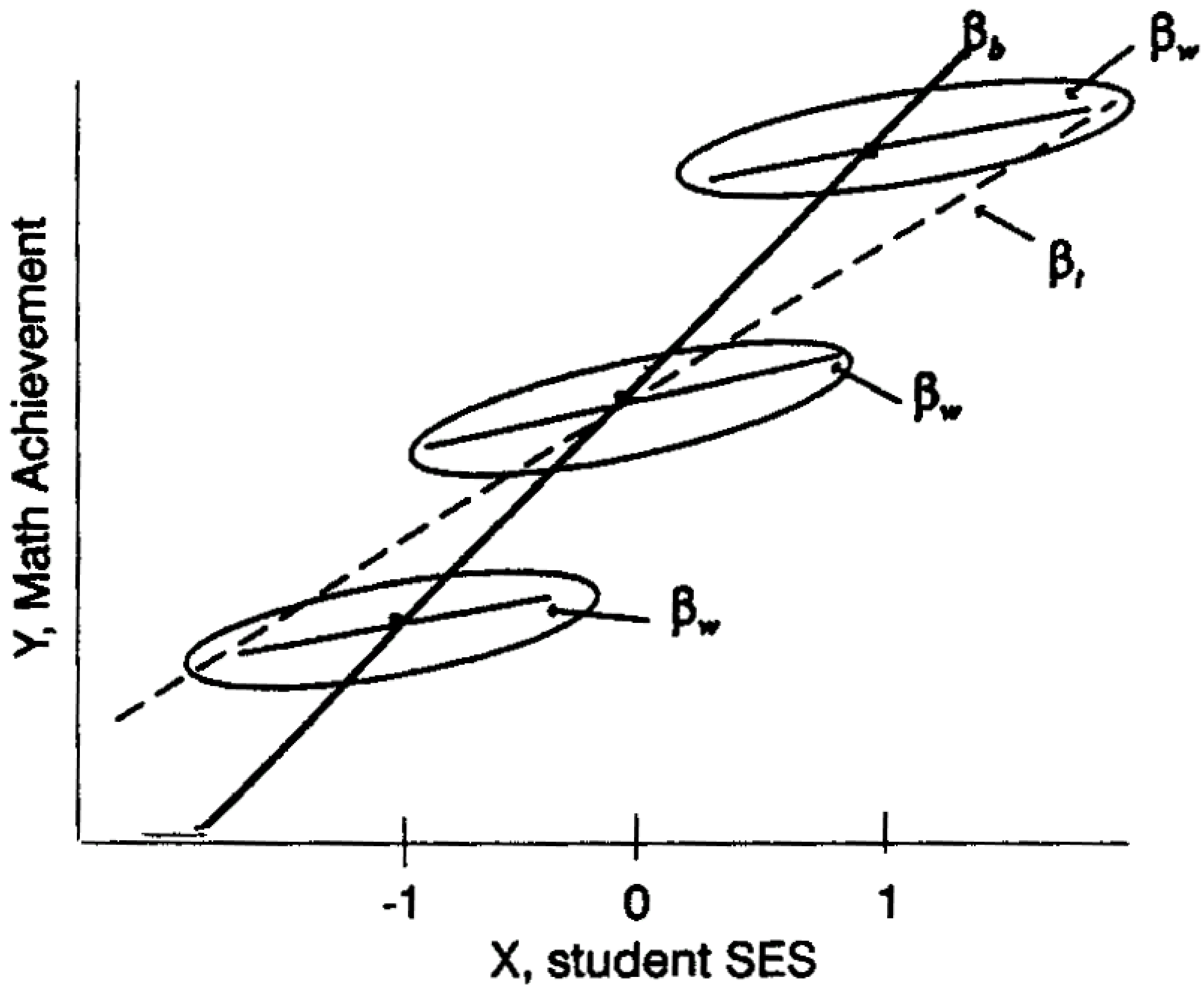
# Statistical Reasoning with Ellipses: The Data Ellipse

*Simple regression, the regression paradox, the anatomy of outliers: fit vs leverage, significance at a glance*

Georges Monette

random@yorku.ca



βb
βw
βt
βw
βw
Y, Math Achievement
-1
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1
X, student SES

