

Tools for General Parametric Splines

2
'spida' package in R developed for
the Summer Program In Data Analysis (SPIDA)
at York University

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Flexible tools to explore general parametric splines:

- Splines are conceptually simple but generally awkward to use in statistical analyses
- Goal: make splines as easy to use in regression as categorical variables
- “General parametric spline:”
 - Polynomials of different degrees in each interval
 - Different degrees of smoothness at each knot – including possible discontinuity
- Natural parametrization with interpretable coefficients:
 - can use regular expression to test groups of coefficients
- Can estimate and test features of spline: slope, curvature, salti.
- Can interact with other variables (numerical or categorical), other splines, etc.
- Limitation: need to centre and scale variable

Installation

- 1) Install R <http://cran.r-project.org/>

Only once: Install 'car' and 'spida'. After starting R:

```
>1 install.packages('car')  
> install.packages('spida',  
                    repos = 'http://r-forge.r-project.org')
```

- 2) Each time you run R: load spida
> `library(spida)`

¹ The '>' sign is the prompt in R. Type only what comes after the '>'.

Main Functions:

- gsp** Creates general parametric splines
- wald** Uses regular expressions or hypothesis matrices to estimate coefficients and test hypotheses
- Lmat** Generates hypothesis matrices
- sc** Operates on spline functions to generate hypothesis matrices for spline derivatives: slope, curvature, salti at knots
- smsp** Non-parametric splines using mixed models

Example:

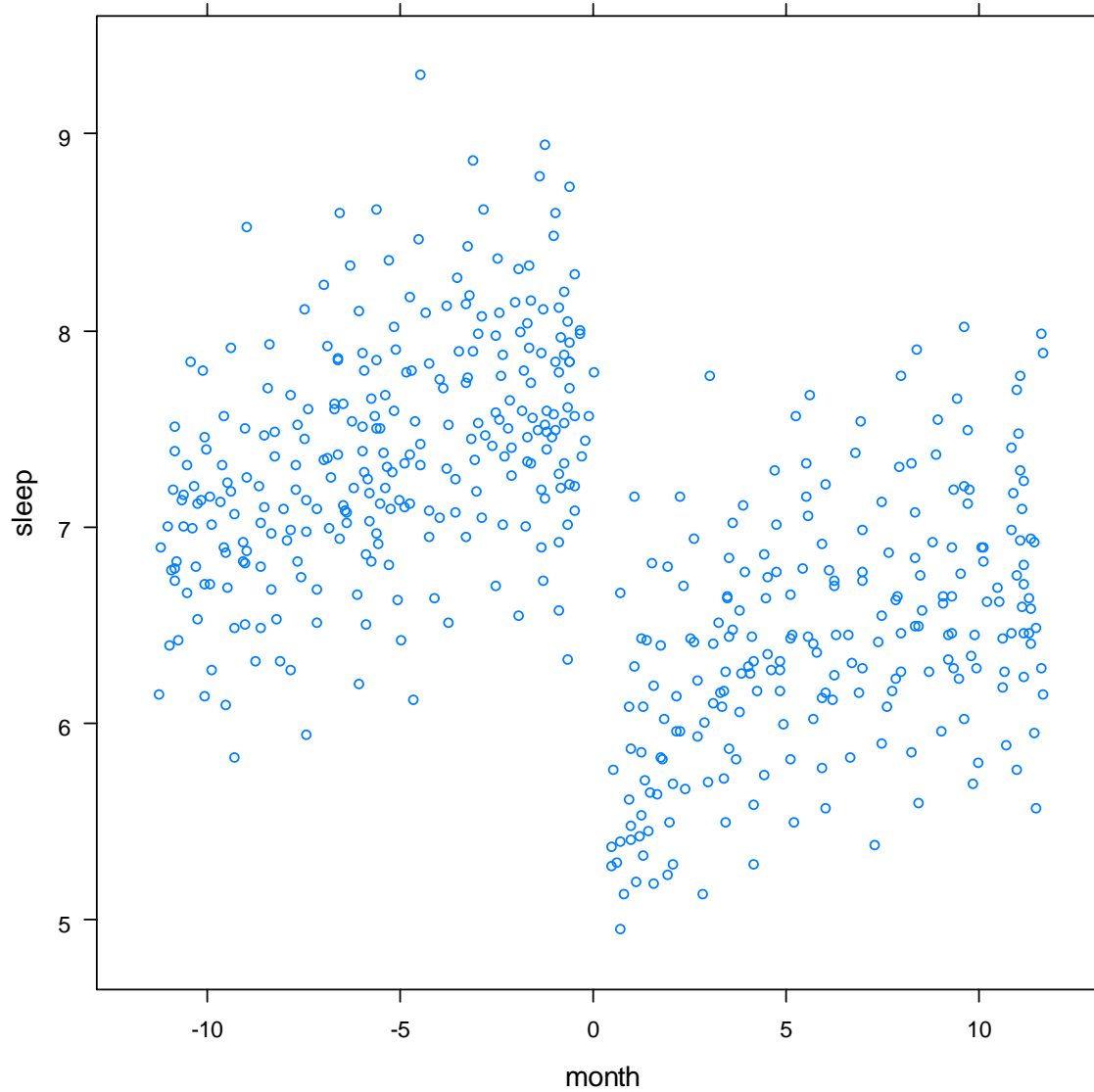
Simulated sleep data before and after birth.

Inspired by Kunkel, Reitav, Monette (2010)

