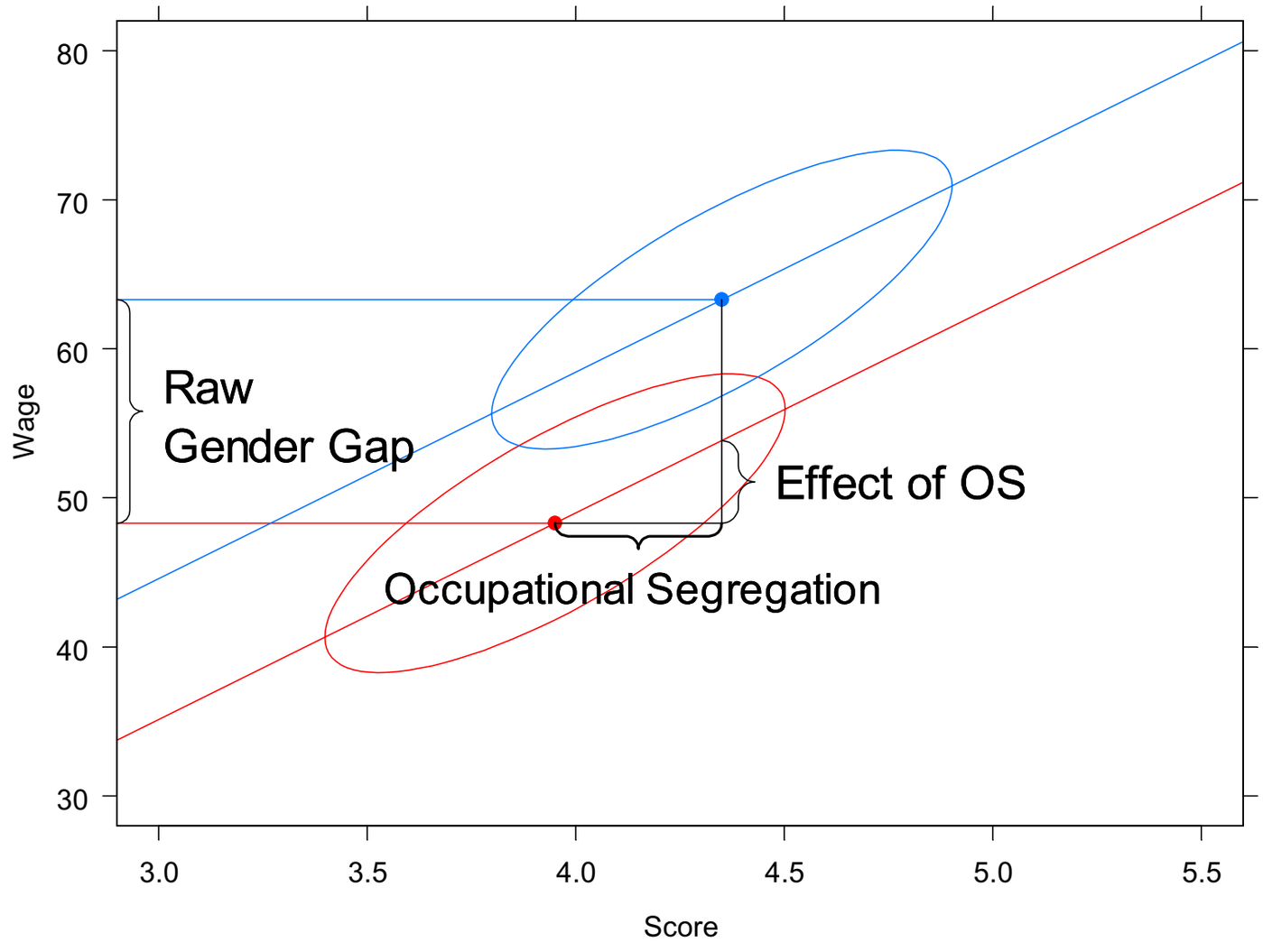

80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score

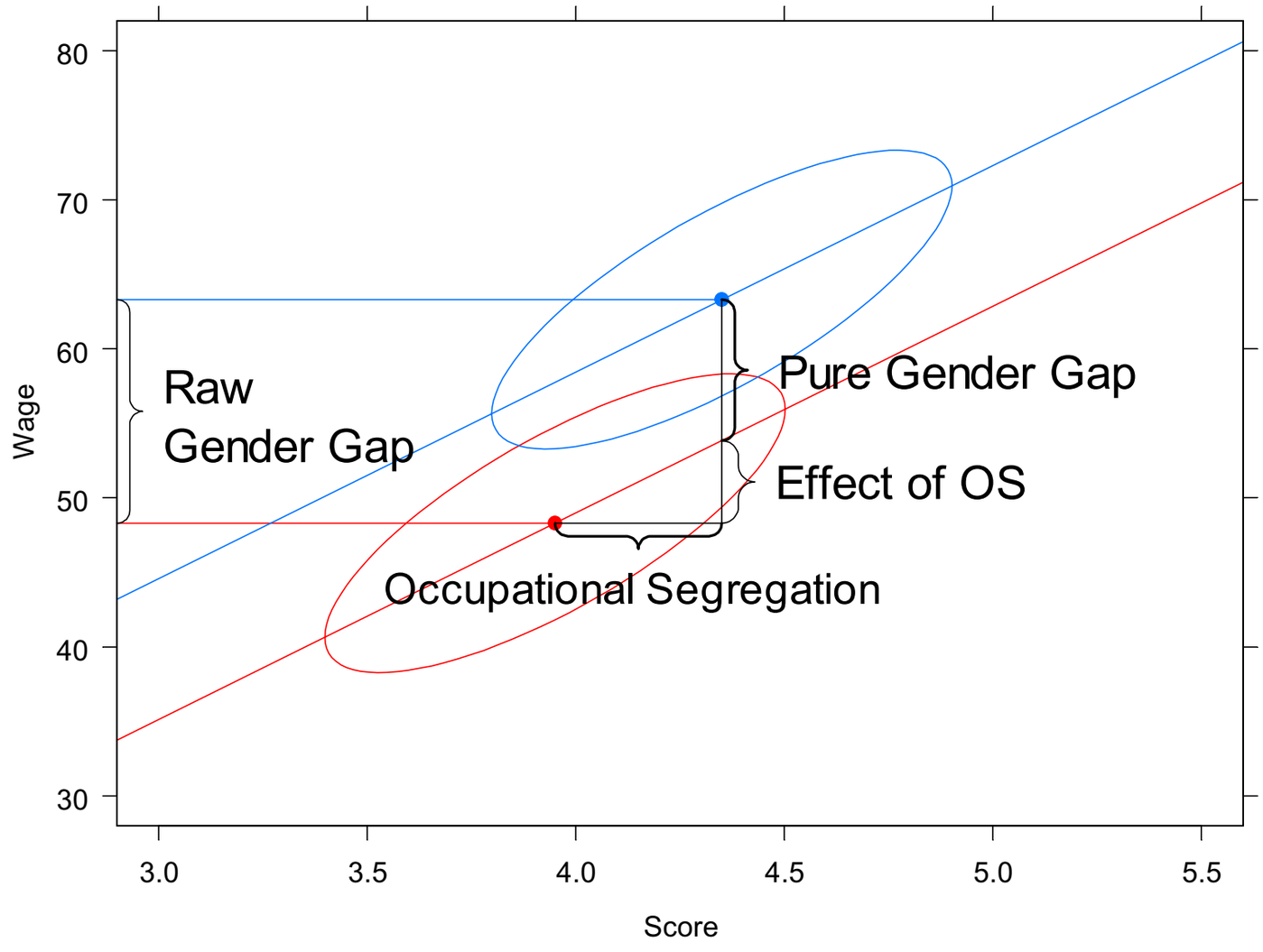

80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score

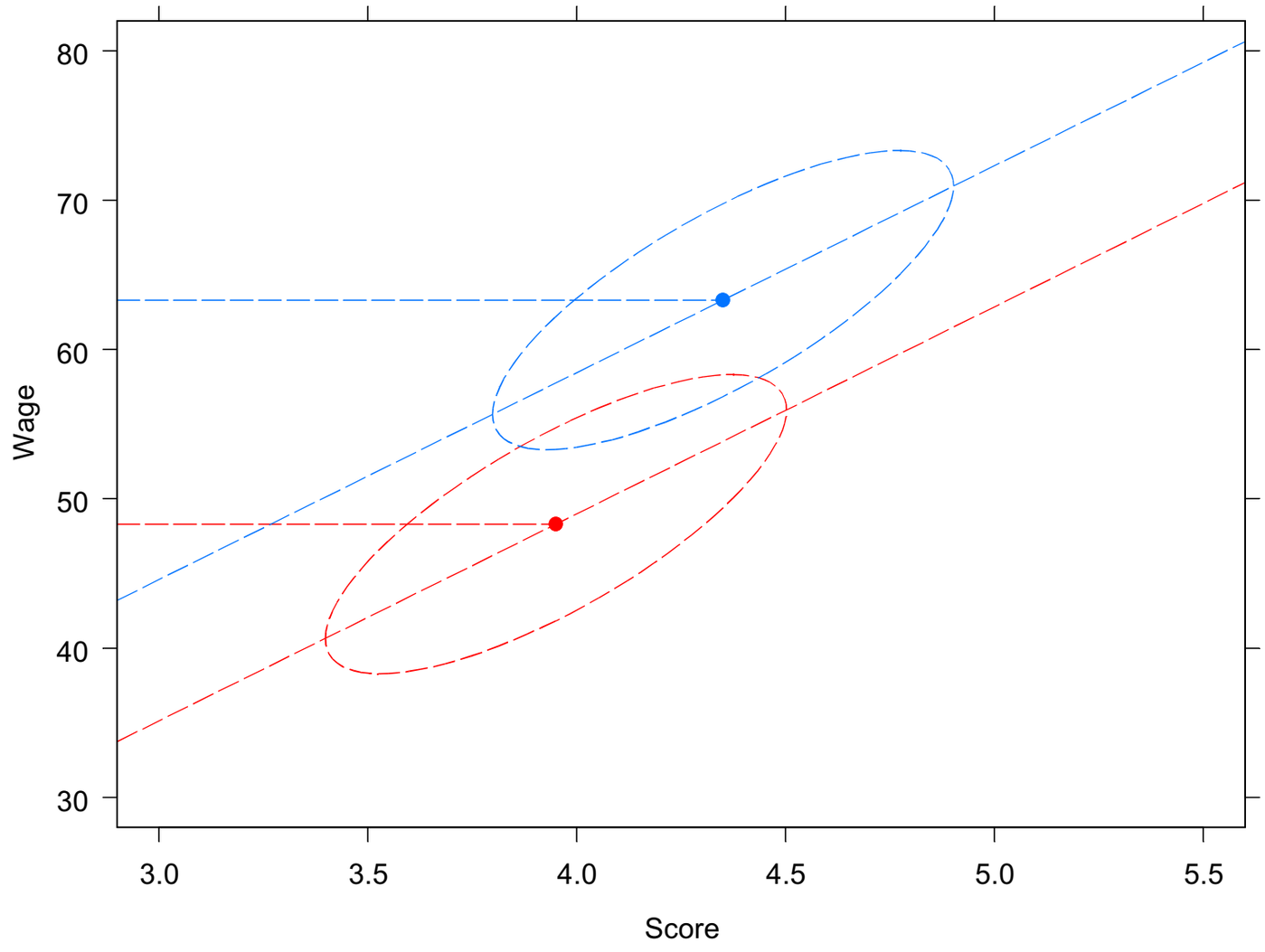

80
70
60
50
40
30
Wage
Raw
Gender Gap
3.0
3.5
4.0
4.5
5.0
5.5
Score

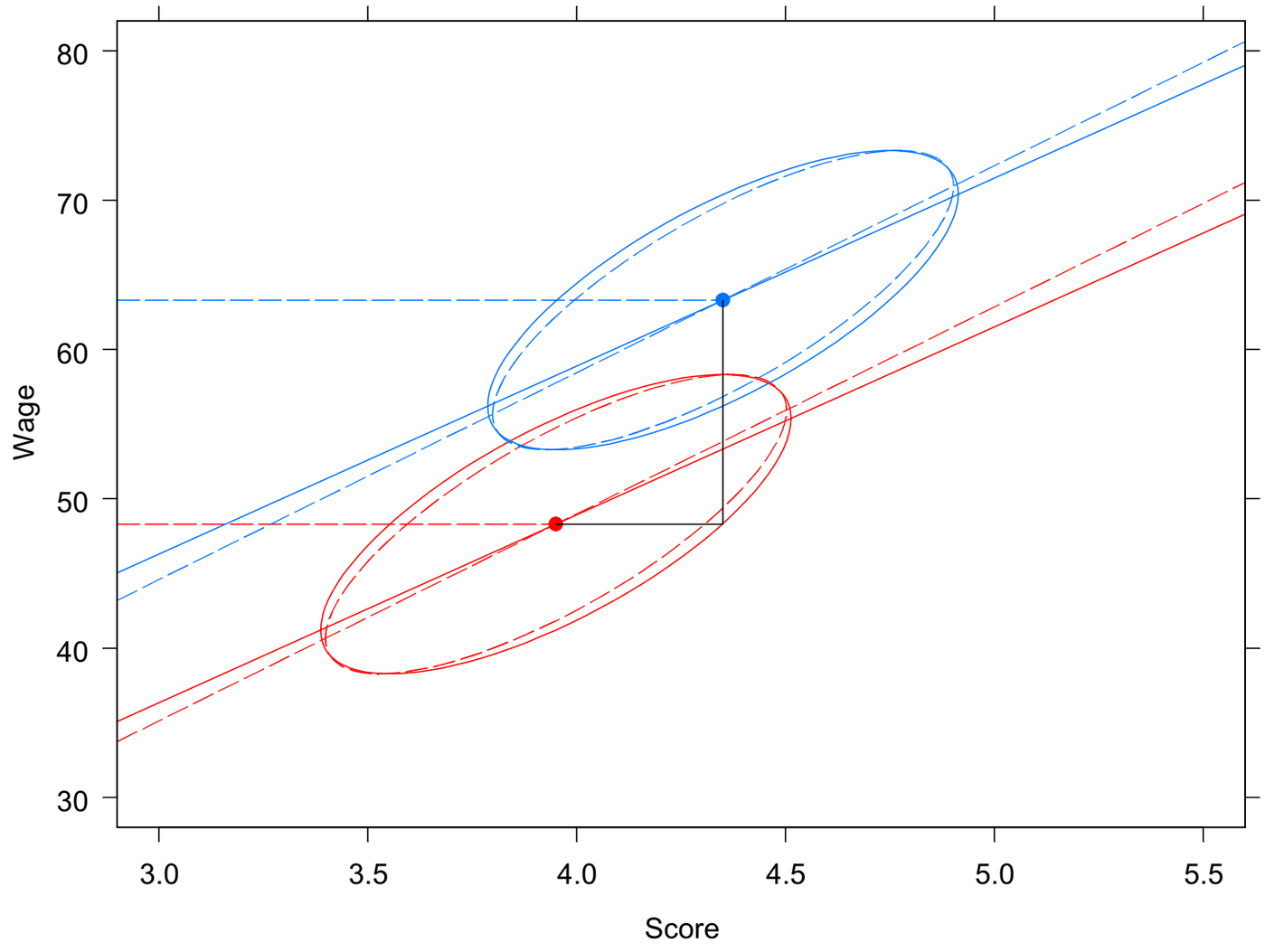

80
70
60
50
40
30
Wage
Raw
Gender Gap
3.0
3.5
4.0
4.5
5.0
5.5
Score

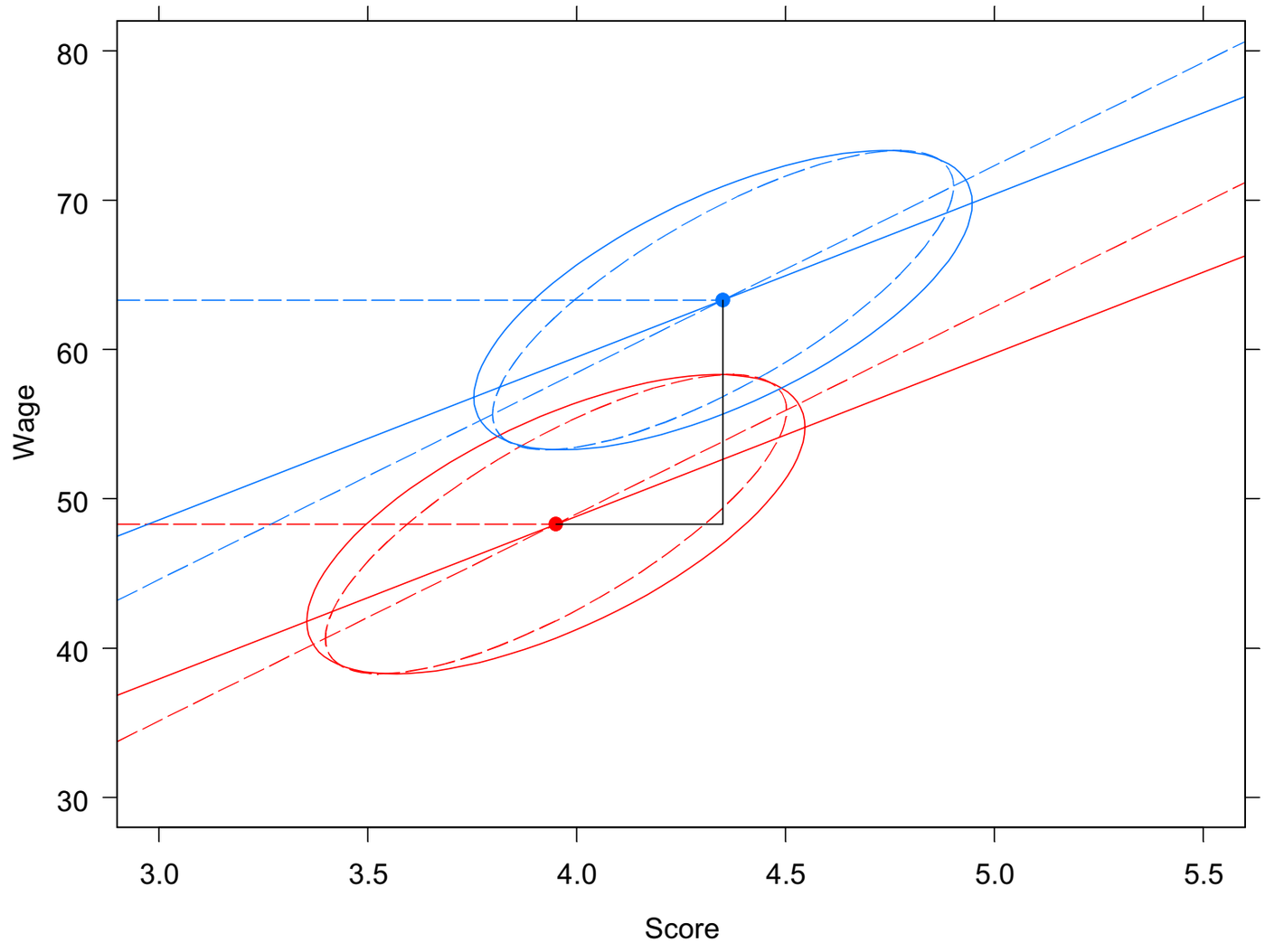