***Example: Origins of regression***

Galton and Pearson studied the inheritance of height from father to son

To make things simple we use simulated data with adult heights in two generations where there is no drift:

| | |
|---|---|
| Fathers: | mean 68″ sd 3″ |
| Sons (adult) | mean 68″ sd 3″ |

Father's height  Son's expected height

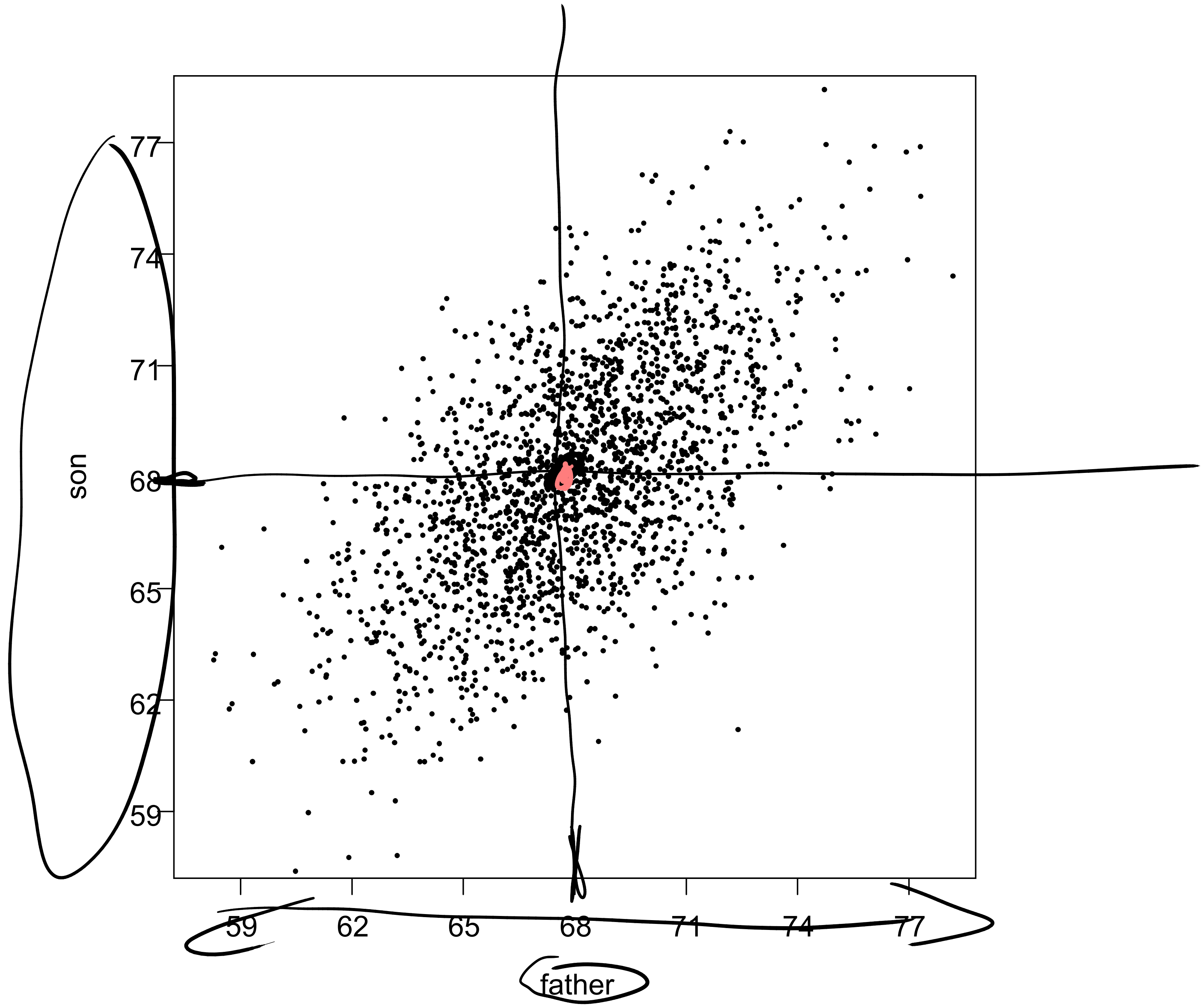
68

| Father's height | Son's expected height |
| --- | --- |
| 68 | 68 |

| Father's height | Son's expected height |
|---|---|
| $68$ | $68$ |
| $71 = 68 + 3$ | 71 |