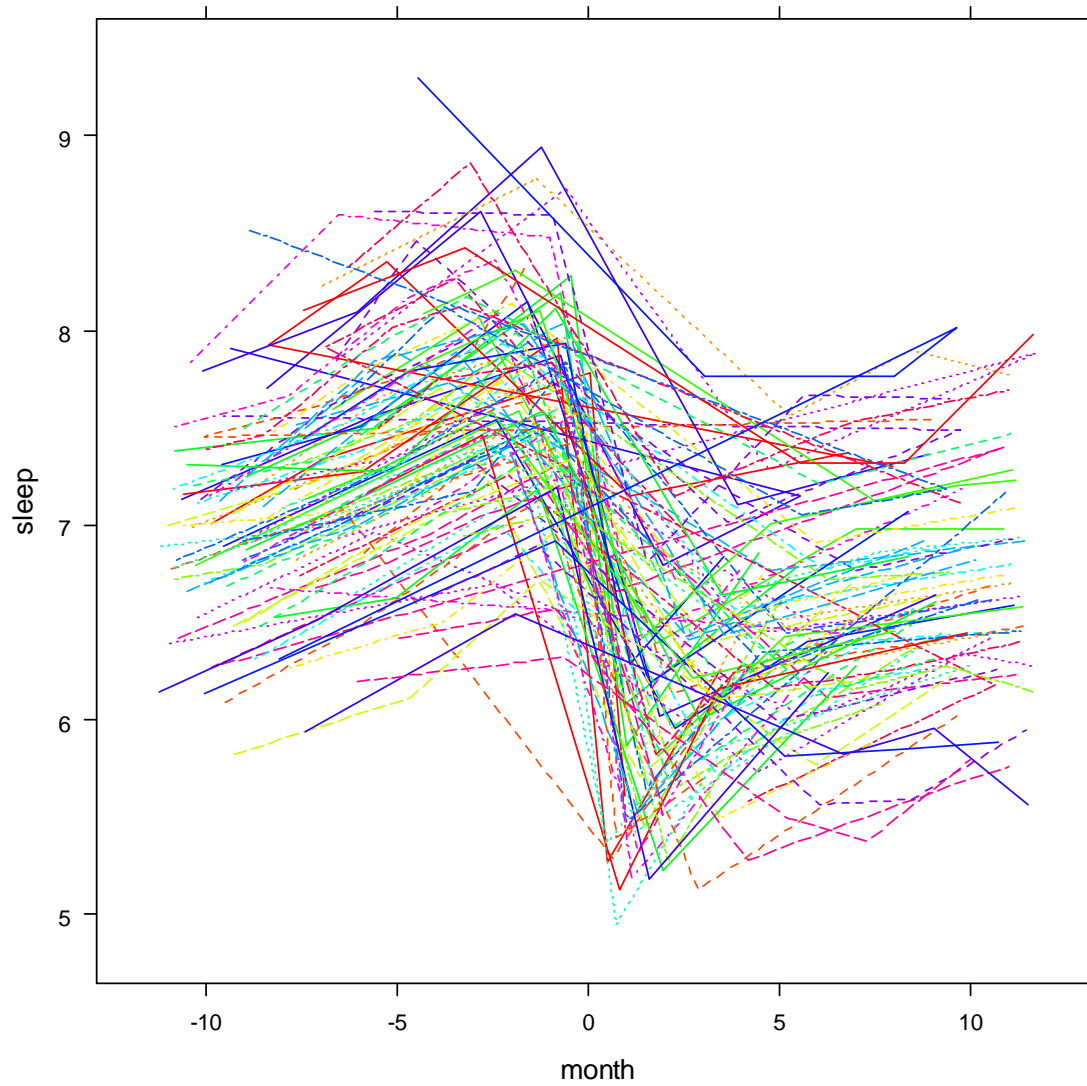
100 women each observed 5 times:

```
> sim.data <- read.csv(
  'http://www.math.yorku.ca/~georges/Data/sim.data.csv')
> dim(sim.data)
[1] 500    4
> head( sim.data)
   X      month id  sleep
1 1 -9.86370968  1 7.008448
2 2 -6.95388563  1 7.336828
3 3 -2.11407625  1 7.642227
4 4  0.05344575  1 7.778133
5 5  0.97390029  1 5.609873
6 6 -7.66649560  2 7.184266
```

```
> xyplot( sleep ~ month, sim.data)
```



```
> td( col = rainbow(20))  
> xyplot( sleep ~ month, sim.data,  
          groups = id, type = 'l')
```



Defining a spline function:



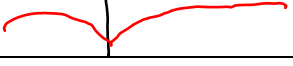

Create your own spline function

```
> sp <- function( x ) gsp( x,  
    knots = c(-5, 0, 5), # position of knots  
    degree = c(2, 2, 2, 2), # quad in each interval  
    smooth = c(1,-1,1)) # smoothness at knot
```

degree parameter:

1,2,3,...	linear, quadratic, cubic, etc.
0	flat
-1	equal to 0

smooth parameter:

1	'smooth': same slope on both sides of knot	
2	same curvature on both sides	
0	continuous (no gap)	
-1	discontinuous	


What `sp` does:

Generates a portion of a model matrix:


```
> sp( c(-5,0,5) )
```

	D1(0)	D2(0)	C(0).0	C(0).1	C(0).2	C(5).2	C(-5).2
f(-5)	-5	12.5	0	0	0.0	0	0
f(0)	0	0.0	0	0	0.0	0	0
f(5)	5	12.5	1	5	12.5	0	0


linear
term



saltus
in level
at 0



change
in quad.
term at 5



Using your **sp** in a linear multilevel mixed model:

```
> library( nlme )
> fit <- lme( sleep ~ sp(month), sim.data,
             random = ~ 1 | id )
> summary( fit )
Linear mixed-effects model fit by REML
Data: sim.data
      AIC      BIC    logLik
384.8794 426.8642 -182.4397

Random effects:
Formula: ~1 | id
      (Intercept)  Residual
StdDev:  0.4922676 0.2439248

. . . .
```

Fixed effects: sleep ~ sp(month)

	Value	Std.Error	DF	t-value	p-value
(Intercept)	7.770252	0.07562772	393	102.74344	0.0000
sp(month)D1(0)	0.007316	0.04231428	393	0.17289	0.8628
sp(month)D2(0)	-0.027195	0.01204080	393	-2.25853	0.0245
sp(month)C(0).0	-2.284334	0.10143225	393	-22.52078	0.0000
sp(month)C(0).1	0.295341	0.06867966	393	4.30027	0.0000
sp(month)C(0).2	-0.015195	0.01850084	393	-0.82130	0.4120
sp(month)C(5).2	0.027771	0.02041699	393	1.36018	0.1746
sp(month)C(-5).2	-0.048239	0.01943028	393	-2.48267	0.0135

Standardized Within-Group Residuals:

Min	Q1	Med	Q3	Max
-6.955982485	-0.326223723	-0.009829296	0.320549996	7.518591530

Number of Observations: 500

Number of Groups: 100

$$\begin{array}{l} - - = - \quad \hat{c} \\ - - = + \quad \hat{c} \end{array}$$

```
> round(model.matrix(fit) [1:5,],4)
```

```
(Intercept) sp(month)D1(0) sp(month)D2(0) sp(month)C(0).0
1           1           -9.8637           48.6464           0
2           1           -6.9539           24.1783           0
3           1           -2.1141            2.2347           0
4           1            0.0534            0.0014           1
5           1            0.9739            0.4742           1
sp(month)C(0).1 sp(month)C(0).2 sp(month)C(5).2 sp(month)C(-5).2
1           0.0000           0.0000           0           -11.8278
2           0.0000           0.0000           0           -1.9088
3           0.0000           0.0000           0            0.0000
4           0.0534           0.0014           0            0.0000
5           0.9739           0.4742           0            0.0000
```