80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score
Raw
Gender Gap
Occupational Segregation

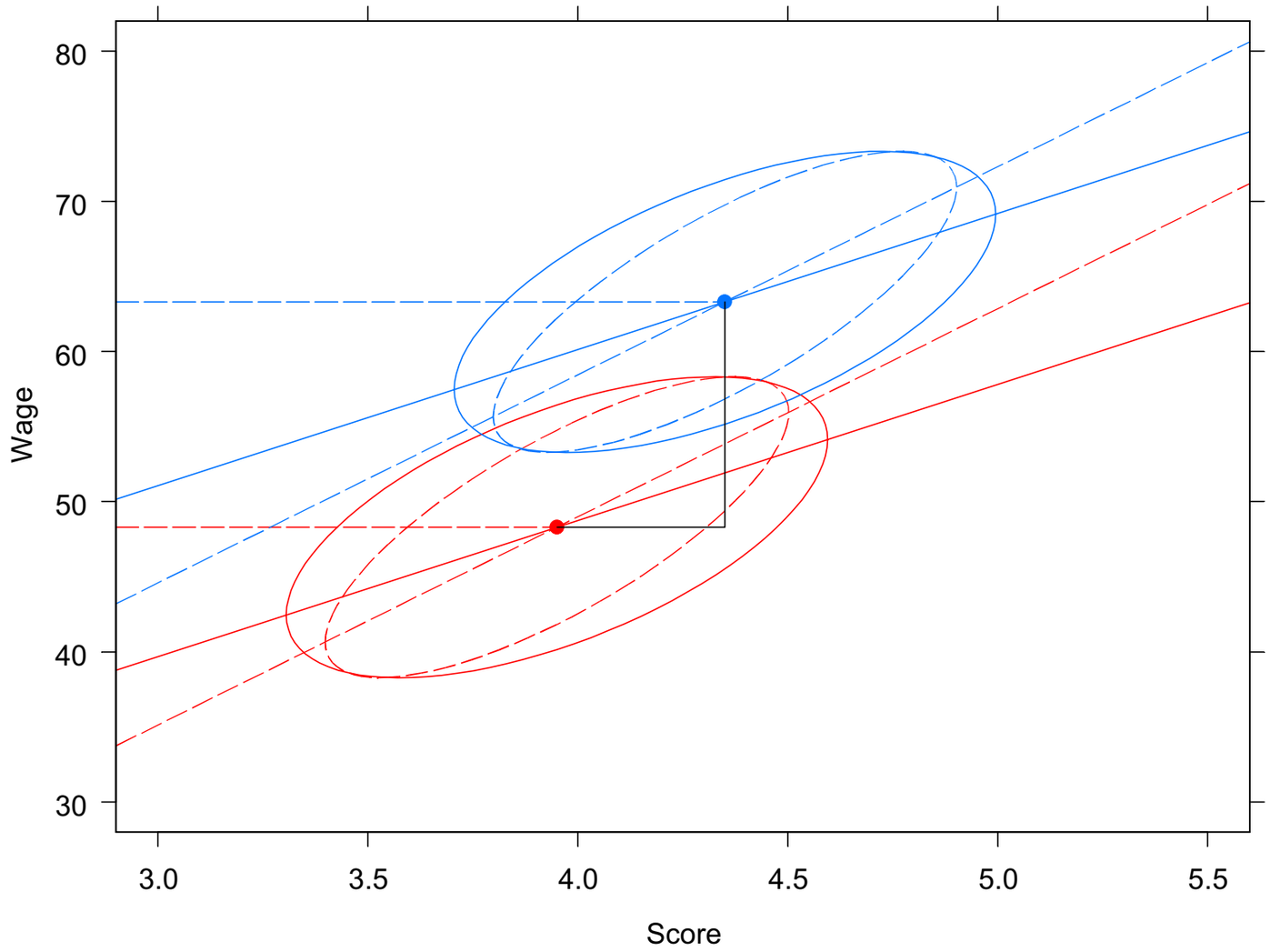

80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score
Raw
Gender Gap
Effect of OS
Occupational Segregation

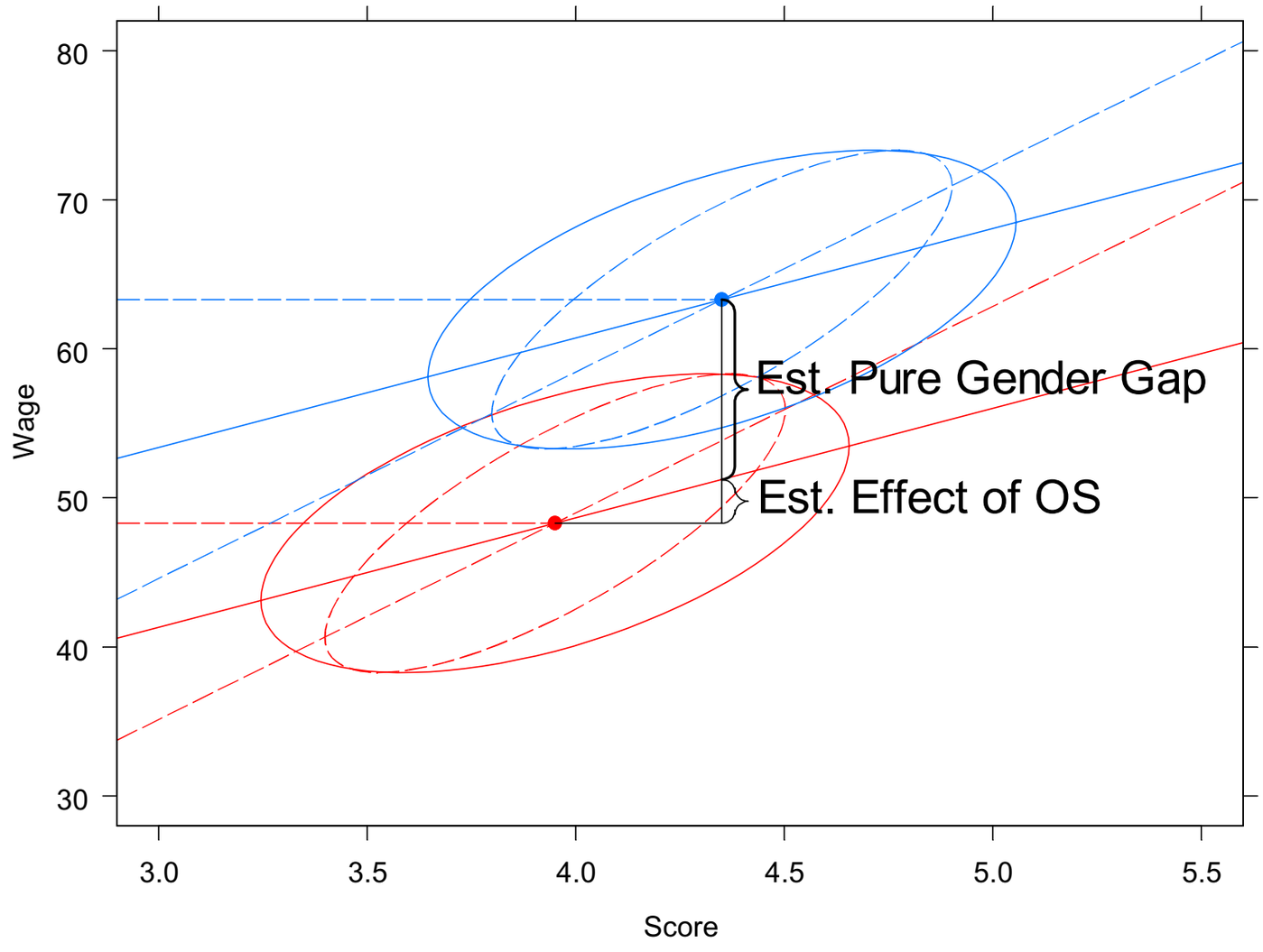

80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score
Raw
Gender Gap
Pure Gender Gap
Effect of OS
Occupational Segregation


