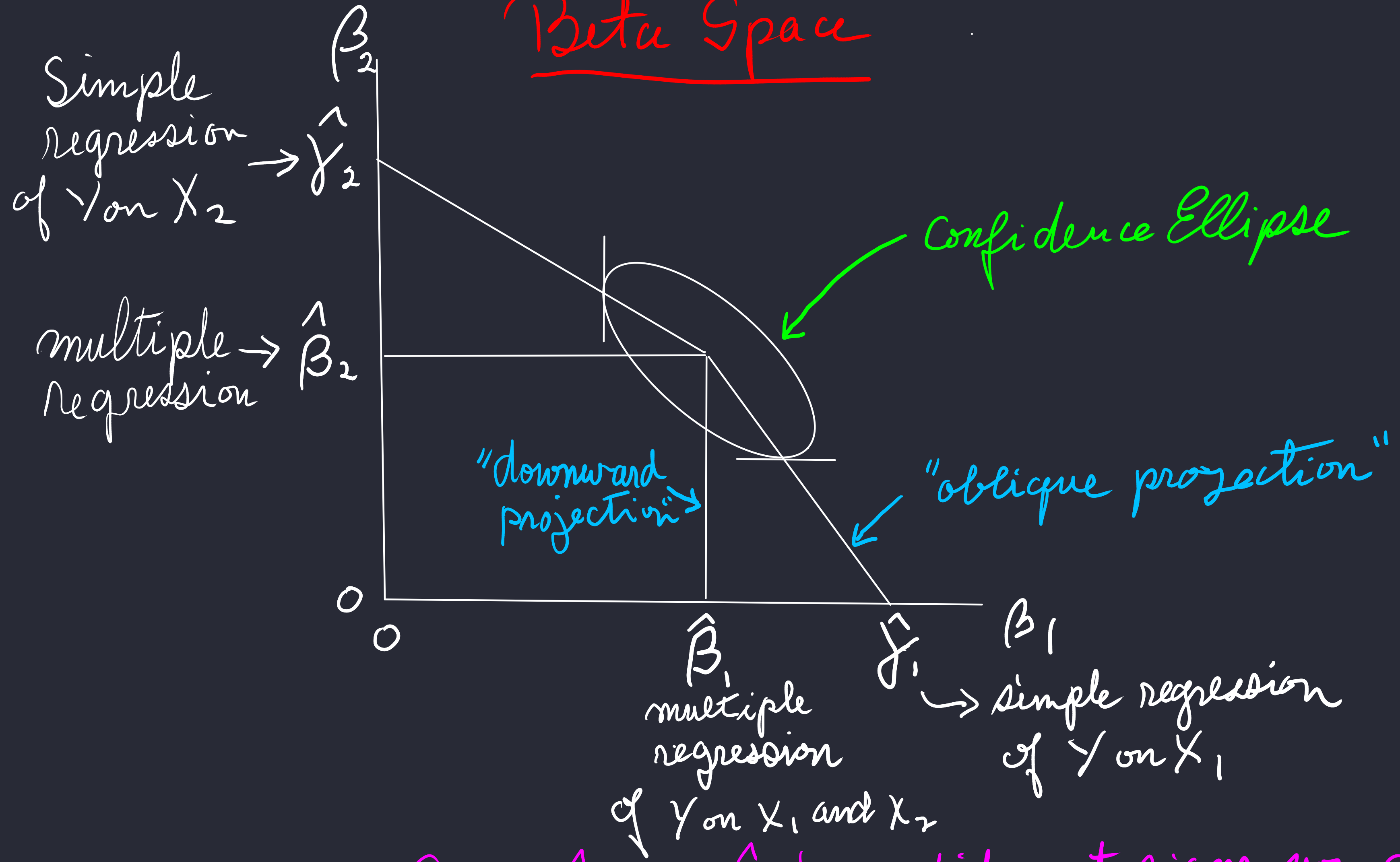
Beta Space

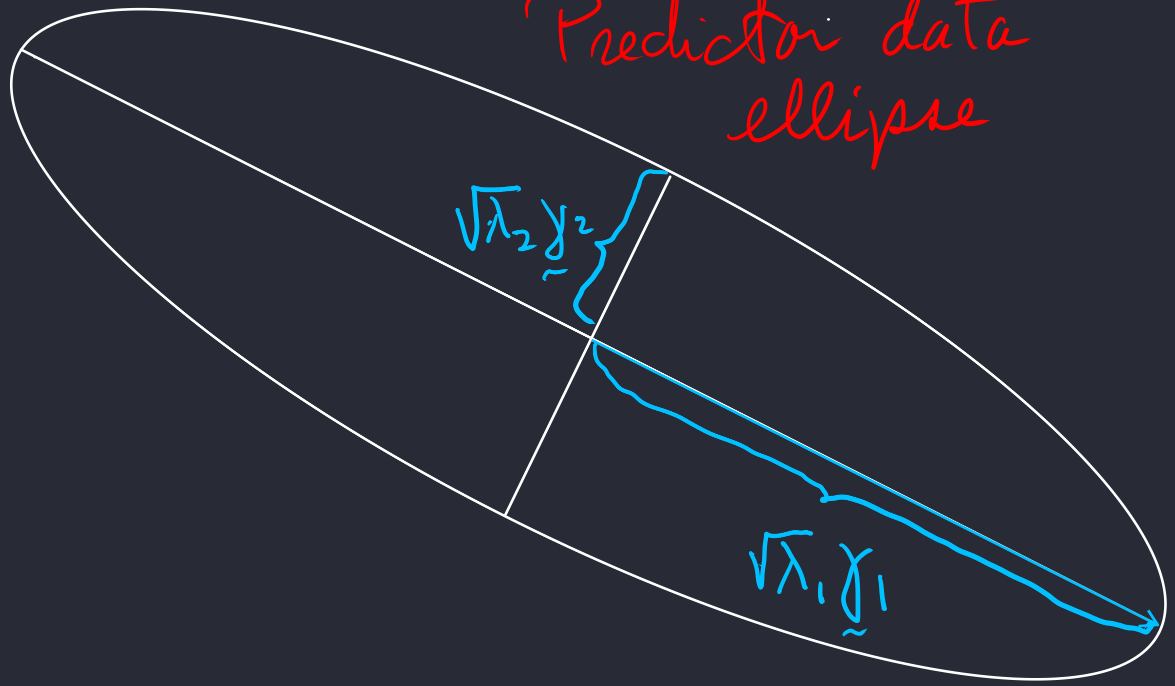


So if $\hat{\beta}_1$ and $\hat{\gamma}_1$ have different signs you get Simpson's Paradox

Here they're both positive, so no Simpson's Paradox.