Natural spline

Linear in the extreme intervals

You can create almost every kind of spline used in regression

```
> spn <- function( x ) gsp( x,  
    knots      = c(-10,-5, 0, 5,10),  
    degree     = c(1, 2, 2, 2, 2, 1),  
    smooth     = c( 1, 1,-1, 1, 1))  
  
> fitn <- lme( sleep ~ spn(month),  
    sim.data, random = ~ 1 | id )  
  
> summary(fitn)  
  
. . .
```

Fixed effects: sleep ~ spn(month)

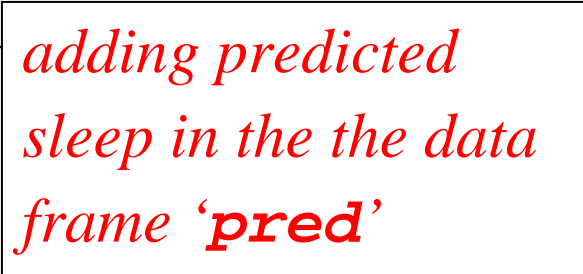
	Value	Std.Error	DF	t-value	p-value
(Intercept)	7.770115	0.07572201	393	102.61369	0.0000
spn(month)D1(0)	0.007163	0.04256467	393	0.16829	0.8664
spn(month)D2(0)	-0.027267	0.01214689	393	-2.24475	0.0253
spn(month)C(0).0	-2.278971	0.10190952	393	-22.36269	0.0000
spn(month)C(0).1	0.290233	0.06946431	393	4.17816	0.0000
spn(month)C(0).2	-0.013372	0.01882470	393	-0.71035	0.4779
spn(month)C(5).2	0.024200	0.02135186	393	1.13338	0.2577
spn(month)C(-5).2	-0.048604	0.01975742	393	-2.46003	0.0143

Note: Although the estimable coefficients have the same interpretation for both splines, the natural spline matrix is necessarily different. There are no estimable coefficients attached to the boundary knots.

Viewing the fitted model:

Create a data frame of values of functional predictors – here just months:

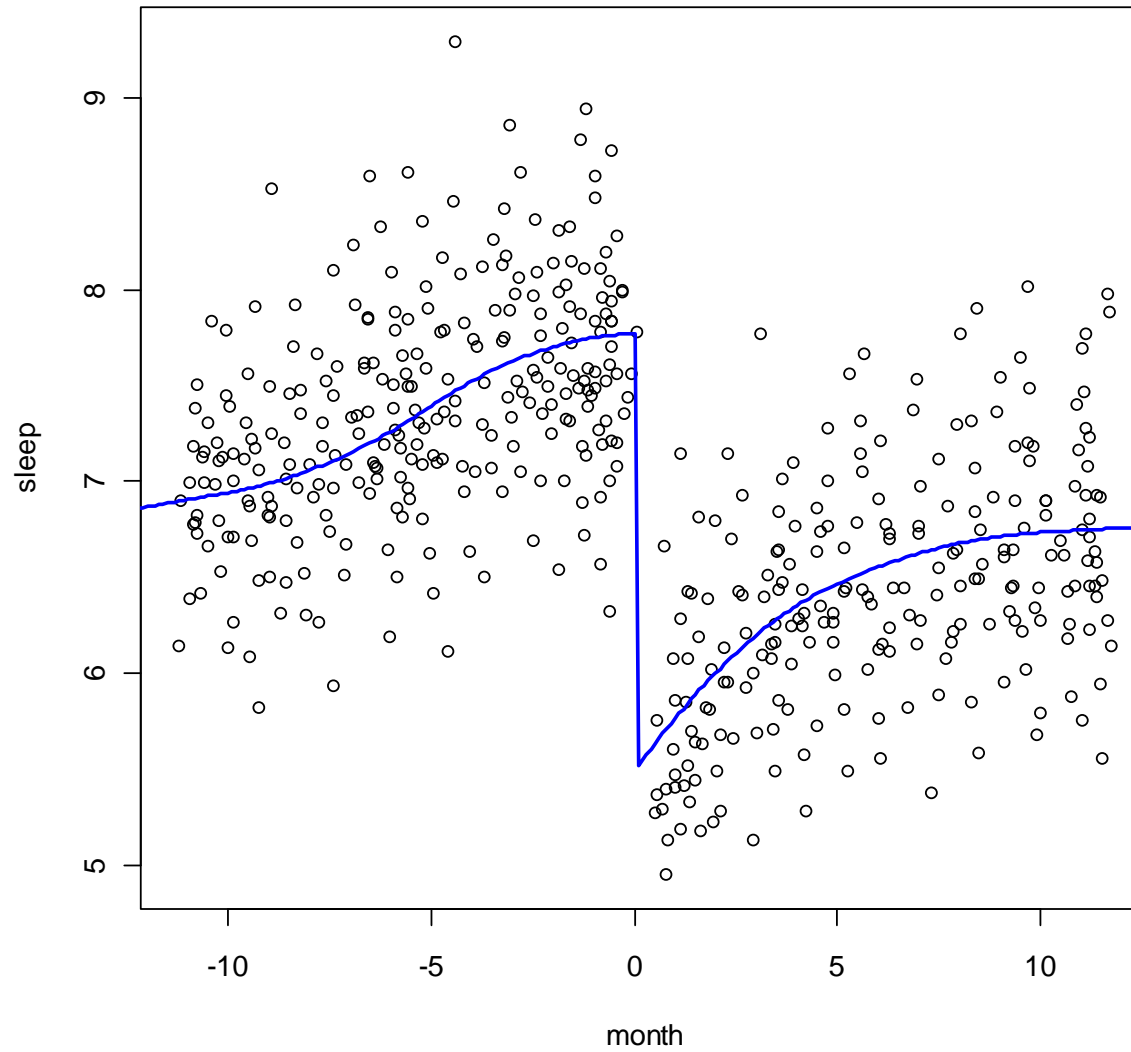
```
> pred <- expand.grid( month = seq(-13,13,.1))2
> pred$sleep <- predict( fitn, pred, level =0)
      # Add predicted sleep to the 'pred' data frame
> head( pred )
  month    sleep
1 -13.0 6.832263
2 -12.9 6.835944
3 -12.8 6.839625
4 -12.7 6.843306
5 -12.6 6.846987
6 -12.5 6.850669
```