80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score


80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score


80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score

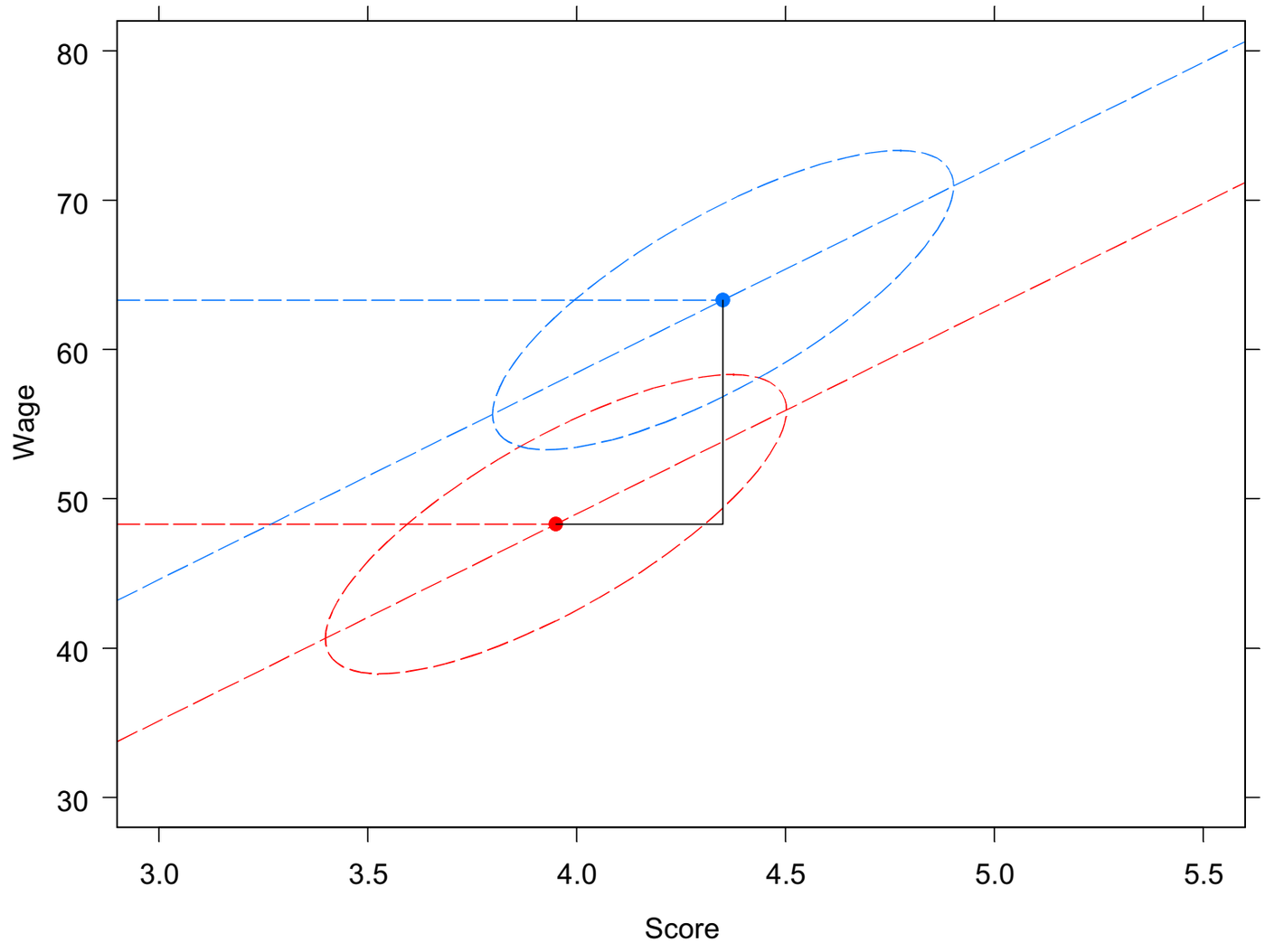

80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score

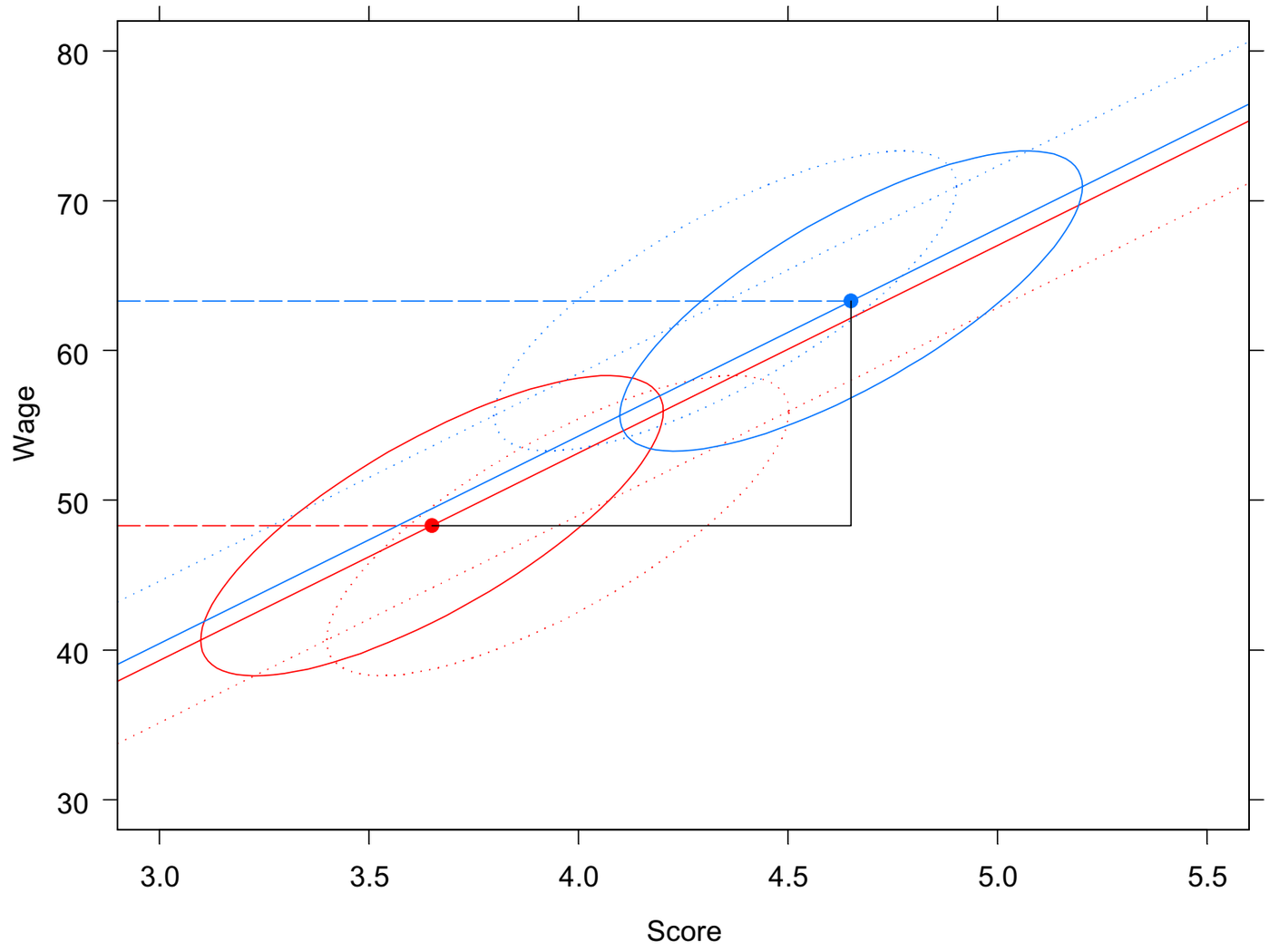

80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score
Est. Pure Gender Gap
Est. Effect of OS

8 Bias in X: Job Evaluation Score

Measurement error leads to overestimation of gender gap

But bias might work in the opposite direction


80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score


80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score

9 Simpson's (Yule's) Paradox

Some types of association between variables:

- Marginal
- Conditional
- Partial
- Ecological

Paradoxes:

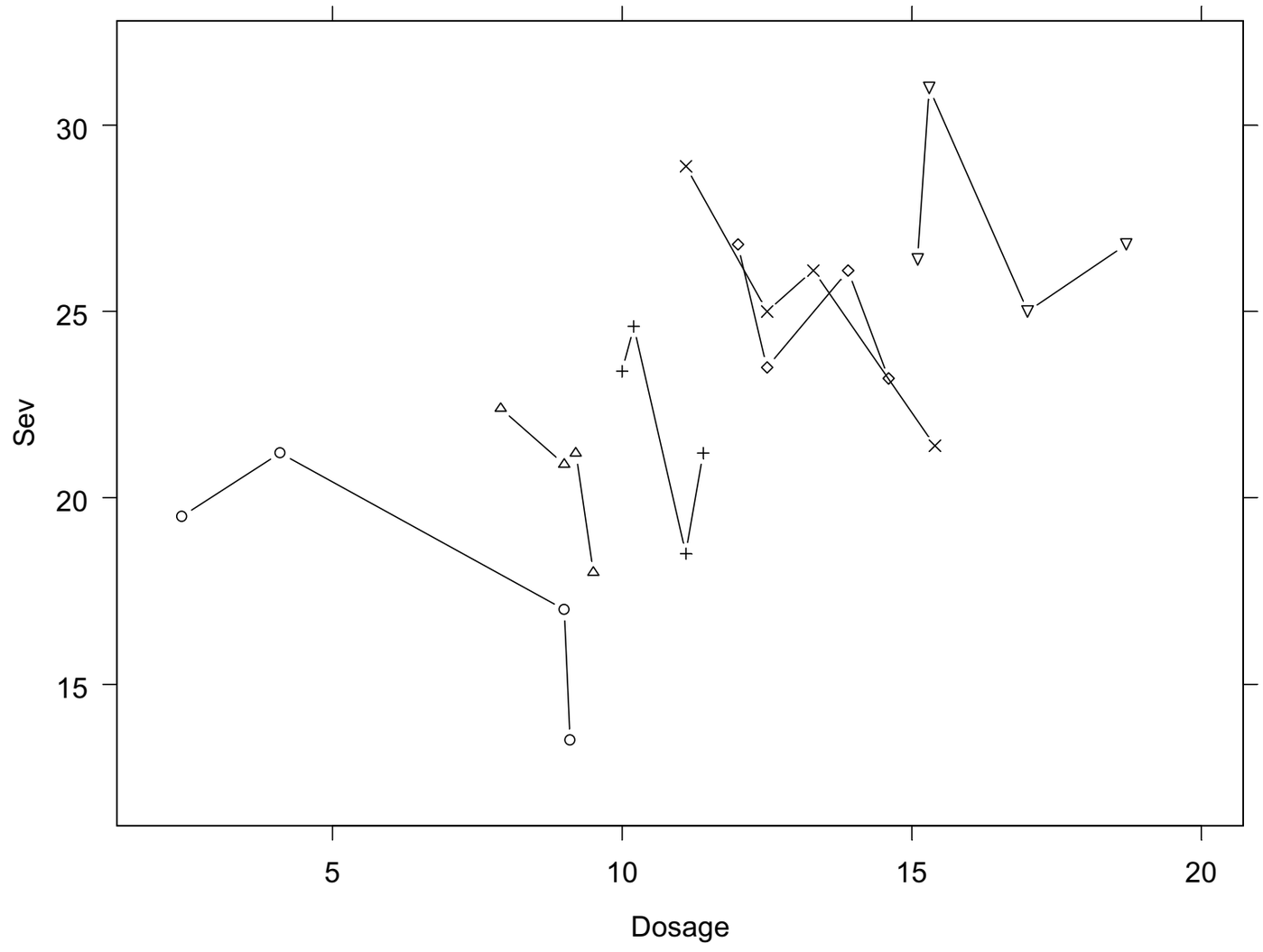
- Simpson's: Marginal and Conditional can have opposite signs
- Robinson's: Ecological and Conditional can have opposite signs

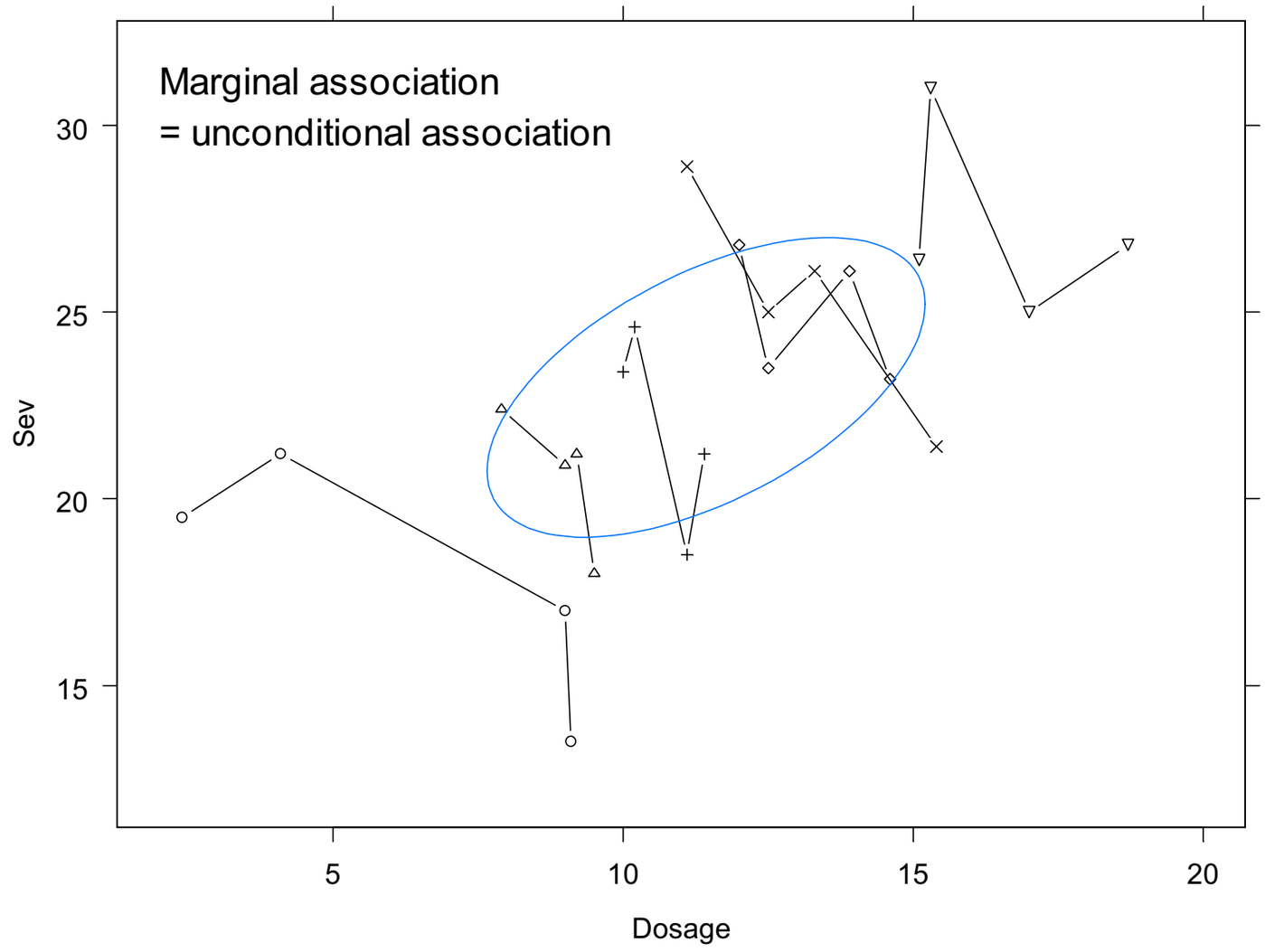
Example:

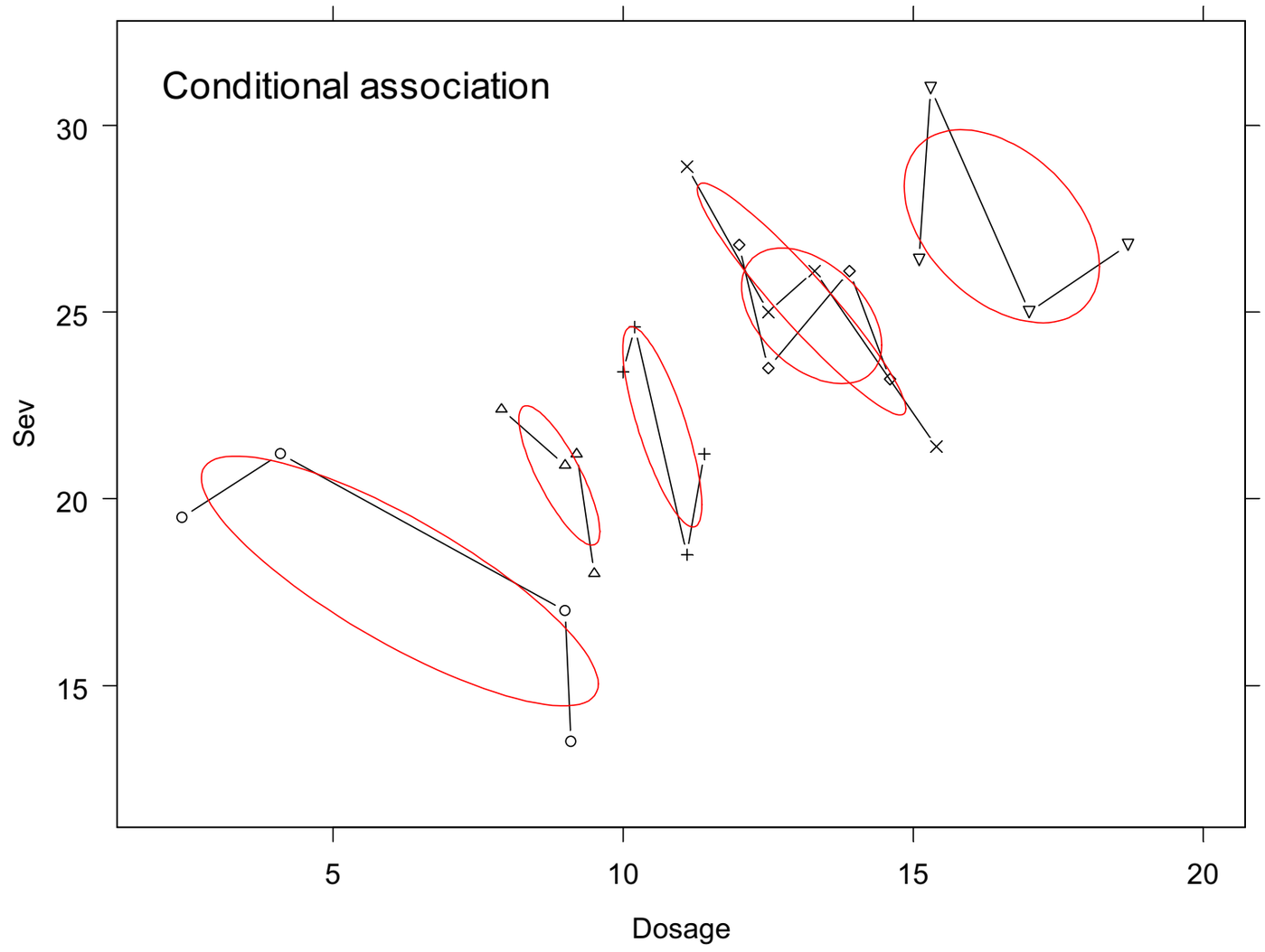
- Symptoms vs. Dosage of Drug: 6 patients at 4 dosages