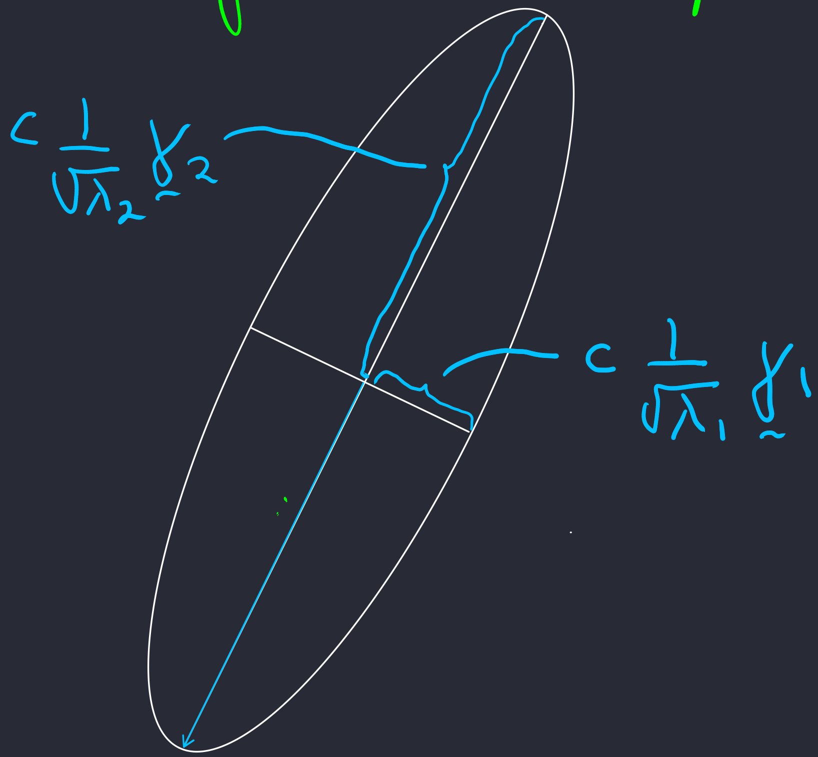
Data space

Predictor data ellipse



Beta space

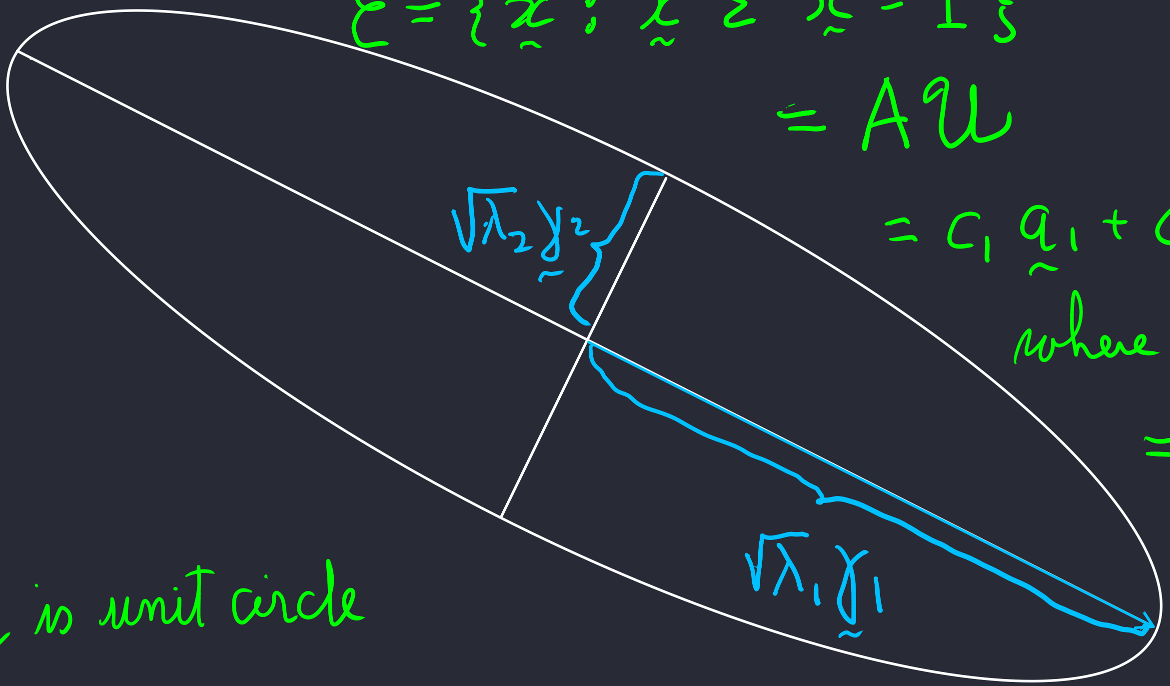
Confidence ellipse



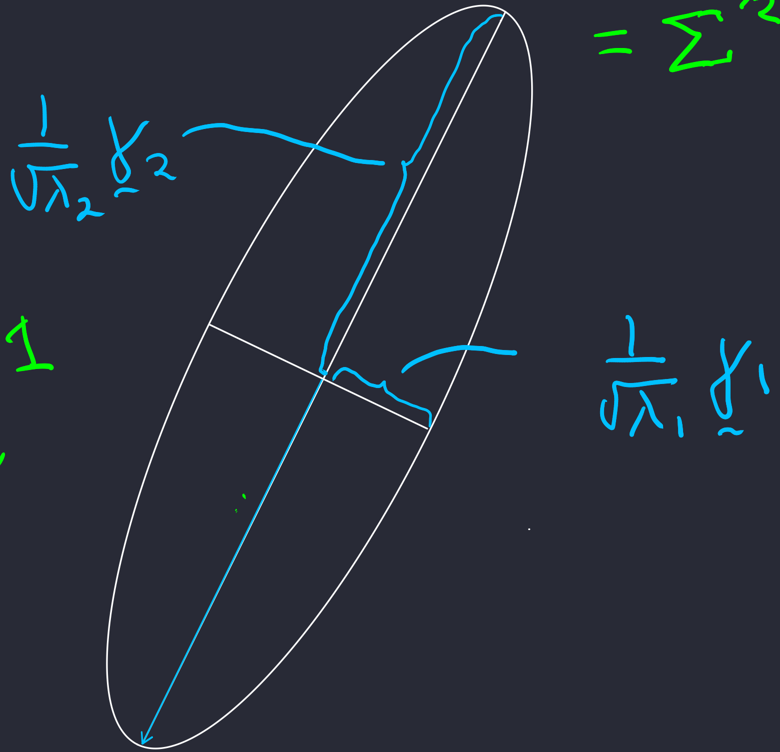
The "inverse" of an ellipse

Let Σ be positive-definite with spectral decomposition $\Sigma = \Gamma \Lambda \Gamma'$. Note that for *any* matrix $A = \Gamma \Lambda^{1/2} G$ with $GG' = I$,
$\Sigma = AA'$, $A = [\underset{\sim}{a}_1 \; \underset{\sim}{a}_2]$.

$$\mathcal{E} = \{\underset{\sim}{x} : \underset{\sim}{x}' \Sigma^{-1} \underset{\sim}{x} = 1\}$$
$$= A\mathcal{U}$$
$$= c_1 \underset{\sim}{a}_1 + c_2 \underset{\sim}{a}_2$$
where $c_1^2 + c_2^2 = 1$
$$= \Sigma^{1/2}\mathcal{U}$$



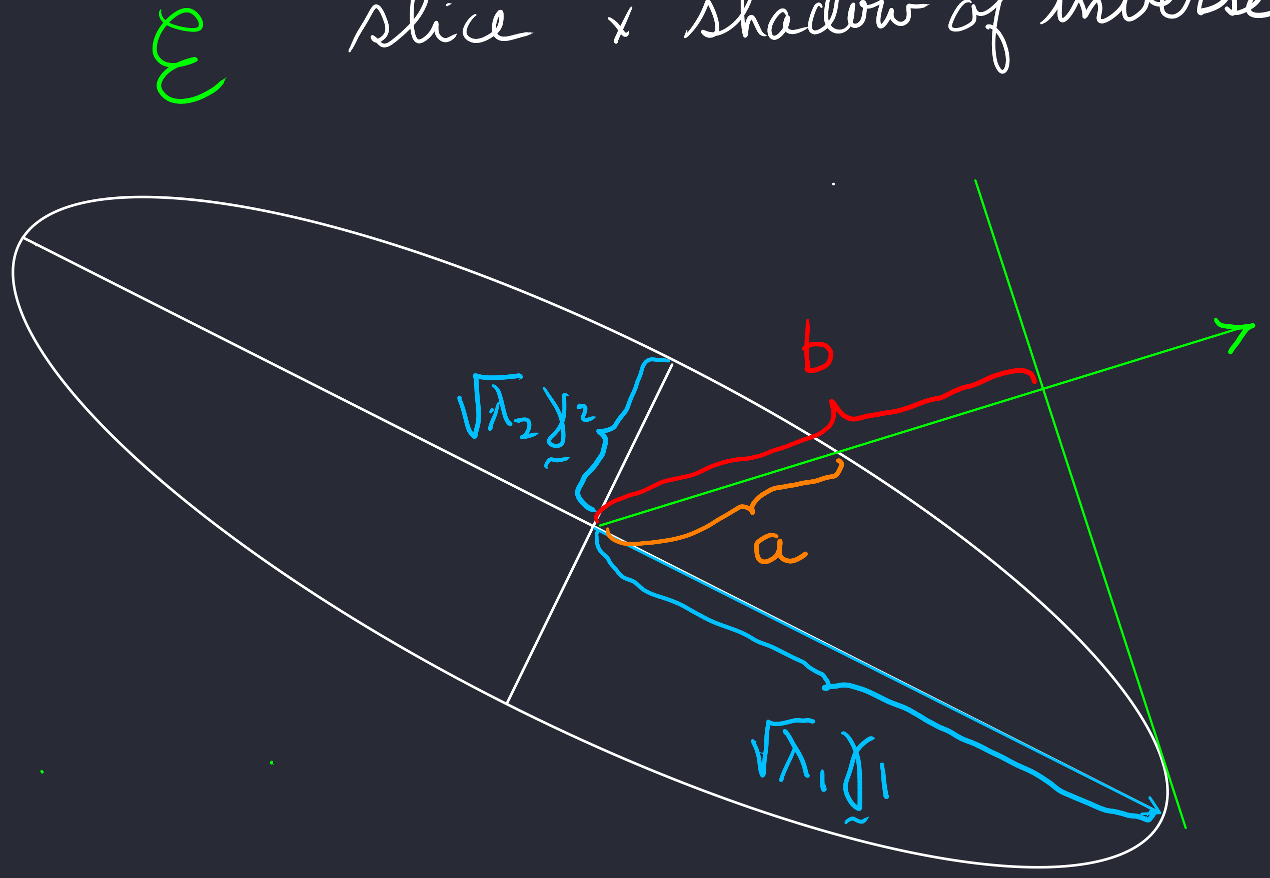
$$\mathcal{E}^{-1} = \{\underset{\sim}{x} : x' \Sigma x = 1\}$$
$$= A'^{-1}\mathcal{U}$$
$$= \Sigma^{-1/2}\mathcal{U}$$