*adding predicted
sleep in the the data
frame 'pred'*

² I used `expand.grid` although it is really needed only for multiple functional predictors

```
> plot( sleep ~ month, sim.data)
> lines( sleep ~ month, pred, col = 'blue', lwd = 2)
```



Estimation and testing:

Overall test for effect of time:

```
> wald( fitn, 'month')  
  
      numDF denDF  F.value p.value  
month      7   393 366.2914 <.00001
```

'month' is a "regular expression" matching names of coefficients

Coefficients	Estimate	Std.Error	DF	t-value	p-value
spn(month)D1(0)	0.007163	0.042565	393	0.168293	0.86644
spn(month)D2(0)	-0.027267	0.012147	393	-2.244749	0.02534
spn(month)C(0).0	-2.278971	0.101910	393	-22.362687	<.00001
spn(month)C(0).1	0.290233	0.069464	393	4.178156	0.00004
spn(month)C(0).2	-0.013372	0.018825	393	-0.710346	0.47791
spn(month)C(5).2	0.024200	0.021352	393	1.133384	0.25774
spn(month)C(-5).2	-0.048604	0.019757	393	-2.460034	0.01432

Can we get rid of knots and just use a quadratic?

```
> wald( fitn, 'C')
```

```
numDF denDF F.value p.value
```

```
C      5    393 402.4999 <.00001
```

No



Coefficients	Estimate	Std.Error	DF	t-value	p-value
spn(month)C(0).0	-2.278971	0.101910	393	-22.362687	<.00001
spn(month)C(0).1	0.290233	0.069464	393	4.178156	0.00004
spn(month)C(0).2	-0.013372	0.018825	393	-0.710346	0.47791
spn(month)C(5).2	0.024200	0.021352	393	1.133384	0.25774
spn(month)C(-5).2	-0.048604	0.019757	393	-2.460034	0.01432

Can we get rid of non-zero knots?

```
> wald( fitn, 'C\\(5|C\\(-5')  
# '(' is a special character and needs to be 'escaped' to  
# have its literal meaning  
# '|' means 'or'  
      numDF denDF  F.value p.value  
C\\(5|C\\(-5      2   393 3.646731 0.02696
```

Coefficients	Estimate	Std.Error	DF	t-value	p-value
spn(month)C(5).2	0.024200	0.021352	393	1.133384	0.25774
spn(month)C(-5).2	-0.048604	0.019757	393	-2.460034	0.01432

Using a hypothesis matrix:

```
> Lm <- rbind("at 5 mos" = c(0,0,0,0,0,0,1,0),
              "at -5 mos" = c(0,0,0,0,0,0,0,1))
> Lm
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
at 5 mos    0    0    0    0    0    0    1    0
at -5 mos    0    0    0    0    0    0    0    1

> wald( fitn, Lm)
  numDF denDF  F.value p.value
1      2   393 3.646731 0.02696

      Estimate Std.Error  DF   t-value p-value
at 5 mos  0.024200  0.021352 393   1.133384 0.25774
at -5 mos -0.048604  0.019757 393  -2.460034 0.01432
```

Generating Standard Errors

Wald tests generate estimated standard errors for coefficients, we need to capture them for plotting

Example: plotting the estimated response

L matrix as model matrix evaluated over a range of predictors

```
> Lest <- with( pred, cbind( 1, spn(month)))
> dim(Lest)
[1] 261    8
> head(Lest)
      D1(0) D2(0) C(0).0 C(0).1 C(0).2 C(5).2 C(-5).2
f(-13)    1 -13.0    80      0      0      0      0    -27.5
f(-12.9)  1 -12.9    79      0      0      0      0    -27.0
f(-12.8)  1 -12.8    78      0      0      0      0    -26.5
f(-12.7)  1 -12.7    77      0      0      0      0    -26.0
f(-12.6)  1 -12.6    76      0      0      0      0    -25.5
f(-12.5)  1 -12.5    75      0      0      0      0    -25.0
```

```

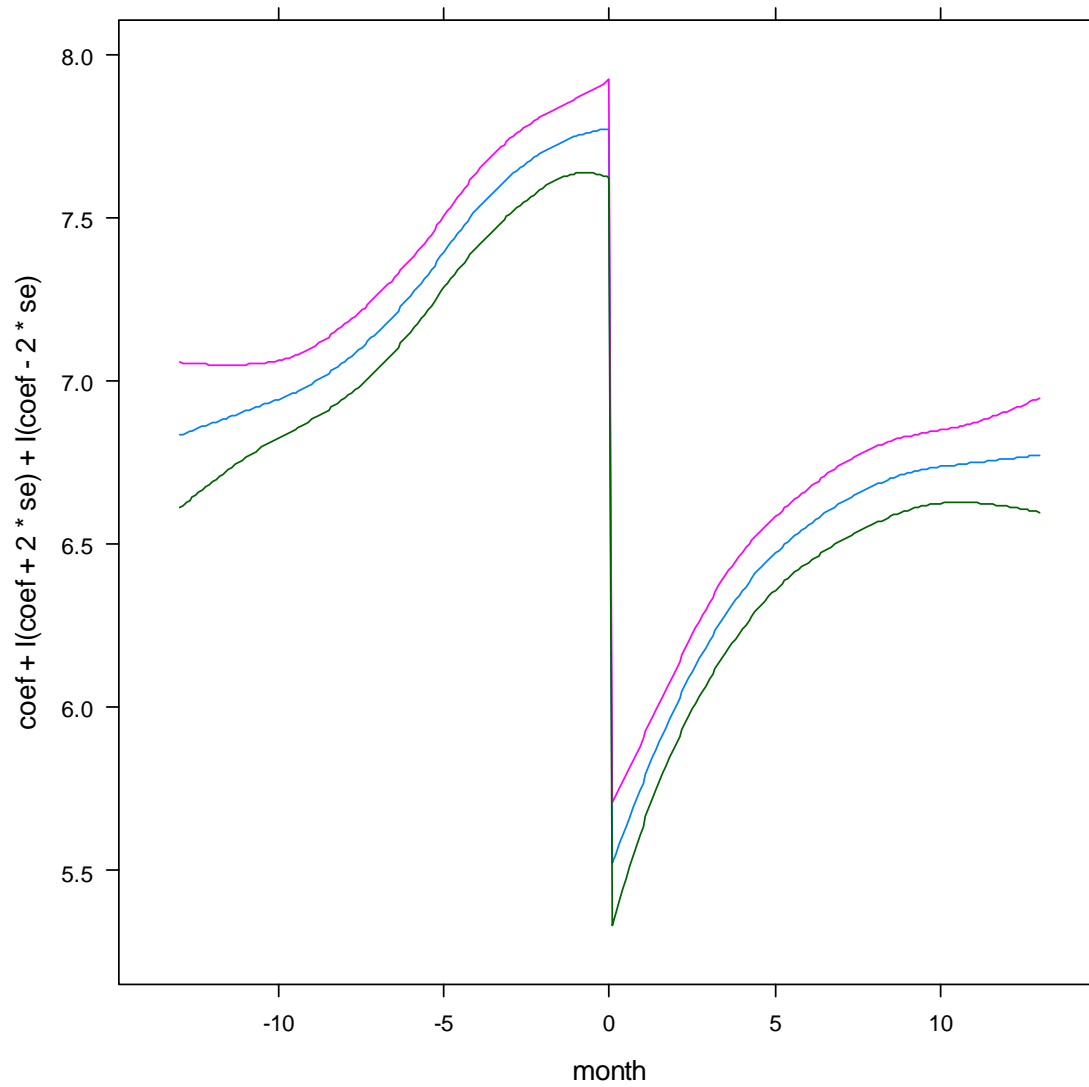
> head( pred )
  month      sleep
1 -13.0  6.832263
2 -12.9  6.835944
3 -12.8  6.839625
4 -12.7  6.843306
5 -12.6  6.846987
6 -12.5  6.850669

> ww <- as.data.frame(wald(fitn, Lest))
      # capture estimates and SEs
> ww <- cbind( ww, pred)
      # combine back with pred
> head( ww )
      coef          se month      sleep
f(-13)  6.832263 0.1111460 -13.0  6.832263
f(-12.9) 6.835944 0.1089242 -12.9  6.835944
f(-12.8) 6.839625 0.1067182 -12.8  6.839625
f(-12.7) 6.843306 0.1045293 -12.7  6.843306
f(-12.6) 6.846987 0.1023583 -12.6  6.846987
f(-12.5) 6.850669 0.1002066 -12.5  6.850669