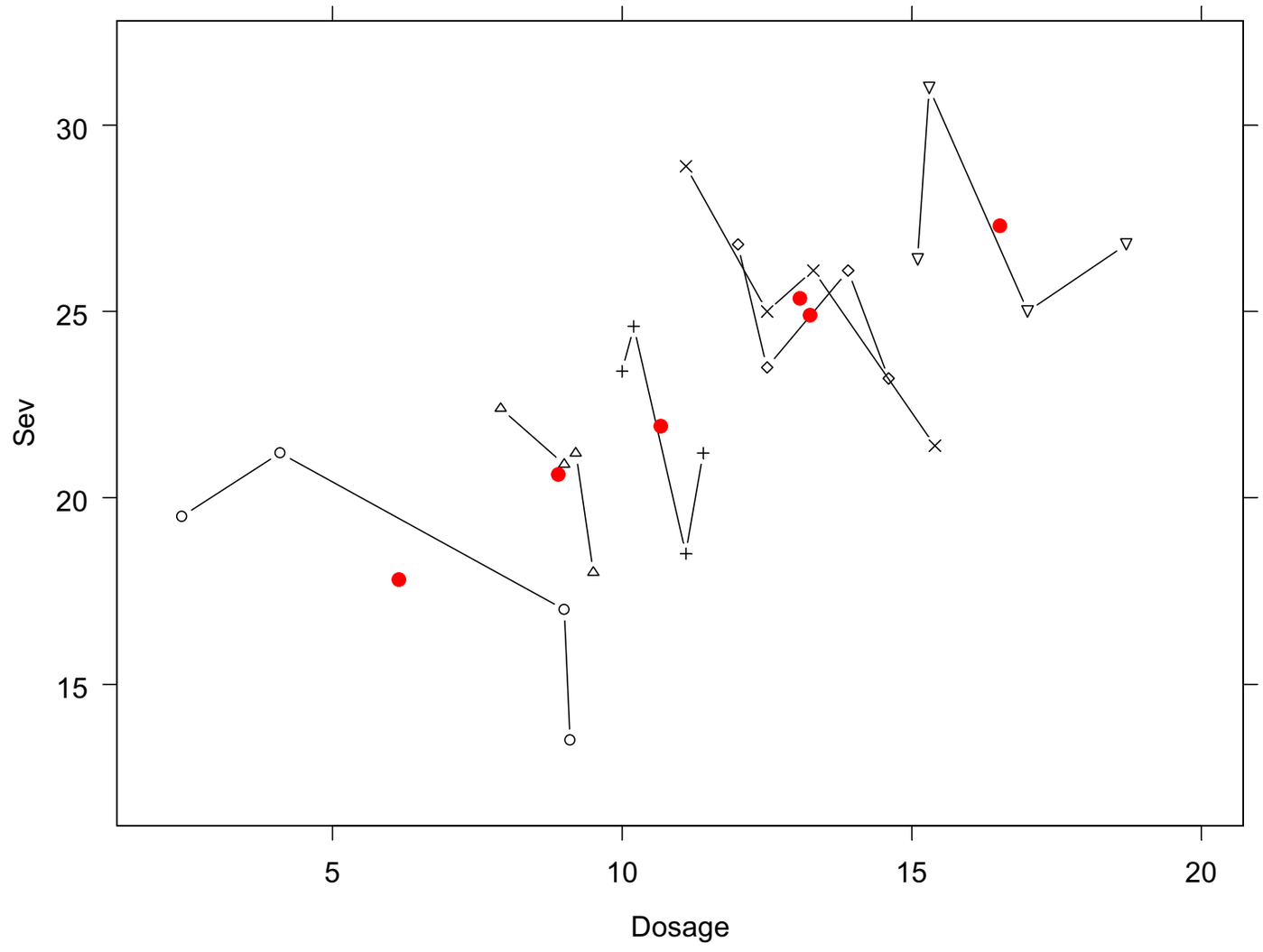
```
> Drug
   ID Dosage  Sev
 1  A   10.3 22.4
 2  A   10.6 22.8
 3  A   11.3 21.7
 4  A   11.3 20.9
 5  B   11.1 23.1
 6  B   11.3 22.7
 7  B   11.3 22.8
 8  B   11.4 22.0
 9  C   11.4 23.3
10  C   11.5 23.6
11  C   11.6 22.1
12  C   11.6 22.8
13  D   11.6 24.7
14  D   11.8 23.8
15  D   11.9 24.0
16  D   12.2 22.8
17  E   11.7 24.2
18  E   11.8 23.4
19  E   12.0 24.0
```

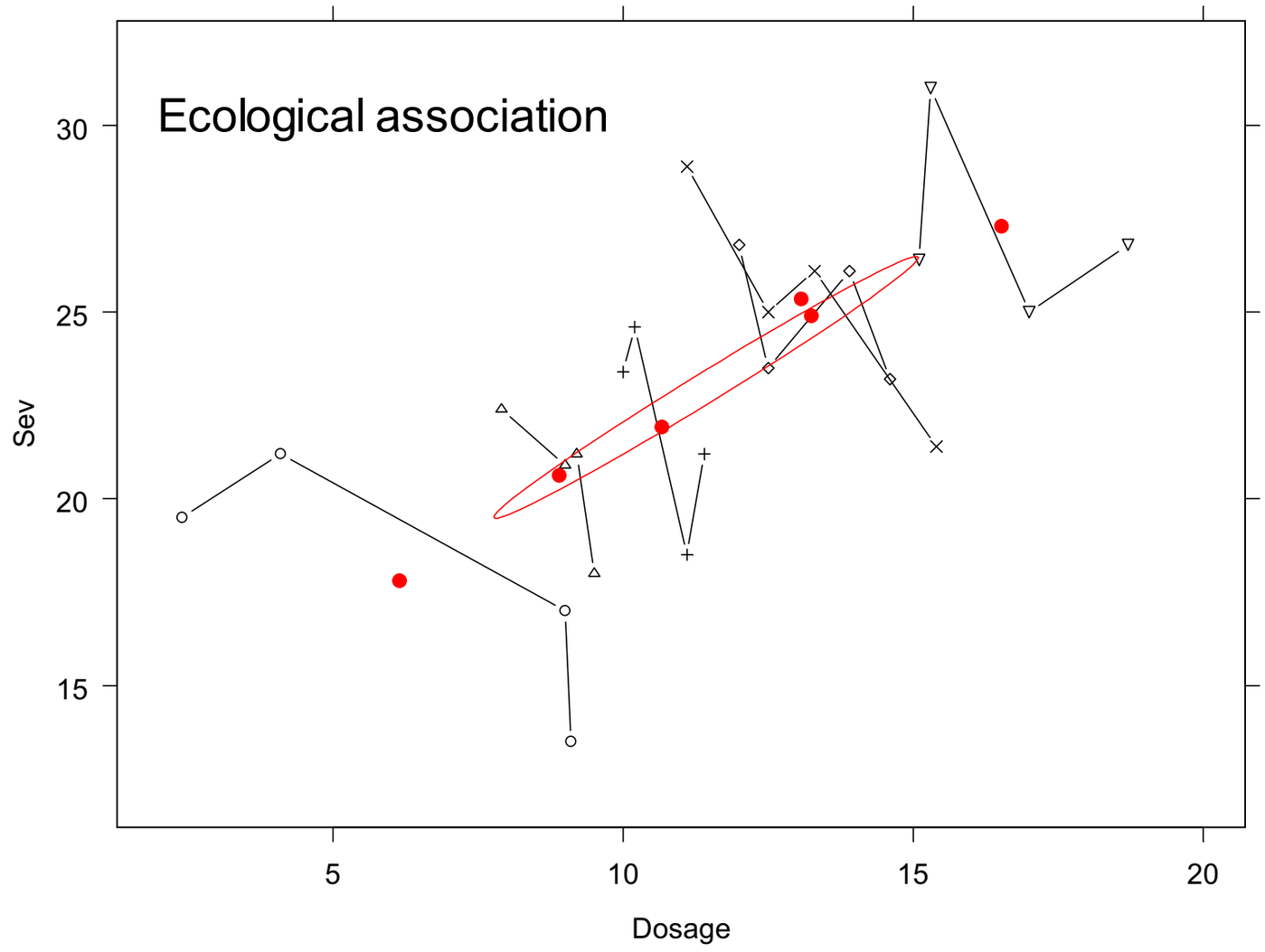
```
20   E    12.1 23.3
21   F    12.2 24.1
22   F    12.2 25.3
23   F    12.4 23.7
24   F    12.7 24.2
```


30
25
20
15
Sev
5
10
15
20
Dosage


Marginal association
= unconditional association
Sev
30
25
20
15
5
10
15
20
Dosage


Conditional association
Sev
Dosage
30
25
20
15
5
10
15
20


30
25
20
15
Sev
5
10
15
20
Dosage


Ecological association
Sev
Dosage
30
25
20
15
5
10
15
20

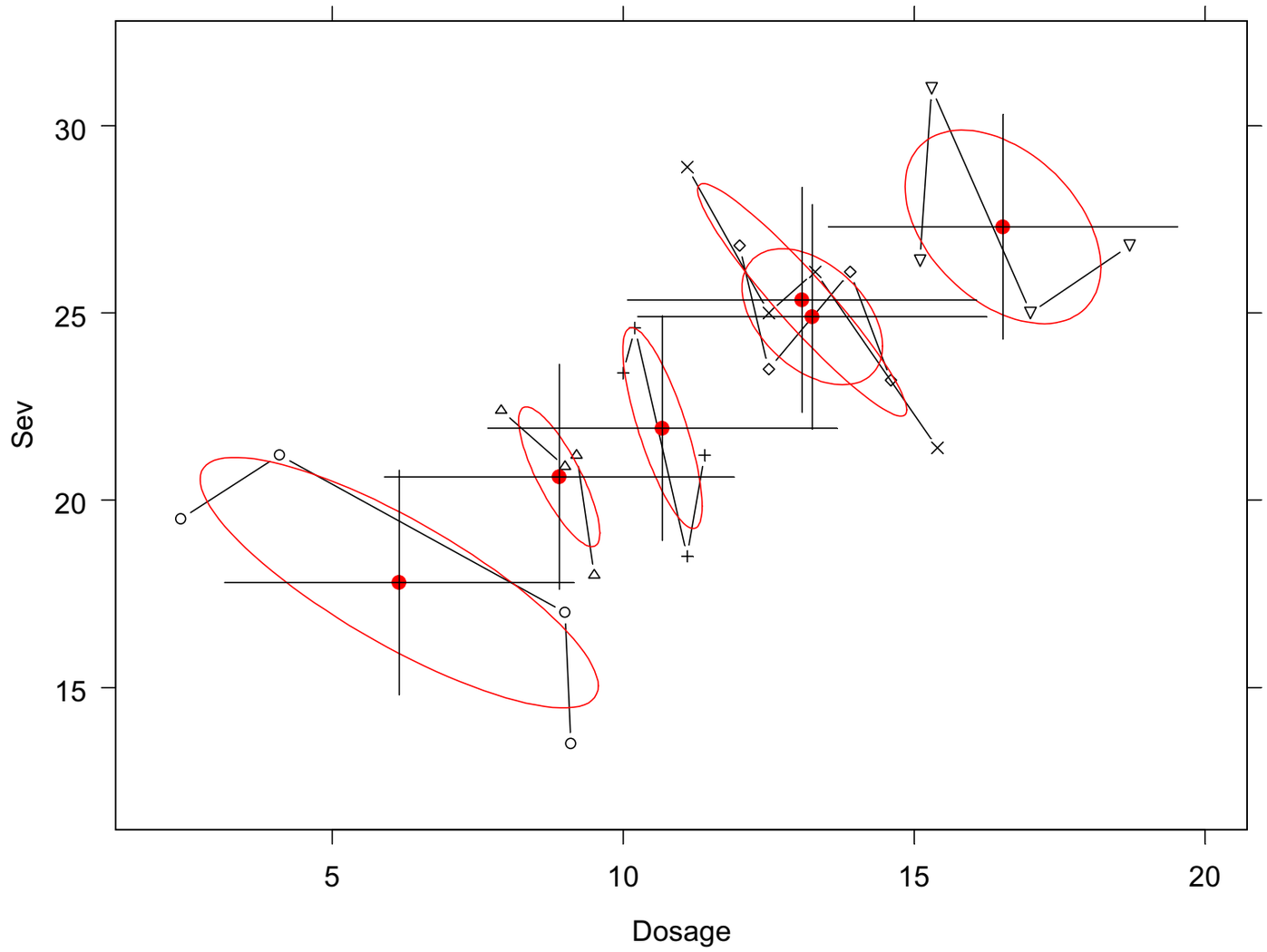
Simple regression = marginal association

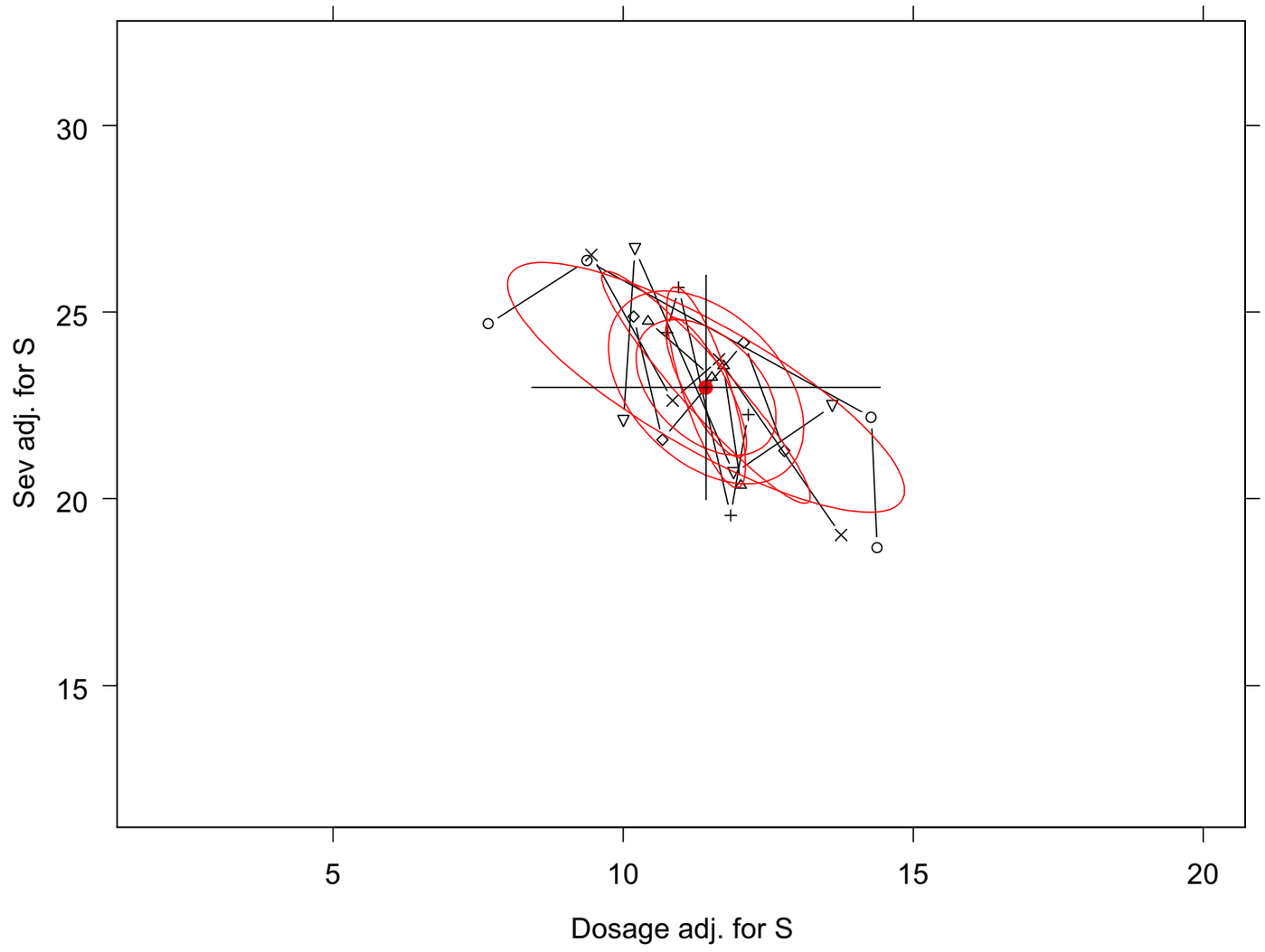
```
> fit <- lm(Sev ~Dosage, Drug)
> summary(fit)

Call: lm(formula = Sev ~Dosage, data = Drug)
Residuals:
    Min      1Q Median    3Q   Max
 -8.103 -1.688 0.5618 1.902 6.112

Coefficients:
             Value Std. Error t value Pr(>|t|)
(Intercept) 16.2110  2.2596     7.1744  0.0000
     Dosage  0.5925  0.1881     3.1500  0.0046

Residual standard error: 3.406 on 22 degrees of freedom
Multiple R-Squared: 0.3108
F-statistic: 9.922 on 1 and 22 degrees of freedom,
         the p-value is 0.004648
```