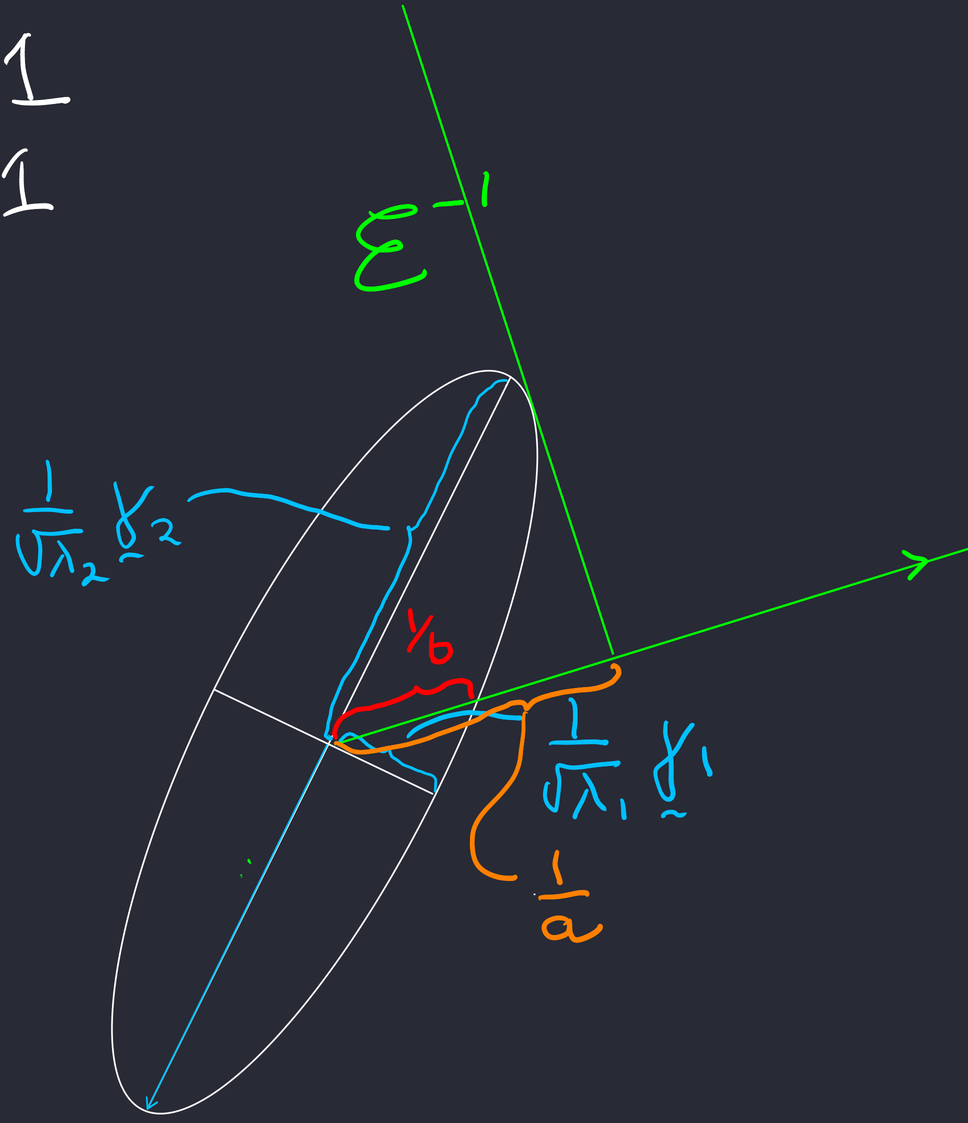


$\mathcal{U}$ is unit circle

shadow × slice of inverse = 1

slice × shadow of inverse = 1





Proof: Let $\underset{\sim}{u}$ be a unit vector.

$$\mathcal{E} = \{ x : x' \Sigma^{-1} x = 1 \}$$

$$\mathcal{E}^{-1} = \{ x : x' \Sigma x = 1 \}$$

Let $a\underset{\sim}{u} \in \mathcal{E}$. i.e $a\underset{\sim}{u}' \Sigma^{-1} a\underset{\sim}{u} = 1$, so $a^2 = 1/\underset{\sim}{u}' \Sigma^{-1} \underset{\sim}{u}$

Consider the perpendicular projection of points in $\mathcal{E}^{-1}$ onto $\underset{\sim}{u}$. They have the form $(\underset{\sim}{u}^T \underset{\sim}{x})\underset{\sim}{u}$ for $\underset{\sim}{x} \in \mathcal{E}^{-1}$.

Consider $\max_{x \in \mathcal{E}^{-1}} \|(\underset{\sim}{u}^T x)\underset{\sim}{u}\| = \max_{x \in \mathcal{E}^{-1}} |\underset{\sim}{u}^T x|$

Now, by Cauchy-Schwarz,

$$|\underset{\sim}{u}^T x|^2 \leq \underset{\sim}{u}^T \Sigma^{-1} \underset{\sim}{u} \cdot x^T \Sigma x$$

$$= \underset{\sim}{u}^T \Sigma^{-1} \underset{\sim}{u} = \frac{1}{a^2} \text{ for } x \in \mathcal{E}^{-1}$$

with equality iff $\underset{\sim}{x}$ & $\Sigma^{-1}\underset{\sim}{u}$ are linearly dependent.

Now we need to exhibit such an $\underset{\sim}{x} \in \mathcal{E}^{-1}$.

Letting $\underset{\sim}{x} = k\Sigma^{-1}\underset{\sim}{u} \Rightarrow 1 = x'\Sigma_1 x = k^2 \underset{\sim}{u}^T \Sigma^{-1}\Sigma\Sigma^{-1}\underset{\sim}{u}$

$$= k^2 \underset{\sim}{u}^T \Sigma^{-1} \underset{\sim}{u}$$

i.e. $\underset{\sim}{x} = \dfrac{1}{\sqrt{\underset{\sim}{u}'\Sigma^{-1}\underset{\sim}{u}}}\Sigma^{-1}\underset{\sim}{u}$

we have $\underset{\sim}{x}'\Sigma\underset{\sim}{x} = 1$ and x' & $\Sigma^{-1}\underset{\sim}{u}$ are collinear

Q.E.D.