```


Easy plotting:

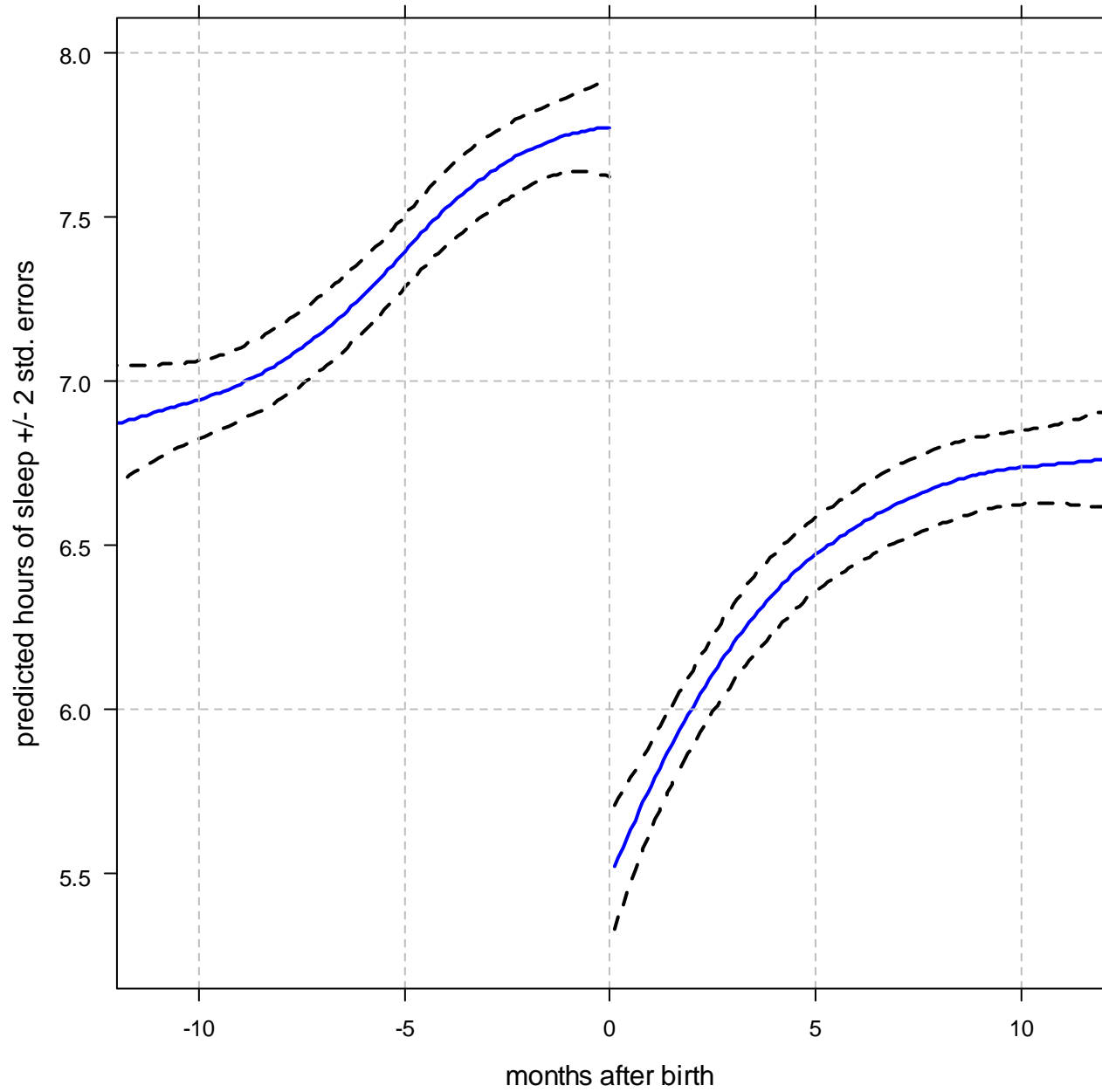
```
> xyplot(coef + I(coef + 2*se) + I(coef - 2*se) ~  
month, ww, type = 'l')
```



Fancier plotting:

```
> td( col = c('blue','black','black'),
      lty = c( 1,2,2), lwd = 2)

> xyplot( coef + I(coef + 2*se) + I(coef - 2*se) ~
          month,
          rbind(ww[1:131,],NA,ww[132:261,]),
          type = 'l',
          panel = function(x,y,...) {
            panel.xyplot( x, y, ...)
            panel.abline( v = c(-10,-5,0,5,10),
                          col = 'gray', lty = 2)
            panel.abline( h = seq(0,10,1),
                          col = 'gray', lty = 2)
          },
          xlim = c(-12,12),
          ylab =
            'predicted hours of sleep +/- 2 std. errors',
          xlab = "months after birth")
```