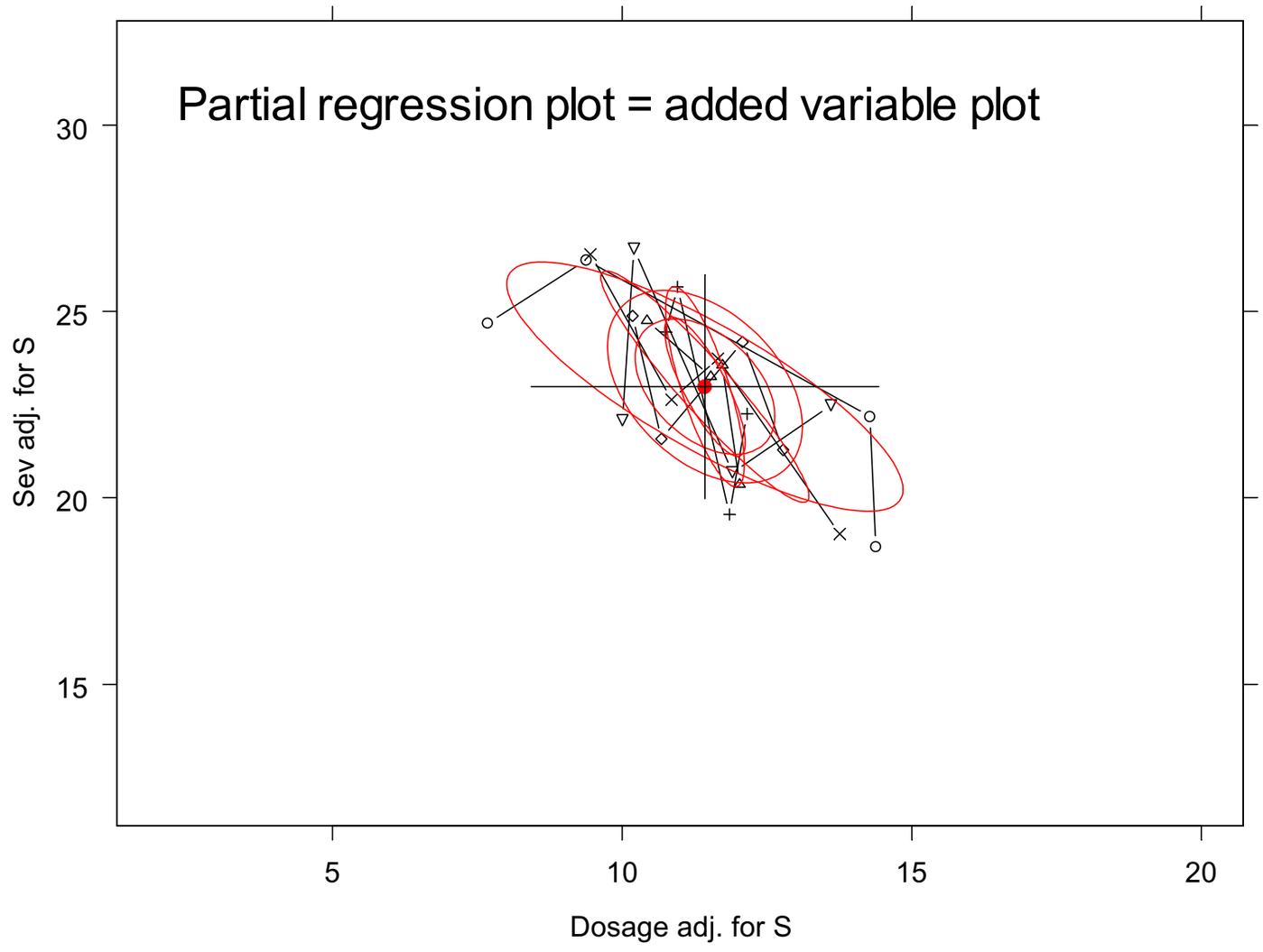
Multiple regression = Analysis of covariance (with these data)

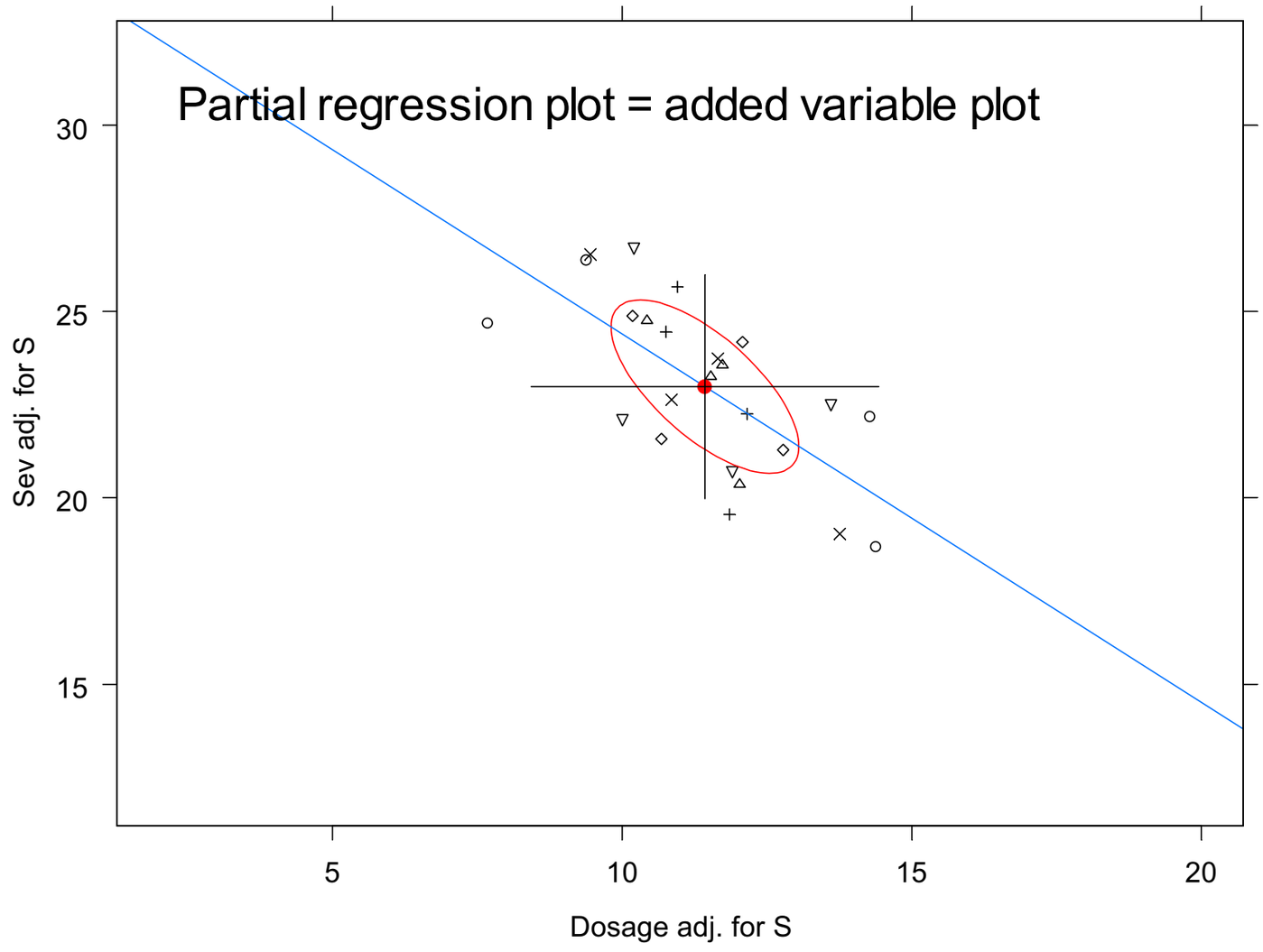
```
> fit <- lm(Sev ~Dosage + ID, Drug)
> summary(fit)

Call: lm(formula = Sev ~Dosage + ID, data = Drug)
Residuals:
    Min      1Q Median    3Q   Max
 -3.005 -1.696  0.519 1.429 2.489

Coefficients:
              Value Std. Error  t value Pr(>|t|)
(Intercept)  34.2840   2.9082   11.7886   0.0000
     Dosage  -0.9888   0.2520   -3.9230   0.0011
        ID1   2.7720   0.7748    3.5776   0.0023
        ID2   1.9424   0.4797    4.0490   0.0008
        ID3   2.4207   0.4005    6.0434   0.0000
```

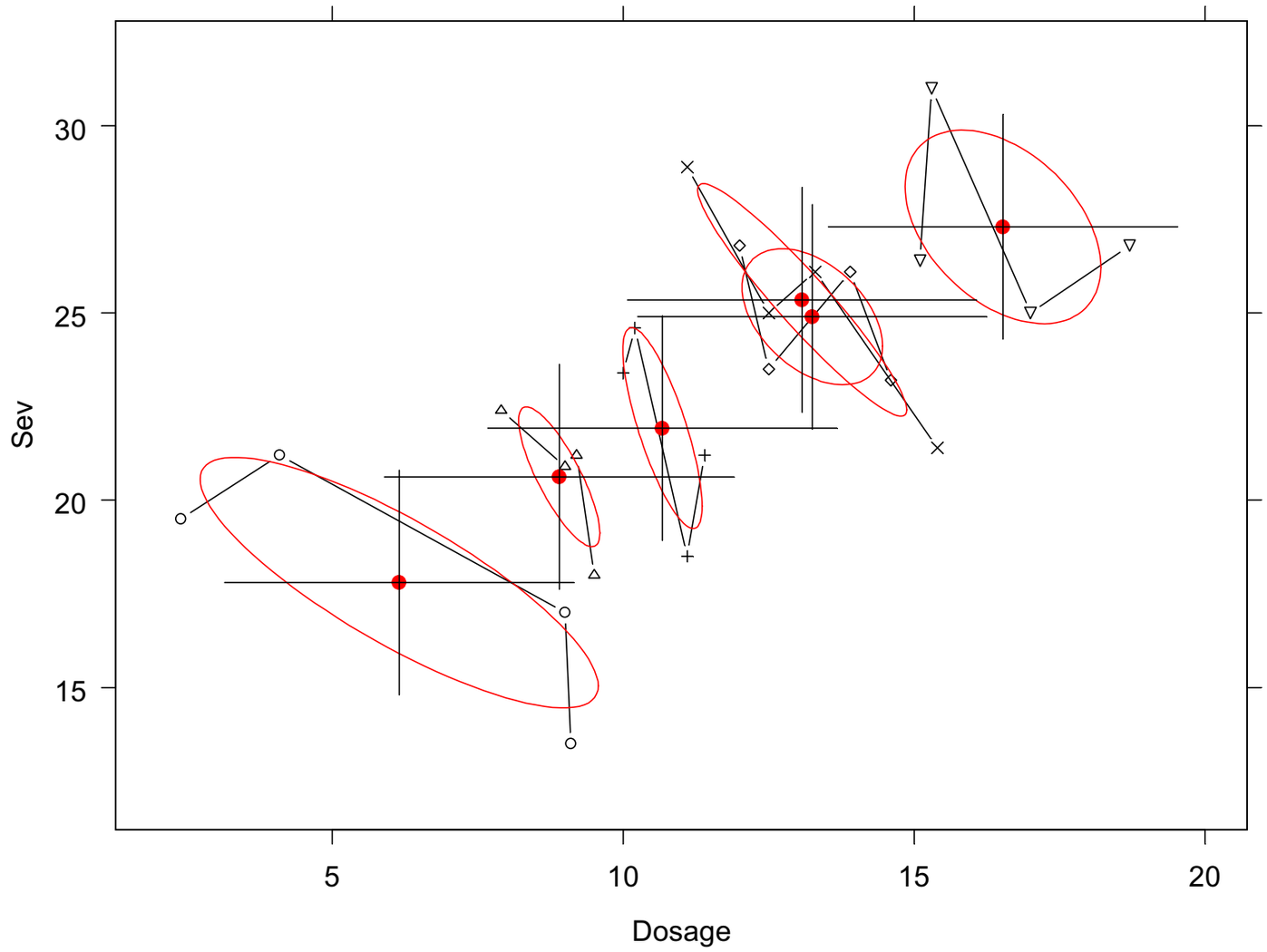
```
        ID4    1.3970    0.2829      4.9377    0.0001
        ID5    1.8710    0.3130      5.9769    0.0000

Residual standard error: 1.96 on 17 degrees of freedom
Multiple R-Squared: 0.8236
F-statistic: 13.23 on 6 and 17 degrees of freedom,
             the p-value is 0.00001393
```