| Father's height | Son's expected height |
| --- | --- |
| $68$ | $68$ |
| $71 = 68 + 3$ | $71 = 68 + 3$ |

| Father's height | Son's expected height |
|---|---|
| $68$ | $68$ |
| $71 = 68 + 3$ | $71 = 68 + 3$ |
| $74 = 68 + 6$ | $74 = 68 + 6$ |
| $65 = 68 - 3$ | $65 = 68 - 3$ |
| $62 = 68 - 6$ | $62 = 68 - 6$ |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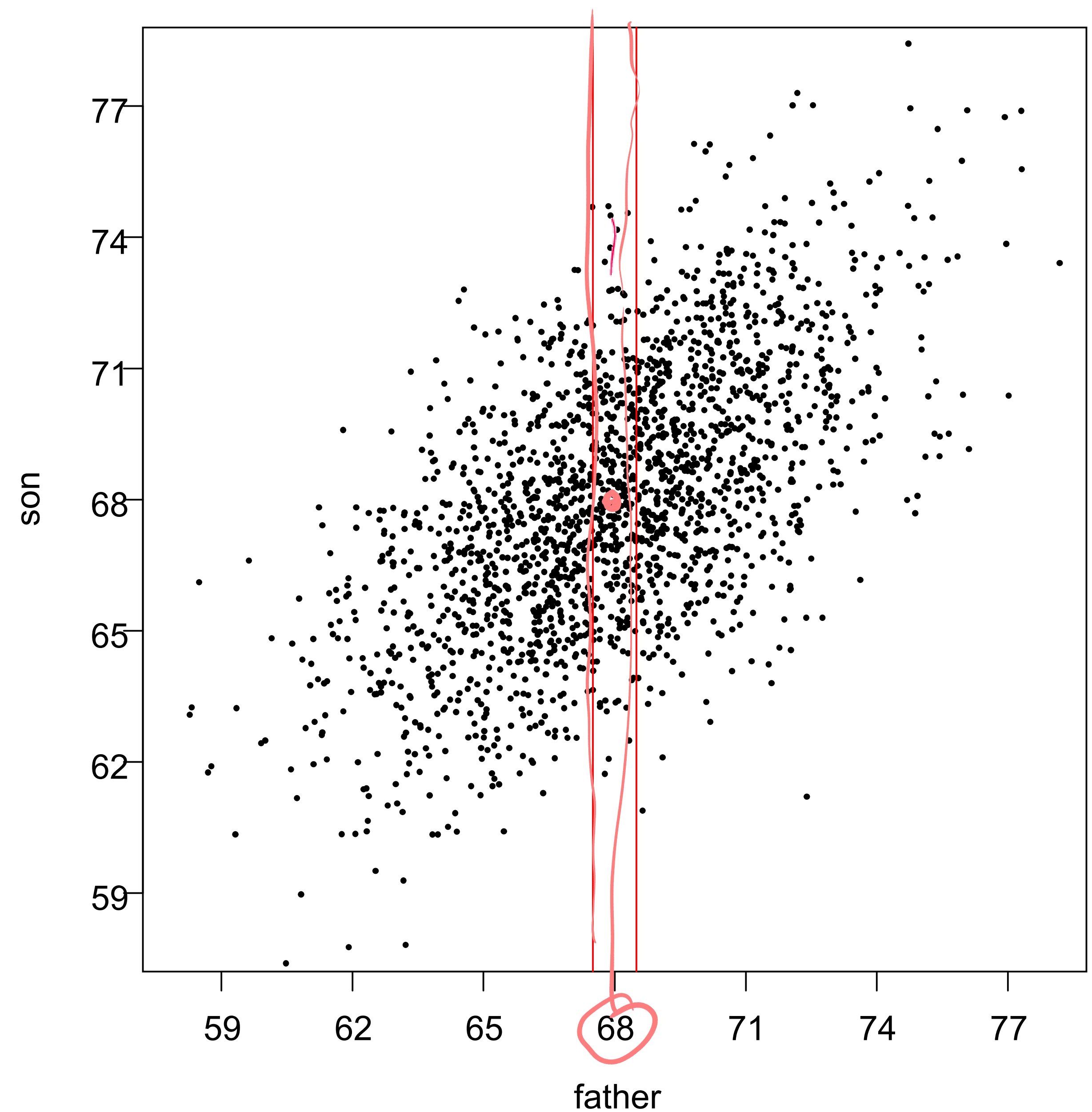