Inference concerning a spline

All fixed effects coefficients:

```
> wald( fitn )
      numDF denDF  F.value p.value
      8     393 2688.578 <.00001

Coefficients      Estimate Std.Error  DF      t-value p-value
(Intercept)       7.770115  0.075722 393  102.613693 <.00001
spn(month)D1(0)   0.007163  0.042565 393   0.168293 0.86644
spn(month)D2(0)  -0.027267  0.012147 393  -2.244749 0.02534
spn(month)C(0).0 -2.278971  0.101910 393 -22.362687 <.00001
spn(month)C(0).1  0.290233  0.069464 393   4.178156 0.00004
spn(month)C(0).2 -0.013372  0.018825 393  -0.710346 0.47791
spn(month)C(5).2  0.024200  0.021352 393   1.133384 0.25774
spn(month)C(-5).2 -0.048604  0.019757 393  -2.460034 0.01432
```

Test saltus at birth (month = 0):

```
> wald( fitn , "C\\(0.*0$")
      numDF denDF  F.value p.value
C\\(0.*0$      1     393 500.0898 <.00001

Coefficients      Estimate Std.Error  DF      t-value p-value
spn(month)C(0).0 -2.278971  0.10191 393 -22.36269 <.00001
```

Better labelled:

```
> wald( fitn , list("discontinuity at 0" = "C\\(0.*0$"))
              numDF denDF  F.value p.value
discontinuity at 0      1    393 500.0898 <.00001
```

Operating on splines: **sc** function

```
sc( spn, x, D, type)
```

generates a hypothesis matrix to estimate the D^{th} power coefficient of the spline **spn** at the points given by **x**. ‘**type**’ only has an effect when the value of **x** corresponds to a knot. In that case, **type** determines whether the evaluation is to the left, to the right or across the knot.

sc parameters:		
D	0	value of spline
	1	first derivative
	2,...	quadratic,... terms
type	0	limit from the left of knot
	1	limit from the right of knot
	2	saltus: limit from the right minus limit from the left

Example:

Estimate value, slope and quadratic component at -1 months:

```
> sc(spn, c(-1,-1,-1), D = c(1,2,3) )
```

	D1(0)	D2(0)	C(0).0	C(0).1	C(0).2	C(5).2	C(-5).2
D1(-1)	1	-1	0	0	0	0	0
D2(-1)	0	1	0	0	0	0	0
D3(-1)	0	0	0	0	0	0	0

```
> wald( fitn, cbind(c(1,0,0,0),  
+ sc(spn, c(-1,-1,-1,-1), D = c(0,1,2,3) )))
```

	numDF	denDF	F.value	p.value
1	3	393	6937.729	<.00001

	Estimate	Std.Error	DF	t-value	p-value
g(-1)	7.749318	0.057332	393	135.165234	<.00001
D1(-1)	0.034430	0.030741	393	1.120008	0.26339
D2(-1)	-0.027267	0.012147	393	-2.244749	0.02534
D3(-1)	0.000000	0.000000	Inf	NaN	NaN

Using the sc function to test a saltus (type = 2):

```
> Lm <- cbind( 0, sc( spn, 0, D = 0, type = 2))
> Lm
```

	D1(0)	D2(0)	C(0).0	C(0).1	C(0).2	C(5).2	C(-5).2
g(0+)-g(0-)	0	0	0	1	0	0	0

```
> wald( fitn, Lm )
```

	numDF	denDF	F.value	p.value
1	1	393	500.0898	<.00001

	Estimate	Std.Error	DF	t-value	p-value
g(0+)-g(0-)	-2.278971	0.10191	393	-22.36269	<.00001

```
> head( model.matrix( fitn))
```

	(Intercept)	spn(month)D1(0)	spn(month)D2(0)	spn(month)C(0).0
1	1	-9.86370968	48.646384300	0
2	1	-6.95388563	24.178262681	0
3	1	-2.11407625	2.234659188	0
4	1	0.05344575	0.001428224	1
5	1	0.97390029	0.474240891	1
6	1	-7.66649560	29.387577401	0

	spn(month)C(0).1	spn(month)C(0).2	spn(month)C(5).2	spn(month)C(-5).2
1	0.00000000	0.000000000	0.000000e+00	-11.827836
2	0.00000000	0.000000000	0.000000e+00	-1.908835
3	0.00000000	0.000000000	0.000000e+00	0.000000
4	0.05344575	0.001428224	-1.441111e-18	0.000000
5	0.97390029	0.474240891	1.717766e-16	0.000000
6	0.00000000	0.000000000	0.000000e+00	-3.555099

```
> L.level <- cbind( 1, spn( c(-10, 10)))
```

```
> L.level
```

		D1(0)	D2(0)	C(0).0	C(0).1	C(0).2	C(5).2	C(-5).2
f(-10)	1	-10	50	0	0	0	0.0	-12.5
f(10)	1	10	50	1	10	50	12.5	0.0

```
> wald( fitn, L.level )
```