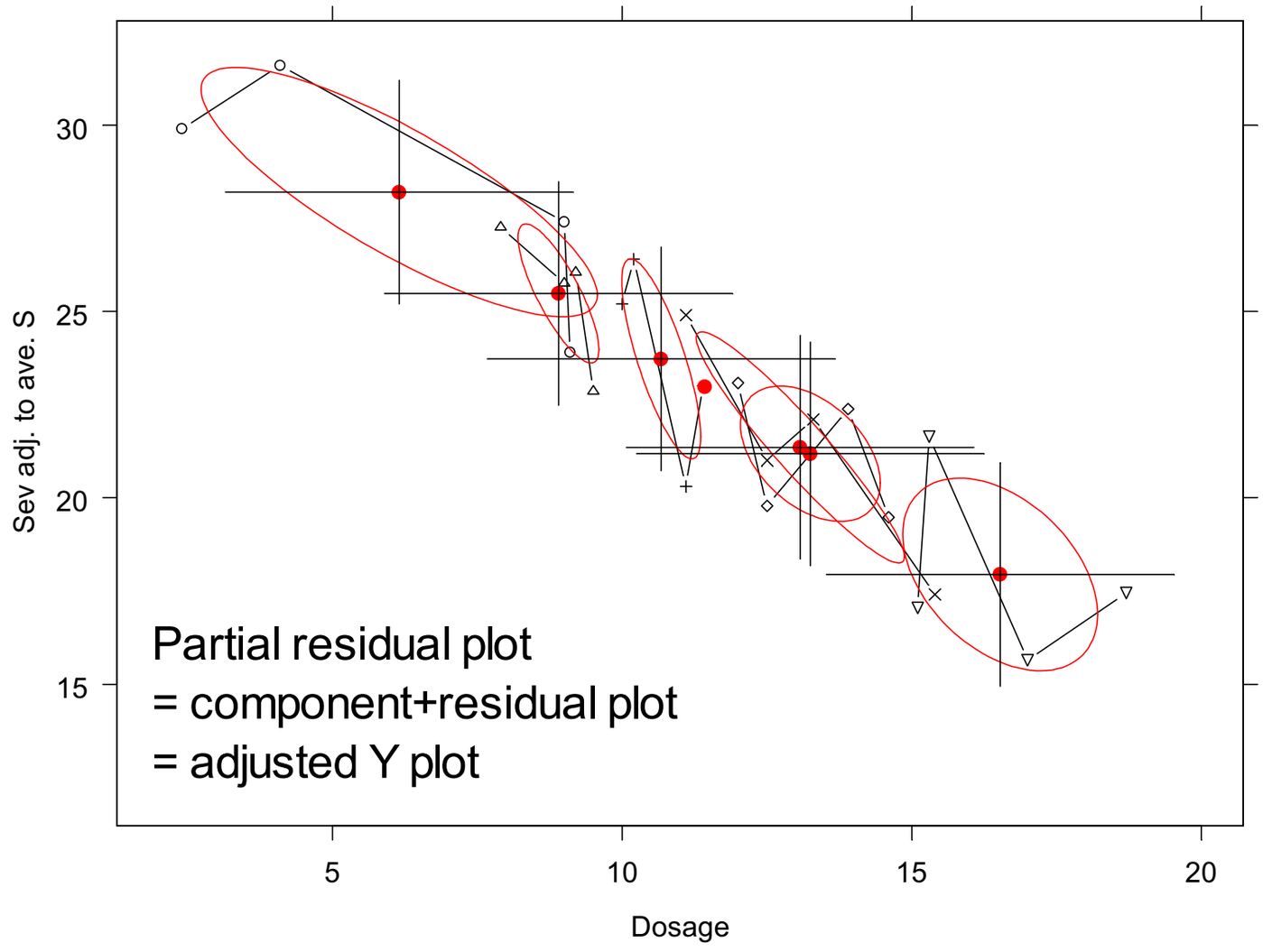

30
25
20
15
Sev
5
10
15
20
Dosage

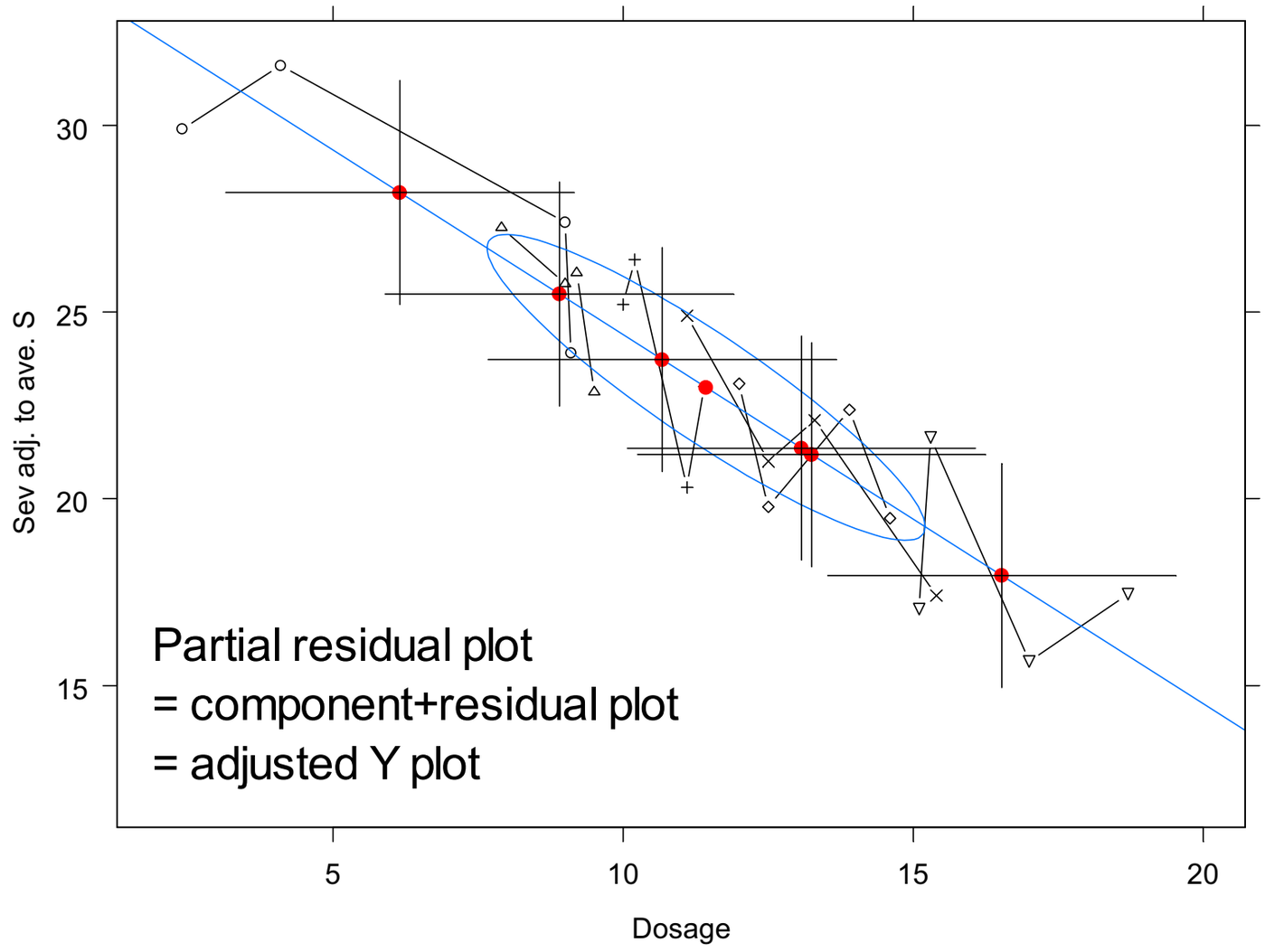

30
25
20
15
Sev adj. for S
5
10
15
20
Dosage adj. for S


Partial regression plot = added variable plot
Sev adj. for S
Dosage adj. for S
30
25
20
15
5
10
15
20


Partial regression plot = added variable plot
Sev adj. for S
Dosage adj. for S
30
25
20
15
5
10
15
20


30
25
20
15
Sev
5
10
15
20
Dosage


Sev adj. to ave. S
30
25
20
15
5
10
15
20
Dosage
Partial residual plot
= component+residual plot
= adjusted Y plot


Sev adj. to ave. S
30
25
20
15
5
10
15
20
Dosage
Partial residual plot
= component+residual plot
= adjusted Y plot

10 Added variable plot

Ideal tool for statistical sleuthing

10.1

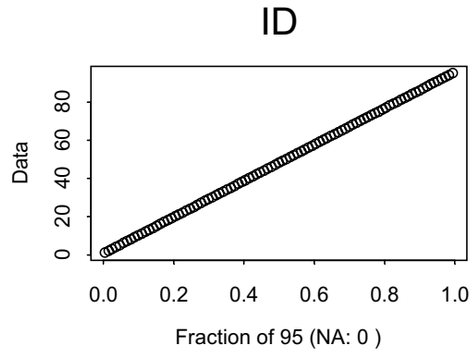
Pay equity in a large law firm

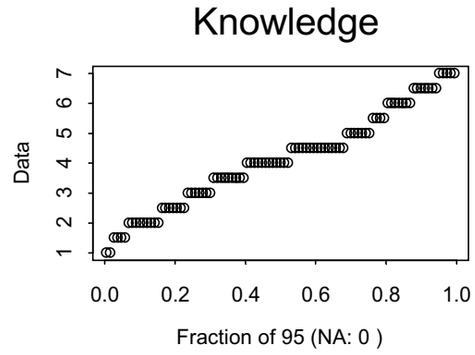
	Title	Knowledge	Experience	Communication	Gender
58	Secretary III	5.0	4.5	4.0	F
70	Admin Asst	6.5	5.5	7.0	M
36	Secretary II	3.5	3.5	2.5	F
74	Secretary III	4.5	5.0	3.5	F
78	Secretary III	3.0	5.5	5.0	F
25	Secretary II	3.0	3.0	2.5	F
19	Admin Asst	7.0	7.0	7.0	M
42	Admin Asst	5.0	6.0	6.5	M
87	Clerk I	1.5	1.0	2.5	F
32	Secretary I	2.5	3.5	2.5	F
66	Admin Asst	4.5	6.0	6.5	F
47	Admin Asst	7.0	7.0	7.0	M
34	Secretary III	4.5	6.5	5.5	F
69	Secretary II	3.0	4.0	4.0	F
21	Secretary III	5.0	5.0	4.5	F
52	Secretary II	3.5	3.5	3.0	F
95	Receptionist	2.5	2.0	4.0	F

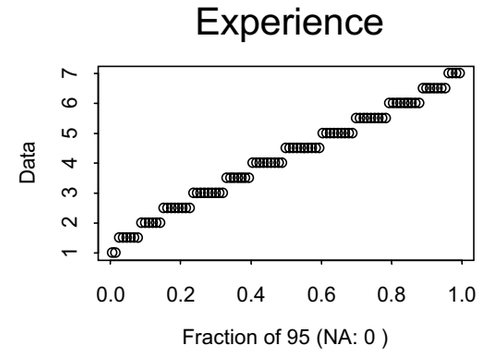
```
89     Admin Asst        5.0          5.5              7.0       M

...    95 positions
```

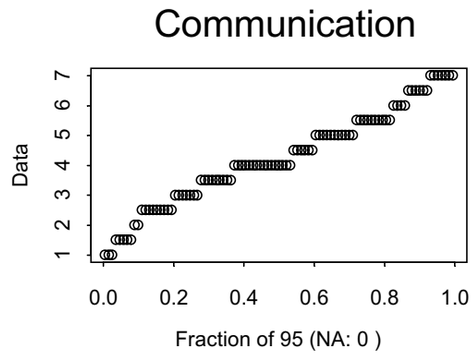
Summary

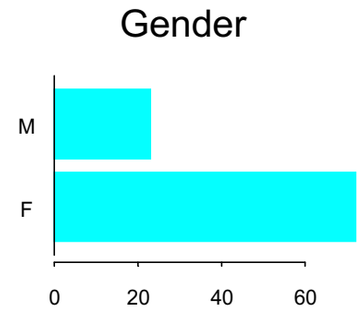
```
> summary(Law)
       ID          Knowledge       Experience   Communication
 Min.: 1.0      Min.:1.000      Min.:1.000      Min.:1.000
1st Qu.:24.5   1st Qu.:3.000   1st Qu.:3.000   1st Qu.:3.000
 Median:48.0    Median:4.000    Median:4.500    Median:4.000
   Mean:48.0      Mean:4.116      Mean:4.184      Mean:4.237
3rd Qu.:71.5   3rd Qu.:5.000   3rd Qu.:5.500   3rd Qu.:5.500
   Max.:95.0      Max.:7.000      Max.:7.000      Max.:7.000

 Gender         Female                  Title         Wage
 F:72        Min.:0.0000   Secretary III:24      Min.:19350
 M:23     1st Qu.:1.0000    Secretary II:24   1st Qu.:39580
           Median:1.0000      Admin Asst:17    Median:50010
             Mean:0.7579         Clerk I:12      Mean:49180
          3rd Qu.:1.0000    Secretary IV: 6   3rd Qu.:58650
             Max.:1.0000     Secretary I: 5      Max.:84190
                                (Other): 7
```

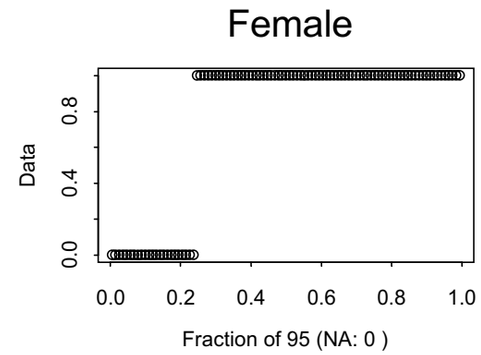

ID
Data
0 20 40 60 80
0.0 0.2 0.4 0.6 0.8 1.0
Fraction of 95 (NA: 0)

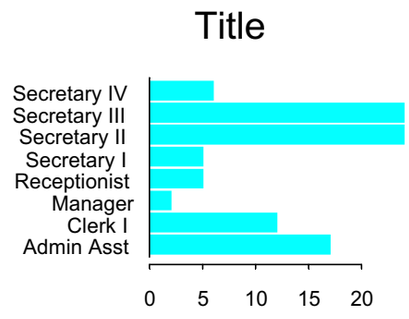

Knowledge
Data
1 2 3 4 5 6 7
0.0 0.2 0.4 0.6 0.8 1.0
Fraction of 95 (NA: 0)

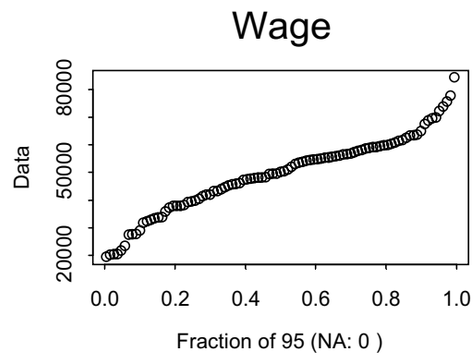

Experience
Data
1 2 3 4 5 6 7
0.0 0.2 0.4 0.6 0.8 1.0
Fraction of 95 (NA: 0)

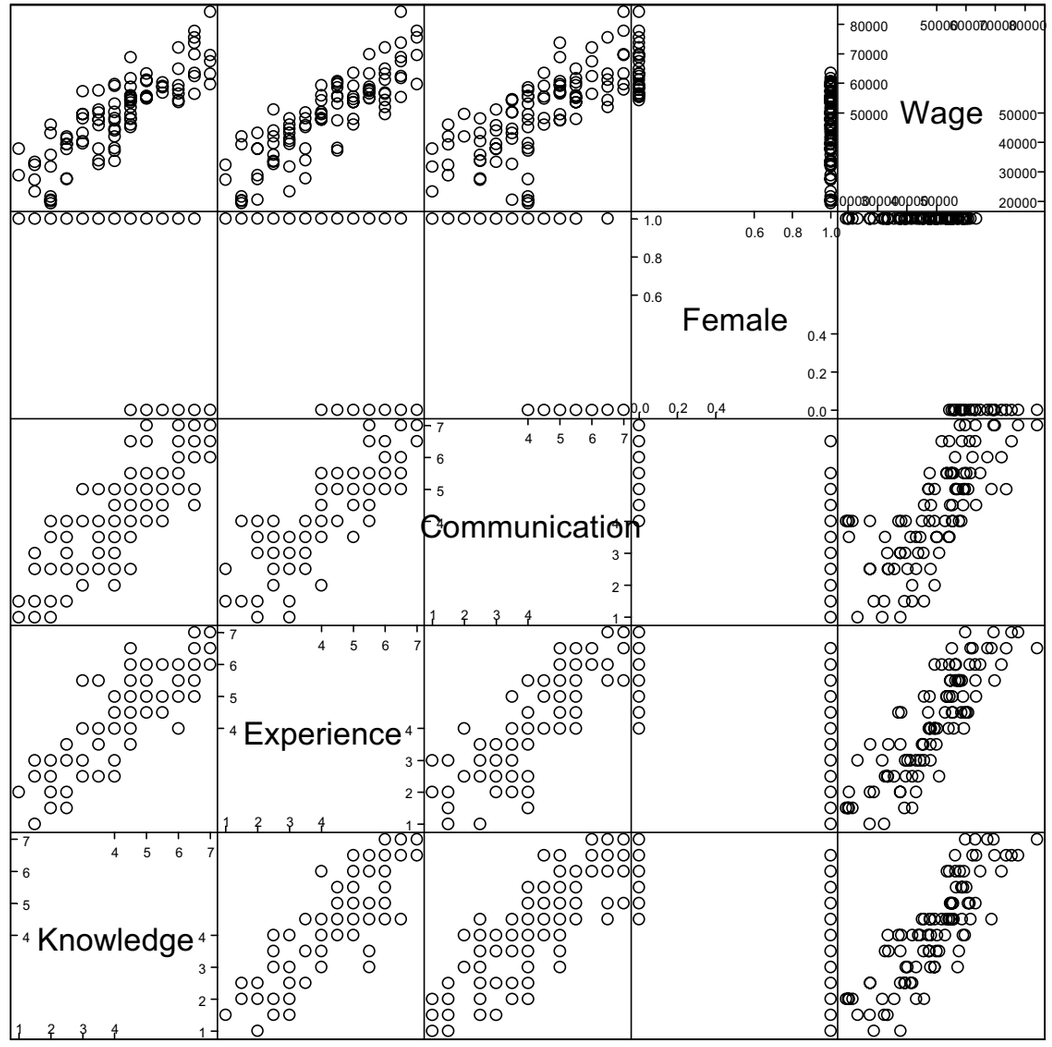

Communication
Data
1 2 3 4 5 6 7
0.0 0.2 0.4 0.6 0.8 1.0
Fraction of 95 (NA: 0)