	numDF	denDF	F.value	p.value
1	2	393	8216.843	<.00001

	Estimate	Std.Error	DF	t-value	p-value	Lower 0.95	Upper 0.95
f(-10)	6.942695	0.058820	393	118.0330	<.00001	6.827054	7.058337
f(10)	6.735666	0.055838	393	120.6283	<.00001	6.625887	6.845445


```

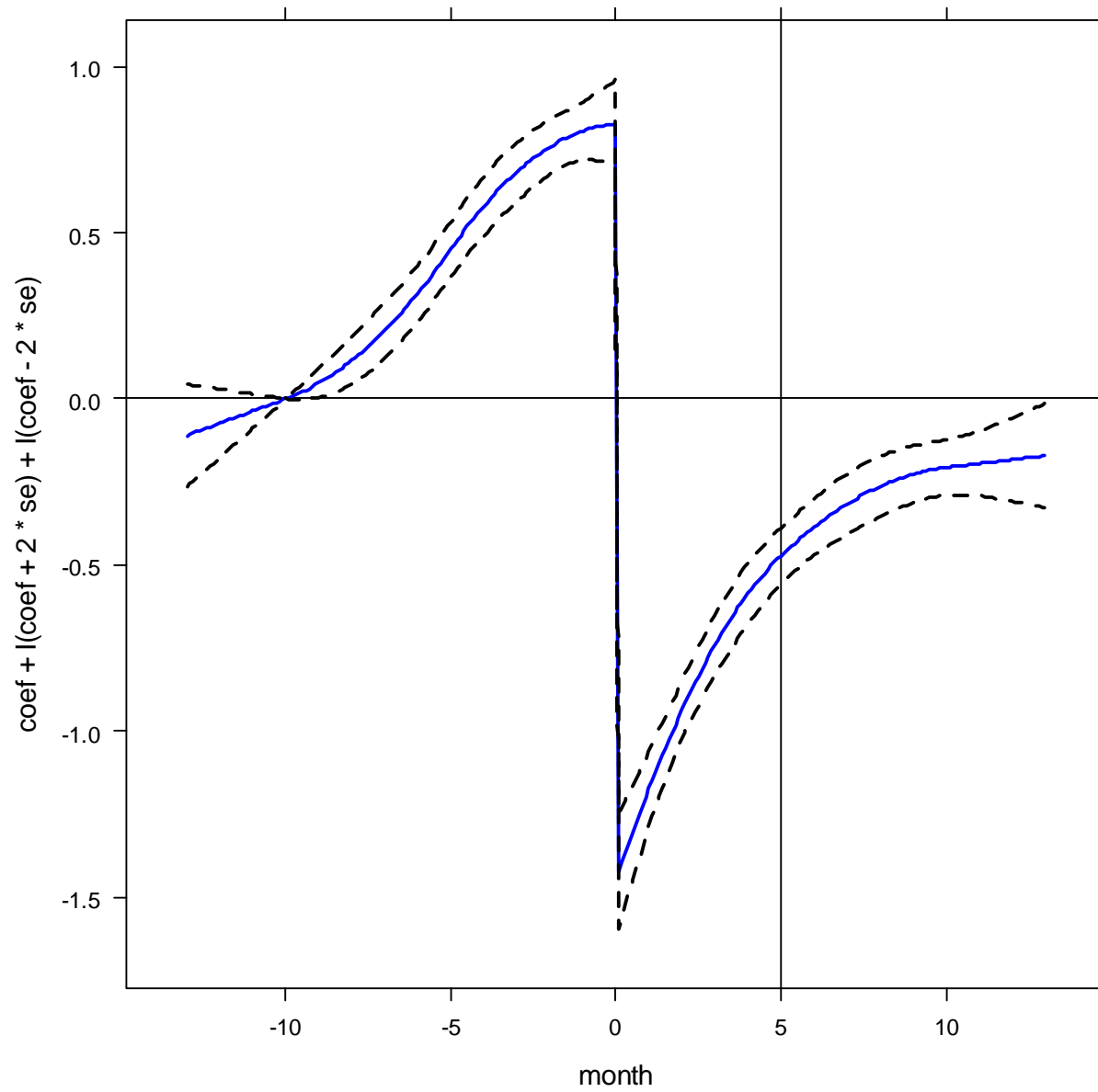
> wald( fitn, L.level[2,] - L.level[1,] )
      numDF denDF F.value p.value
1         1   393 24.4368 <.00001

      Estimate Std.Error  DF   t-value p-value Lower 0.95 Upper 0.95
Larg -0.20703   0.04188 393 -4.943359 <.00001 -0.289367 -0.124692

> ?sc
> Lslope <- with( pred, cbind( 0 , sc( spn, month, D = 1)))
> ws <- as.data.frame(wald(fitn,Lslope))
> head(ws)
              coef          se
D1(-13)    0.03681075 0.02576518
D1(-12.9)  0.03681075 0.02576518
D1(-12.8)  0.03681075 0.02576518
D1(-12.7)  0.03681075 0.02576518
D1(-12.6)  0.03681075 0.02576518
D1(-12.5)  0.03681075 0.02576518
> ws <- cbind(ws, pred)

```

```
> xyplot( coef + I(coef + 2*se)+I(coef - 2*se) ~ month,  
          ws, type = 'l')
```

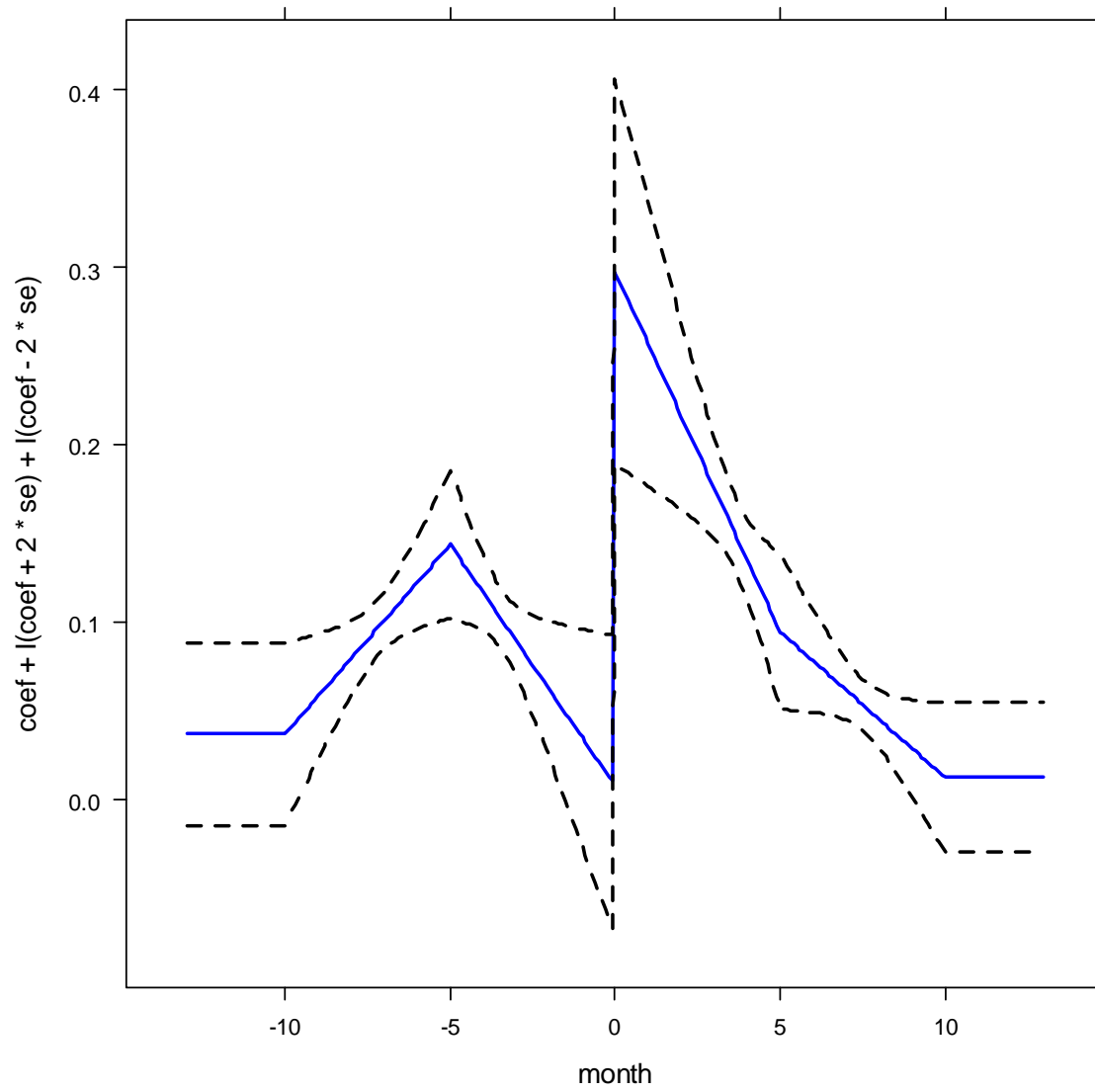


```

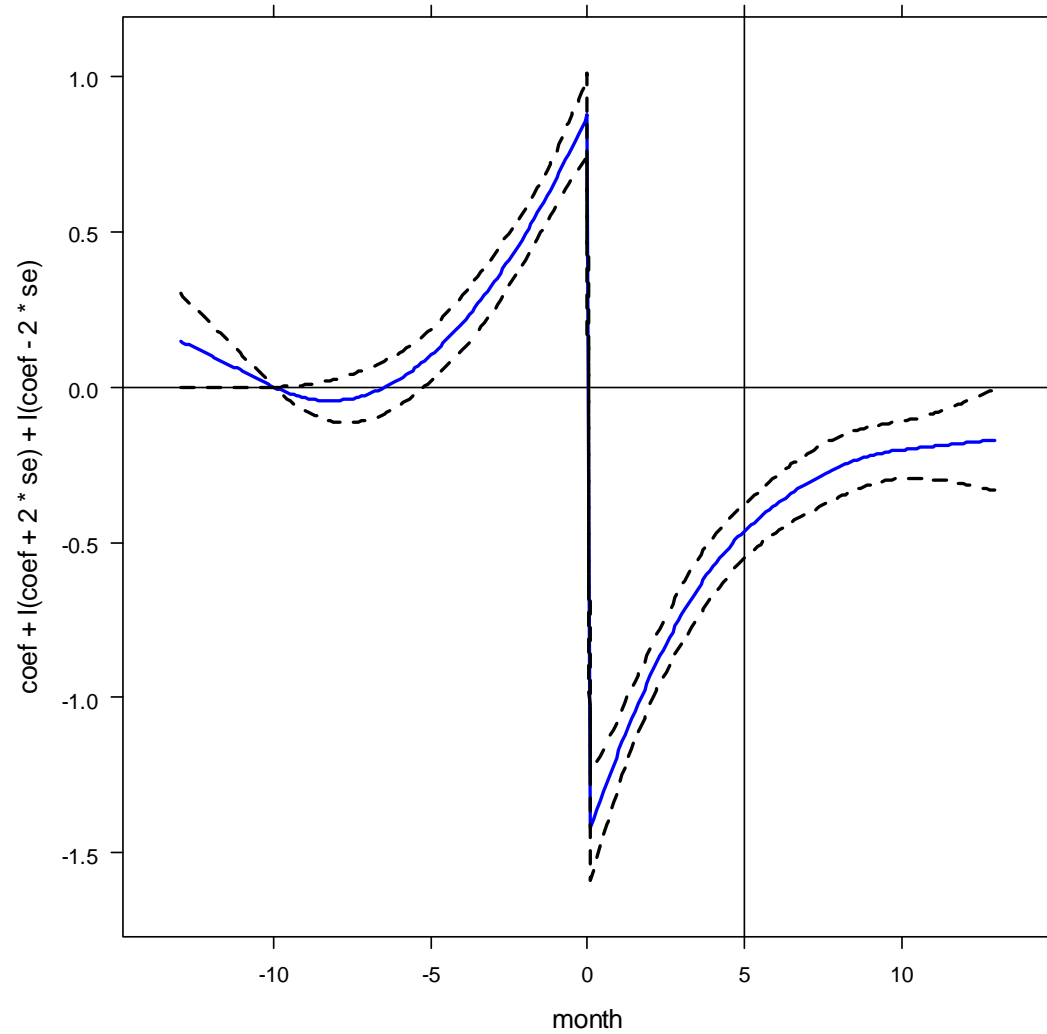
> head( Lest )
      D1(0) D2(0) C(0).0 C(0).1 C(0).2 C(5).2 C(-5).2
f(-13)  1 -13.0   80     0     0     0     0    -27.5
f(-12.9) 1 -12.9   79     0     0     0     0    -27.0
f(-12.8) 1 -12.8   78     0     0     0     0    -26.5
f(-12.7) 1 -12.7   77     0     0     0     0    -26.0
f(-12.6) 1 -12.6   76     0     0     0     0    -25.5
f(-12.5) 1 -12.5   75     0     0     0     0    -25.0
> L.m10 <- cbind( 1, spn(-10))
> L.m10
      D1(0) D2(0) C(0).0 C(0).1 C(0).2 C(5).2 C(-5).2
f(-10)  1  -10   50     0     0     0     0    -12.5
> Ldiff.m10 <- Lest - L.m10[rep(1,nrow(Lpred)),]
> wd <- as.data.frame( wald( fitn, Ldiff.m10))
> wd <- cbind(wd, pred)
> wdr <- as.data.frame( wald( fitnr, Ldiff.m10))
> wdr <- cbind(wdr, pred)

```

```
> xyplot( coef + I(coef + 2*se)+I(coef - 2*se)
~ month, wd, type = 'l',
abline = list(h=0,v=5))
```



```
> xyplot( coef + I(coef + 2*se)+I(coef - 2*se) ~  
          month, wdr, type = 'l',  
          abline = list(h=0, v=5))
```



Caution:

Since `gsp` uses raw parametrization (in contrast with b-splines, for example), you should center and rescale the predictor. A range within -10 to 10 for a spline with no power higher than cubic should be adequate.