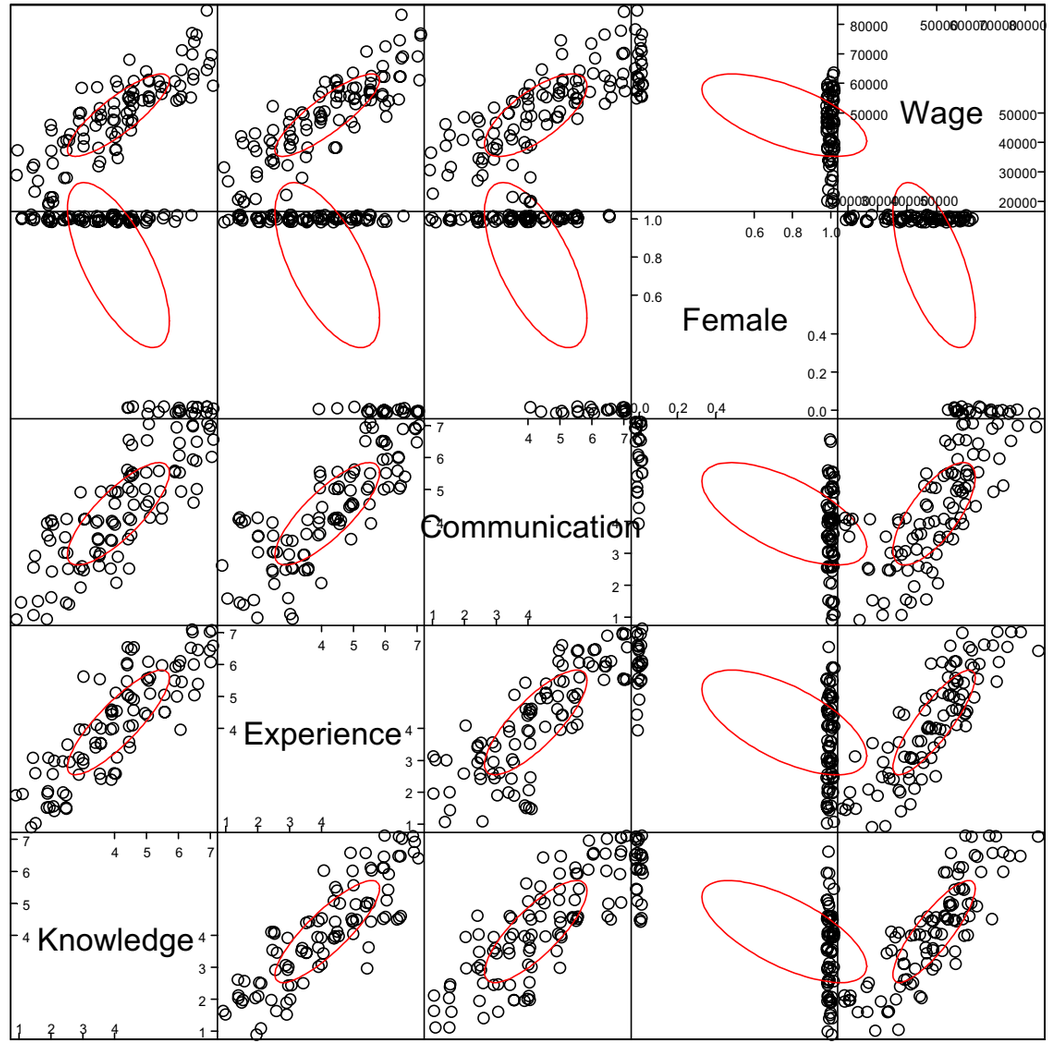

Gender
M
F
0 20 40 60


Female
Data
0.0 0.4 0.8
0.0 0.2 0.4 0.6 0.8 1.0
Fraction of 95 (NA: 0)


Title
Secretary IV
Secretary III
Secretary II
Secretary I
Receptionist
Manager
Clerk I
Admin Asst
0 5 10 15 20


Wage
Data
20000 50000 80000
0.0 0.2 0.4 0.6 0.8 1.0
Fraction of 95 (NA: 0)

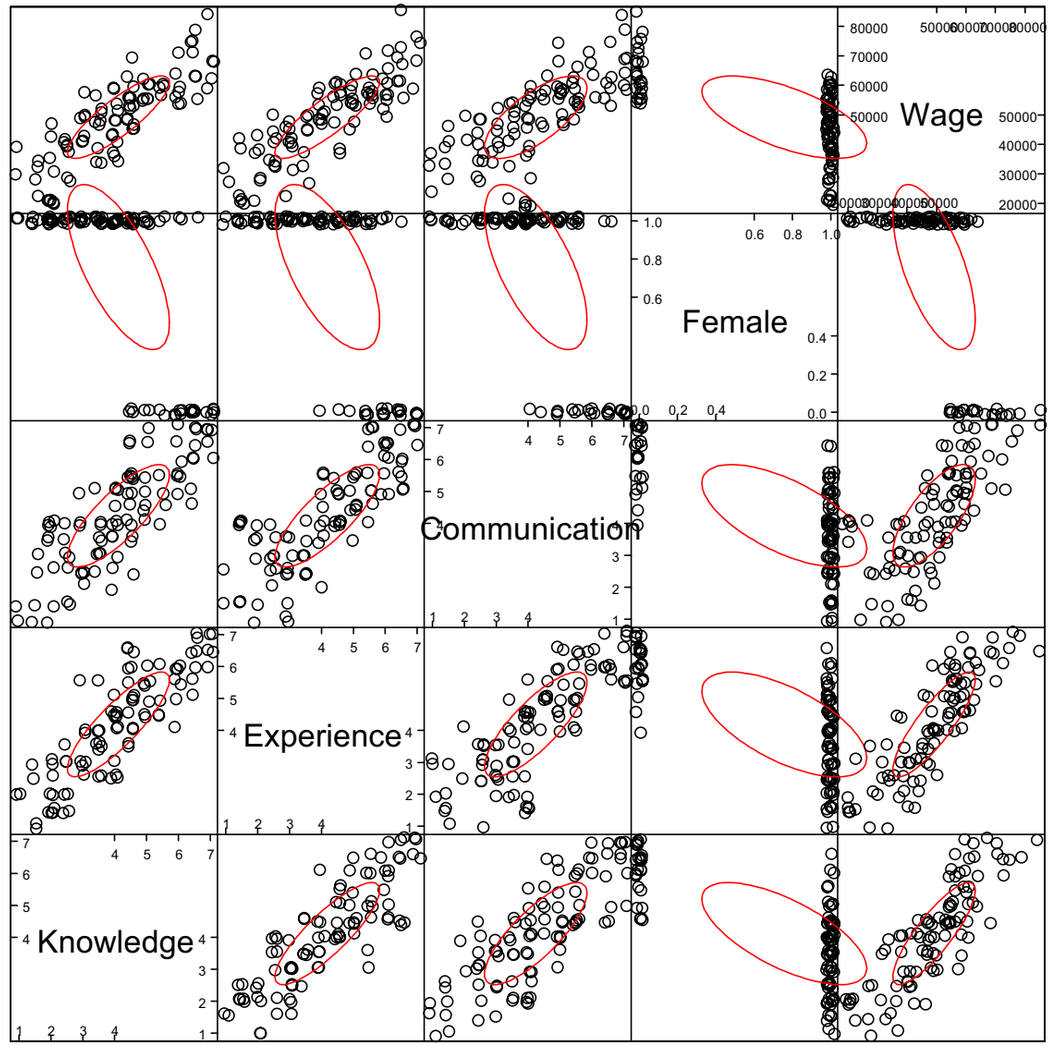

Wage
Female
Communication
Experience
Knowledge

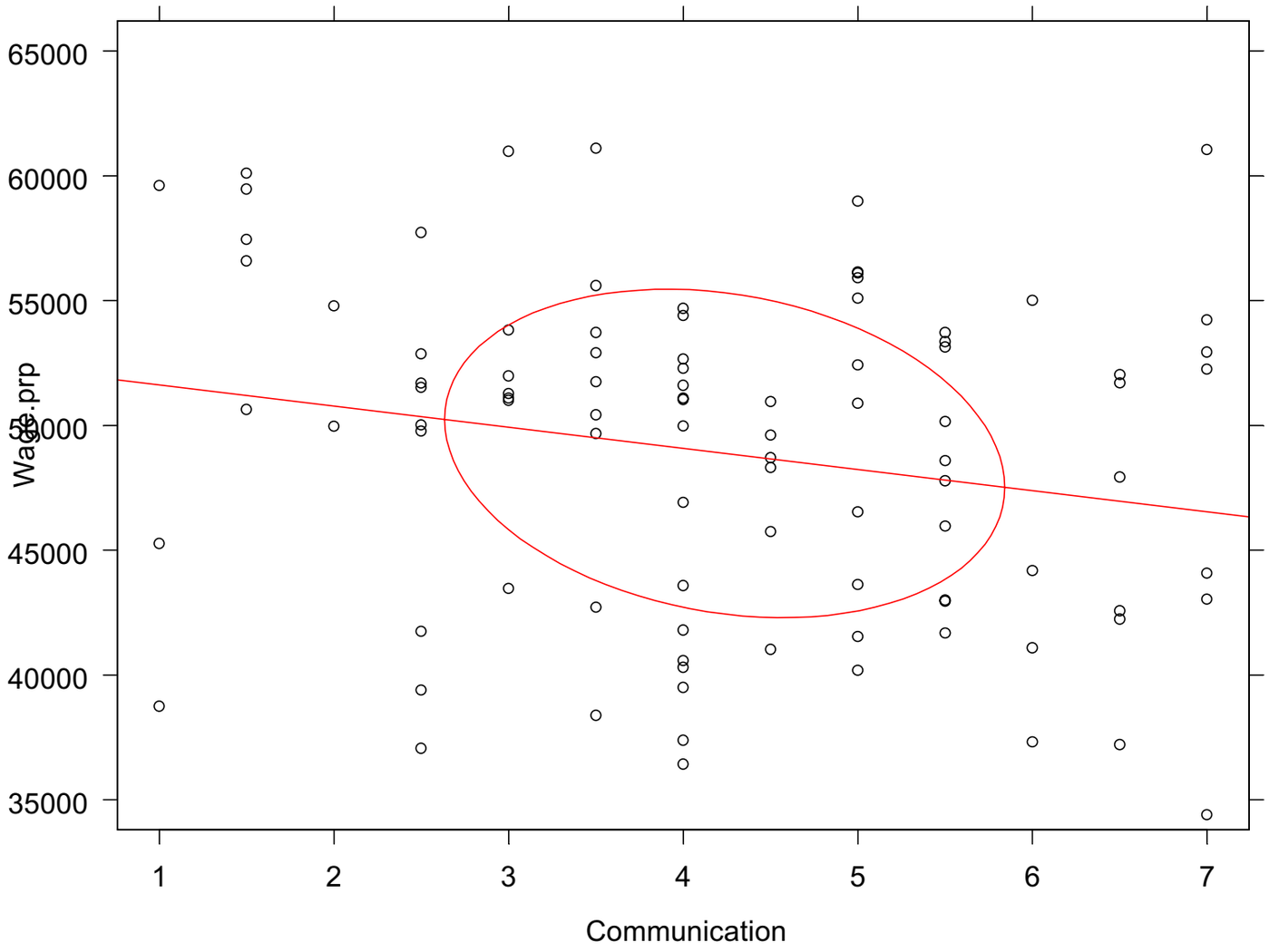

Wage
Female
Communication
Experience
Knowledge

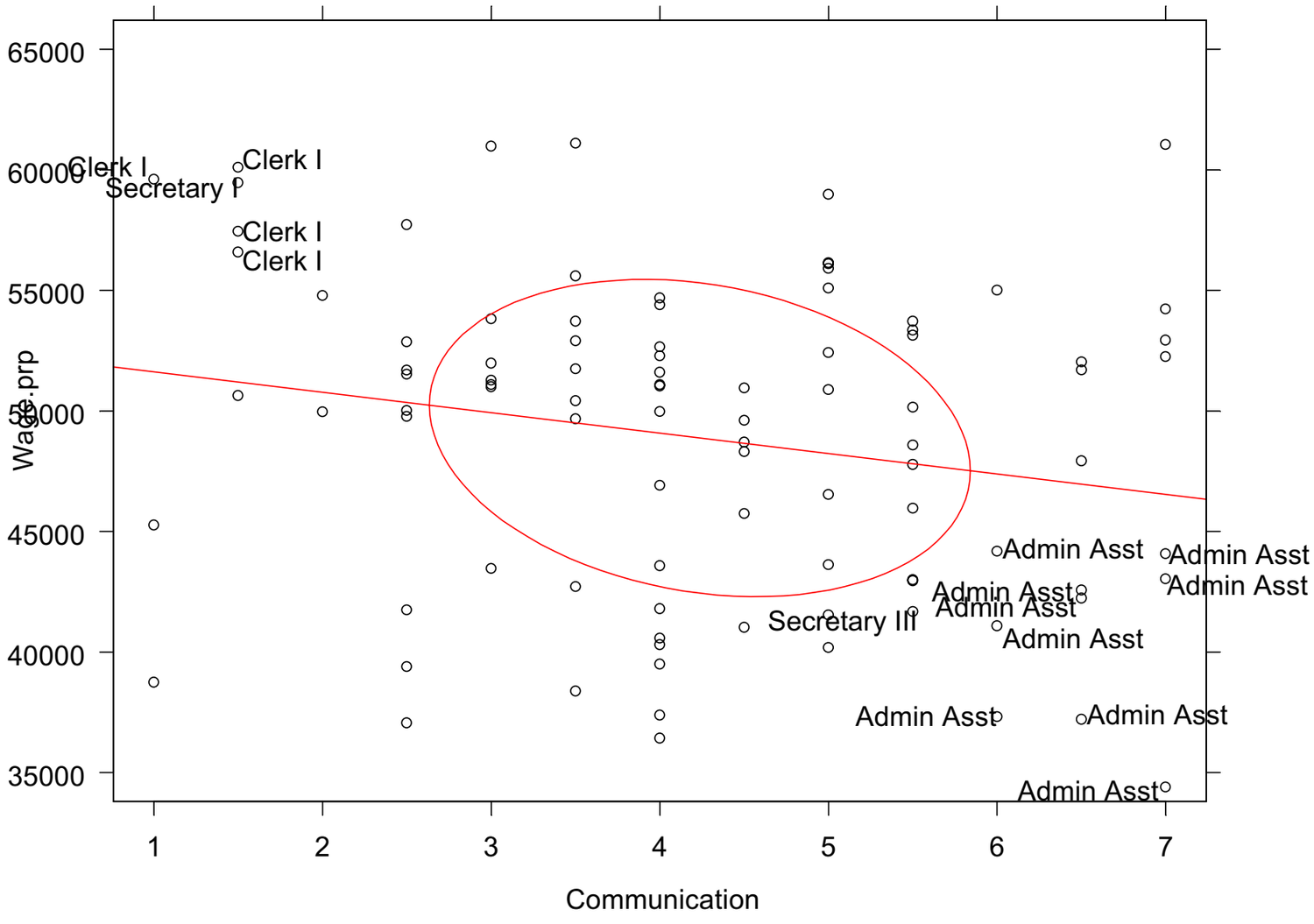
Regression output:

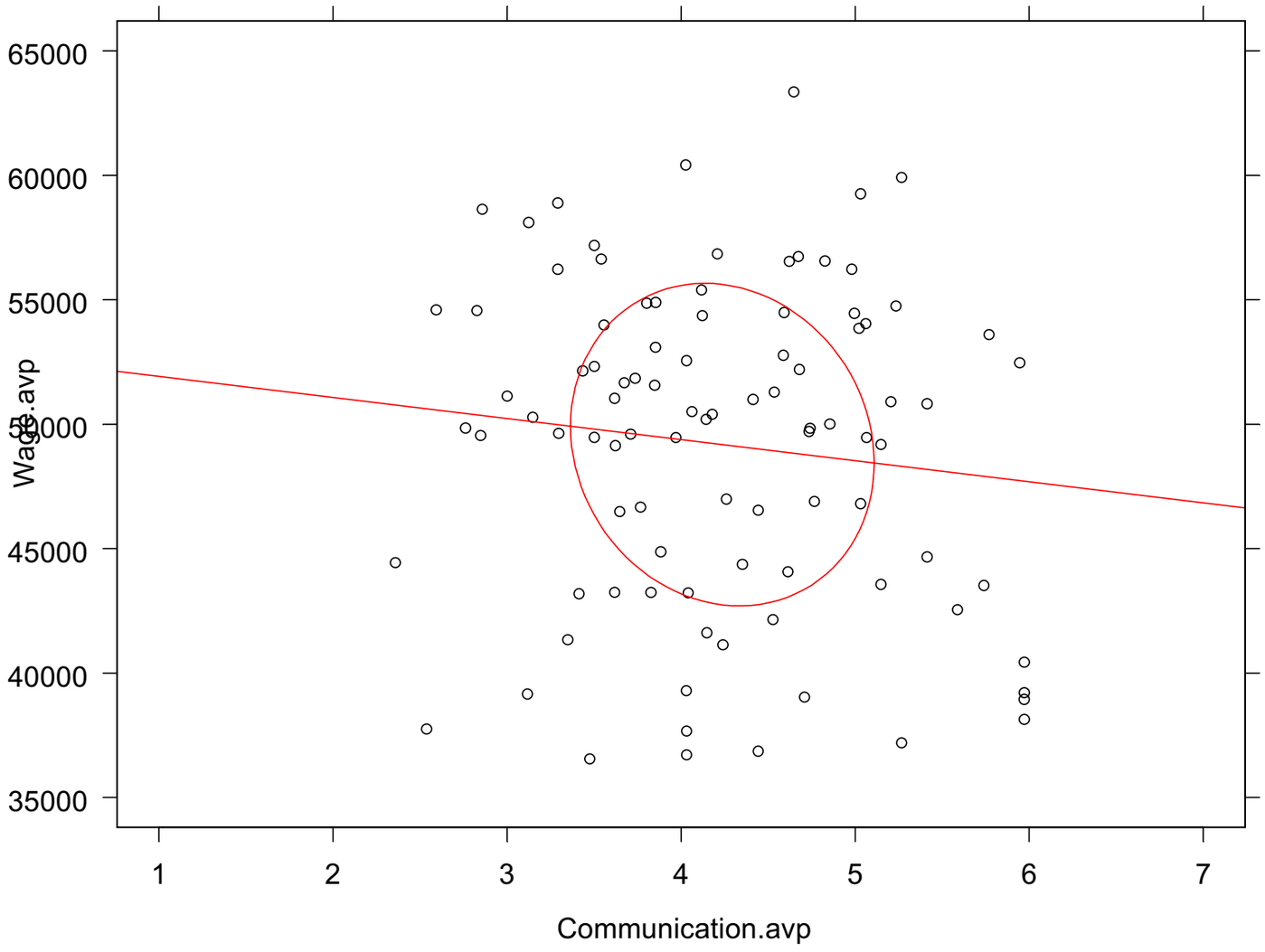
```
> fit <- lm(Wage ~ Knowledge + Experience
                + Communication + Female, data = Law)
> summary(fit)

Call: lm(formula = Wage ~ Knowledge + Experience
              + Communication + Female, data = Law)
Residuals:
    Min     1Q Median    3Q   Max
 -13279  -4824   972.8 5190 14498

Coefficients:
                    Value  Std. Error     t value    Pr(>|t|)
  (Intercept)  20153.7248   4031.6651      4.9989      0.0000
    Knowledge   3825.2482    899.8848      4.2508      0.0001
   Experience   4414.0819    854.7458      5.1642      0.0000
Communication   -846.8698    777.7750     -1.0888      0.2791
       Female  -2113.9833   2206.0361     -0.9583      0.3405

Residual standard error: 6577 on 90 degrees of freedom
```

```
Multiple R-Squared: 0.7864
F-statistic: 82.86 on 4 and 90 degrees of freedom,
the p-value is 0
```


Wage
Female
Communication
Experience
Knowledge


65000
60000
55000
50000
45000
40000
35000
Wage.prp
1
2
3
4
5
6
7
Communication


65000
60000
55000
50000
45000
40000
35000
Wage.prp
1
2
3
4
5
6
7
Communication
Clerk I
Clerk I
Secretary I
Clerk I
Clerk I
Admin Asst
Admin Asst
Admin Asst
Admin Asst
Admin Asst
Secretary III
Admin Asst
Admin Asst
Admin Asst
Admin Asst


65000
60000
55000
50000
45000
40000
35000
Wage.avp
1
2
3
4
5
6
7
Communication.avp

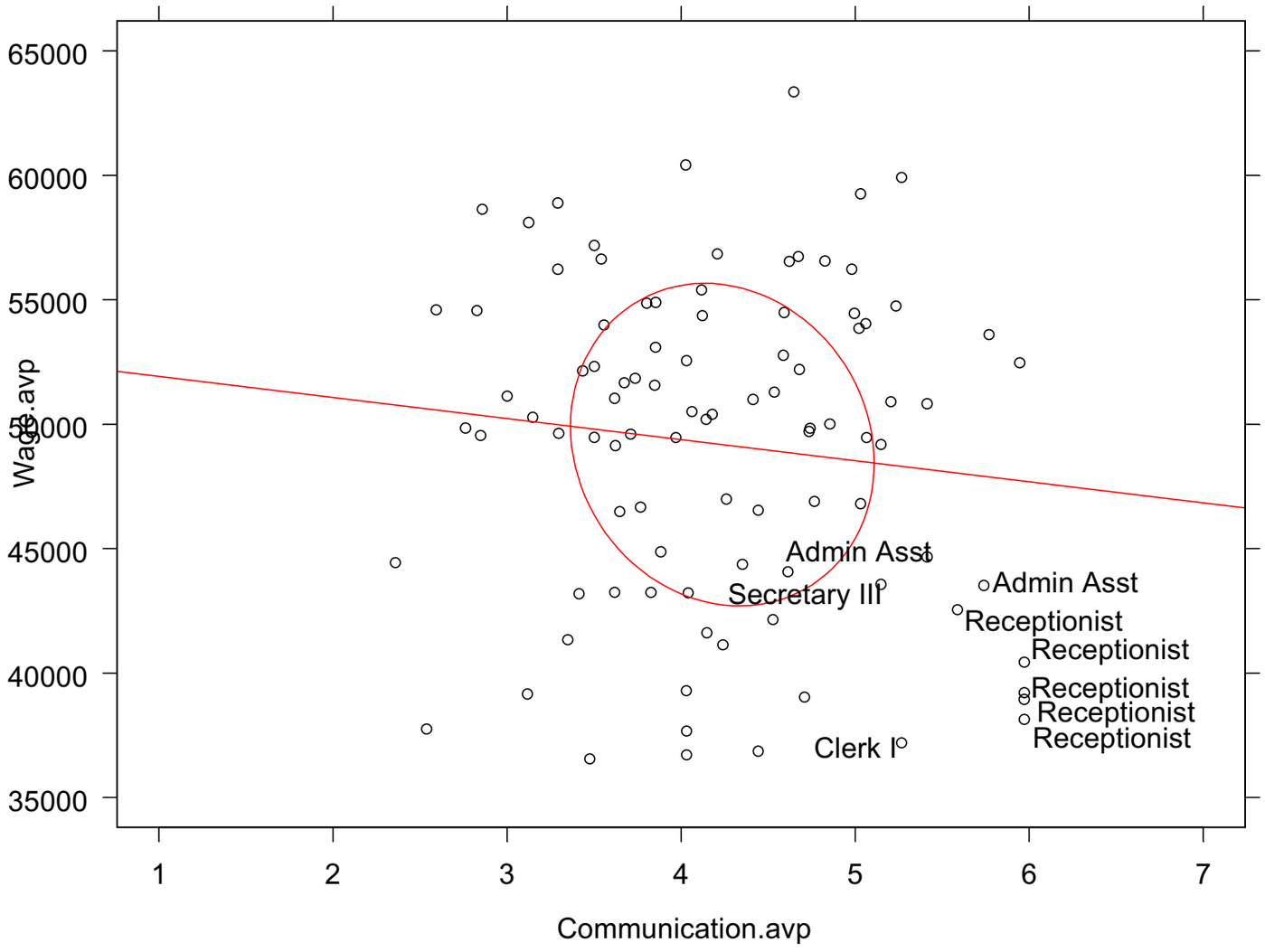
Note:

Horizontal width of ellipse in PRP $= s_X$

Horizontal width of ellipse in AVP $= s_{X \cdot (\text{other Xs})}$

$$\sqrt{VIF} = \frac{s_X}{s_{X \cdot (\text{other Xs})}}$$

So showing the PRP and the AVP side-by-side on the same horizontal scale can be used to convey information on multicollinearity.


65000
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Wage.avp
1
2
3
4
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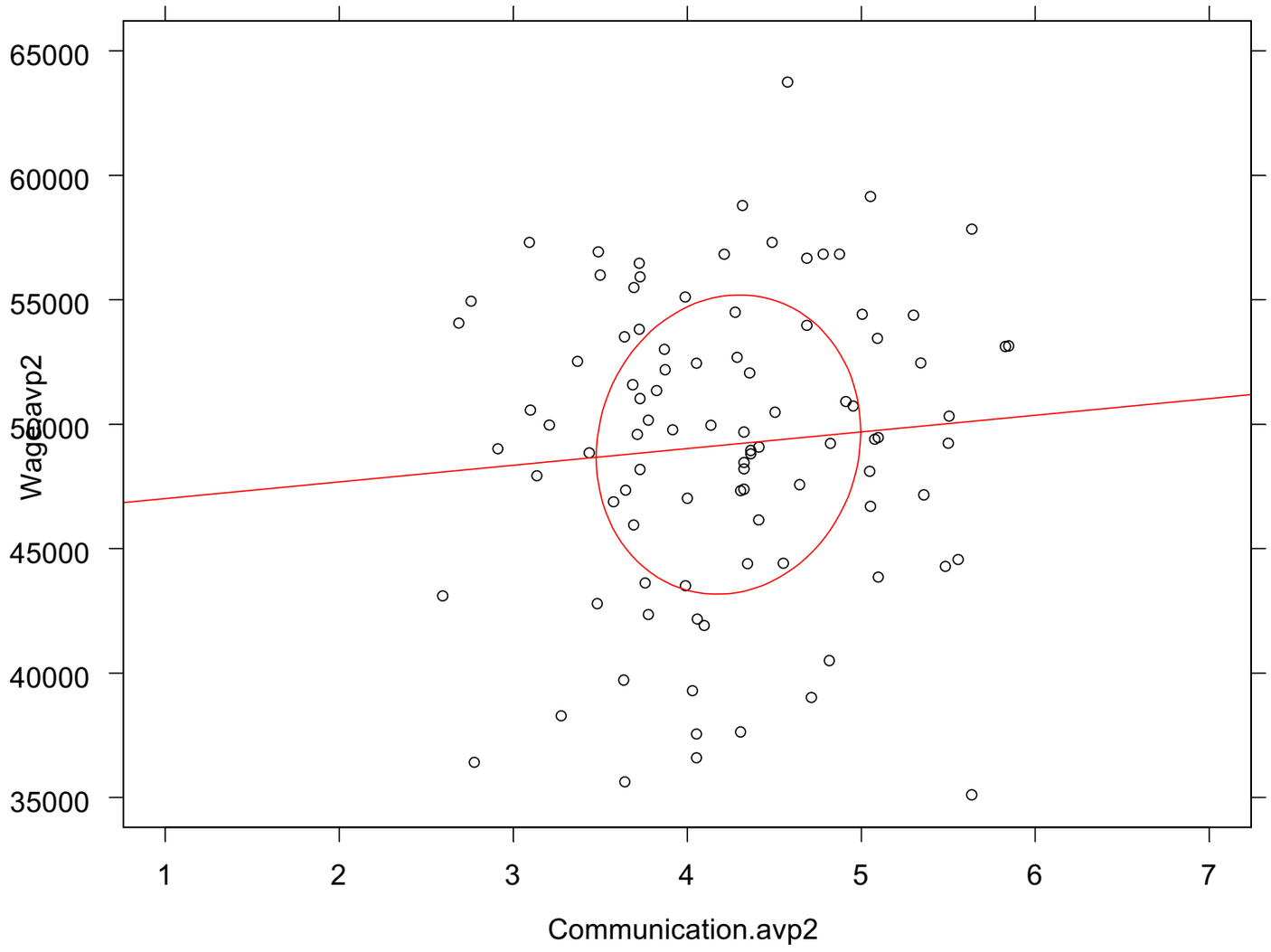
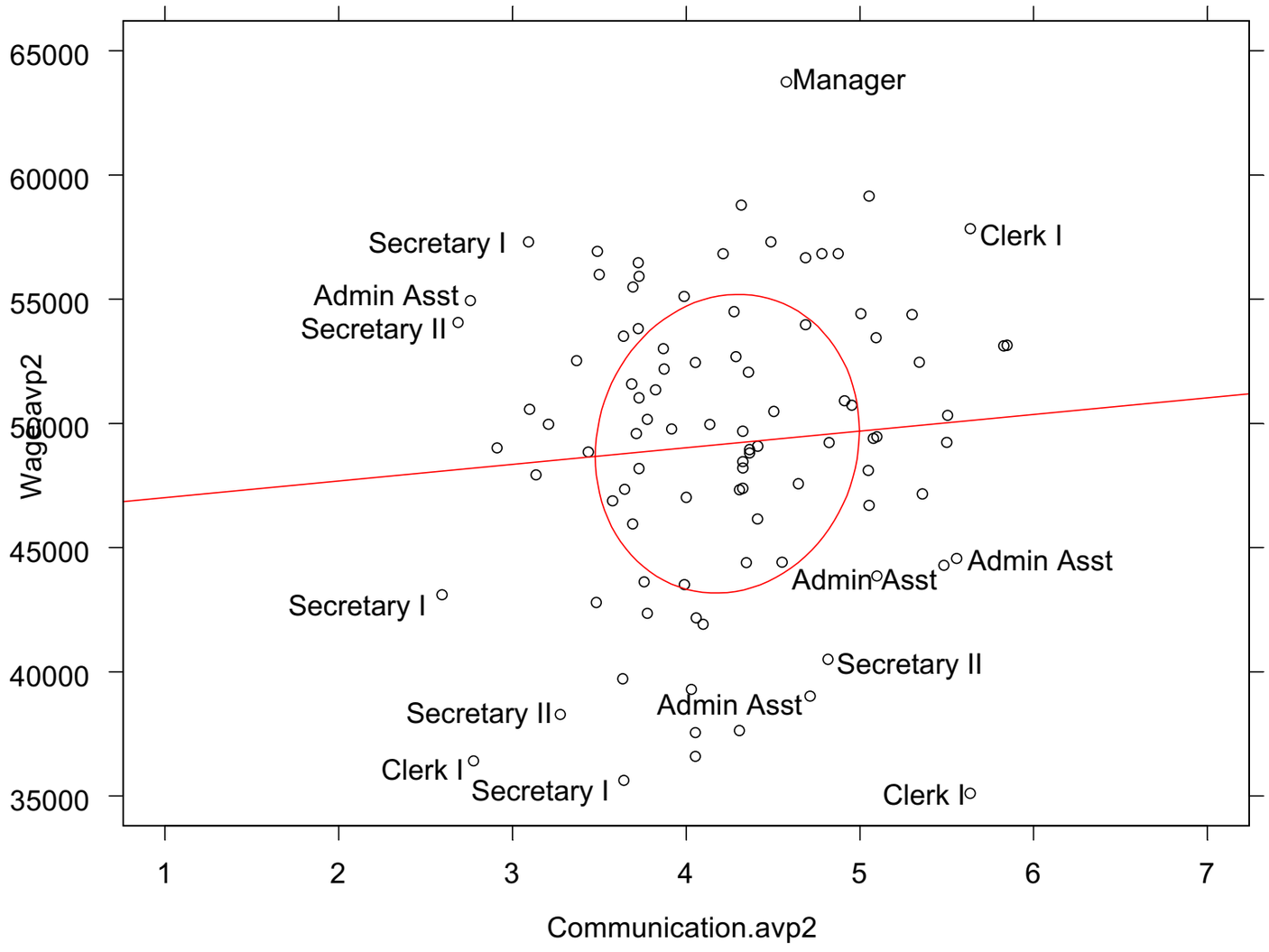
6
7
Communication.avp
Admin Asst
Secretary III
Admin Asst
Receptionist
Receptionist
Receptionist
Receptionist
Receptionist
Clerk I

Model with adjustment for Receptionists:

```
> fit <- lm( Wage ~Knowledge + Experience + Communication +
              Female + Recept, Law)

> summary(fit)

Call: lm(formula = Wage ~Knowledge + Experience + Communication +
                                    Female + Recept, data = Law)
Residuals:
    Min     1Q Median    3Q   Max
 -15008 -3101  441.7 4261 14321

Coefficients:
                    Value  Std. Error     t value    Pr(>|t|)
  (Intercept)  23317.3282   3869.9155      6.0253      0.0000
    Knowledge   3234.6101    857.4269      3.7725      0.0003
   Experience   2955.3080    892.0224      3.3130      0.0013
Communication    671.6964    835.8108      0.8036      0.4237
       Female  -2606.6538   2069.4971     -1.2596      0.2111
       Recept -13098.1539   3539.6990     -3.7004      0.0004
```

```
Residual standard error: 6158 on 89 degrees of freedom
Multiple R-Squared: 0.8149
F-statistic: 78.37 on 5 and 89 degrees of freedom, the p-value is
```


65000
60000
55000
50000
45000
40000
35000
Wage.avp2
1
2
3
4
5
6
7
Communication.avp2


65000
60000
55000
50000
45000
40000
35000
Wage.avp2
1
2
3
4
5
6
7
Communication.avp2
Manager
Clerk I
Secretary I
Admin Asst
Secretary II
Admin Asst
Admin Asst
Secretary I
Secretary II
Secretary II
Admin Asst
Clerk I
Secretary I
Clerk I

17 References:

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