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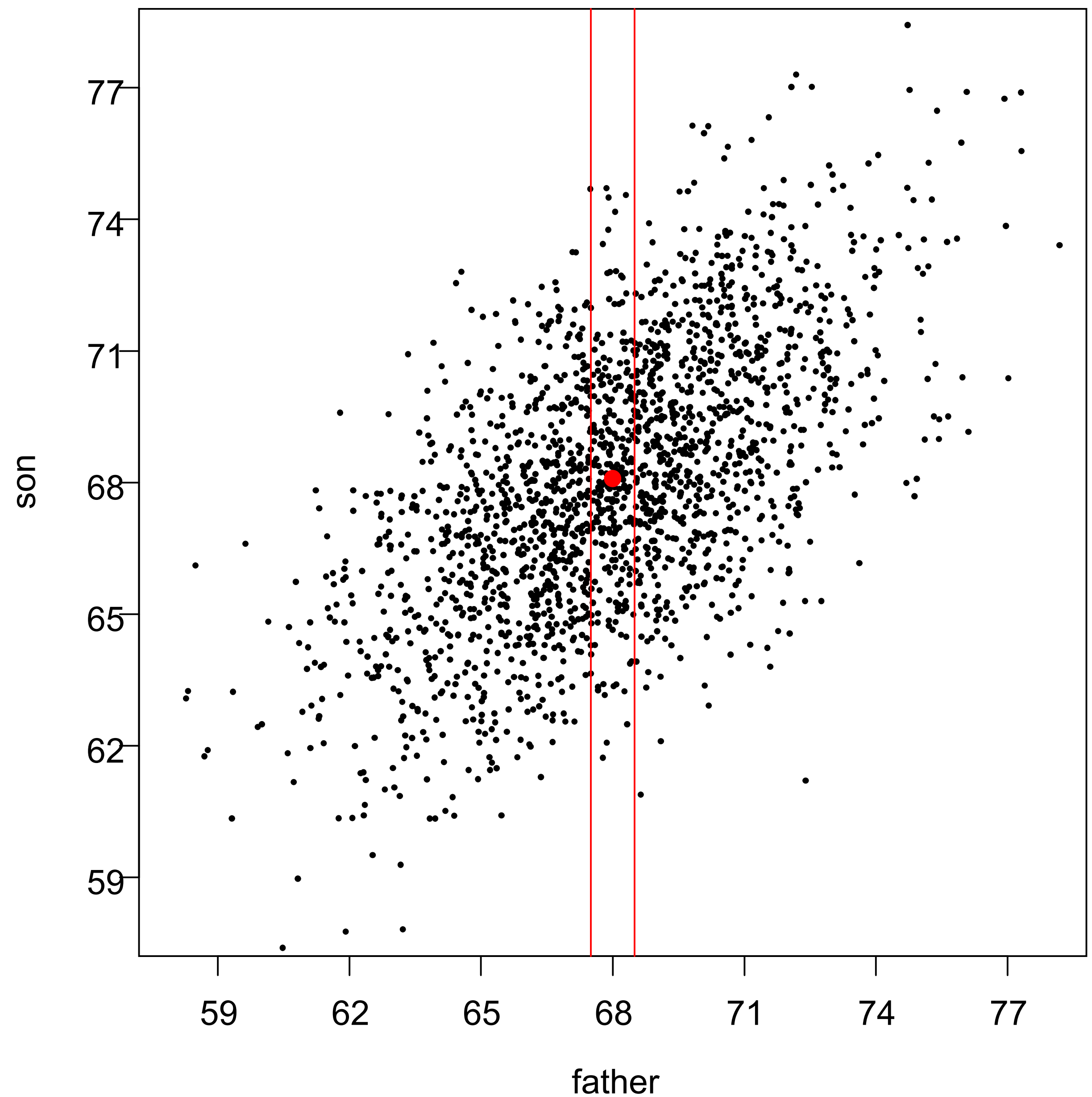
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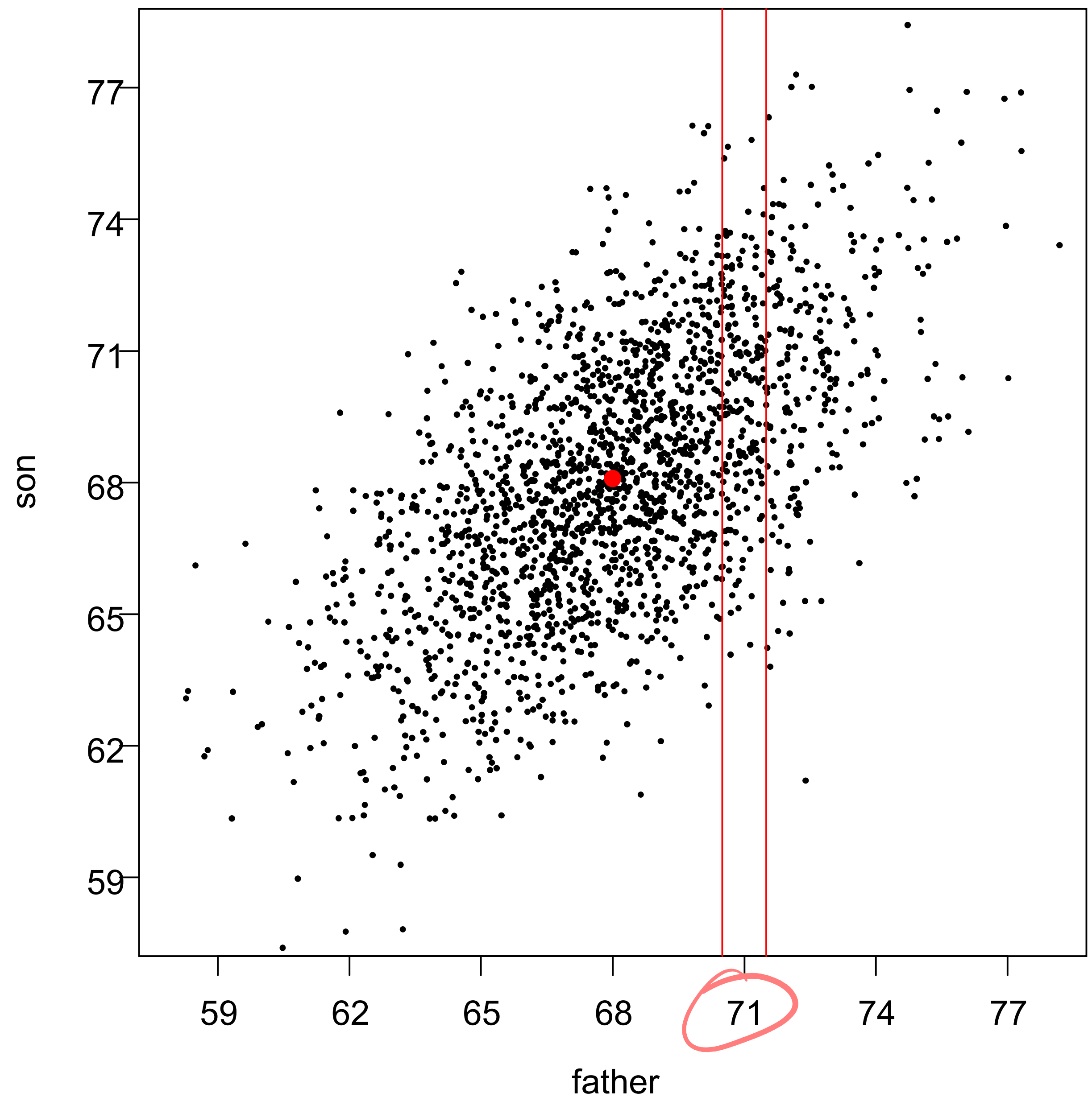
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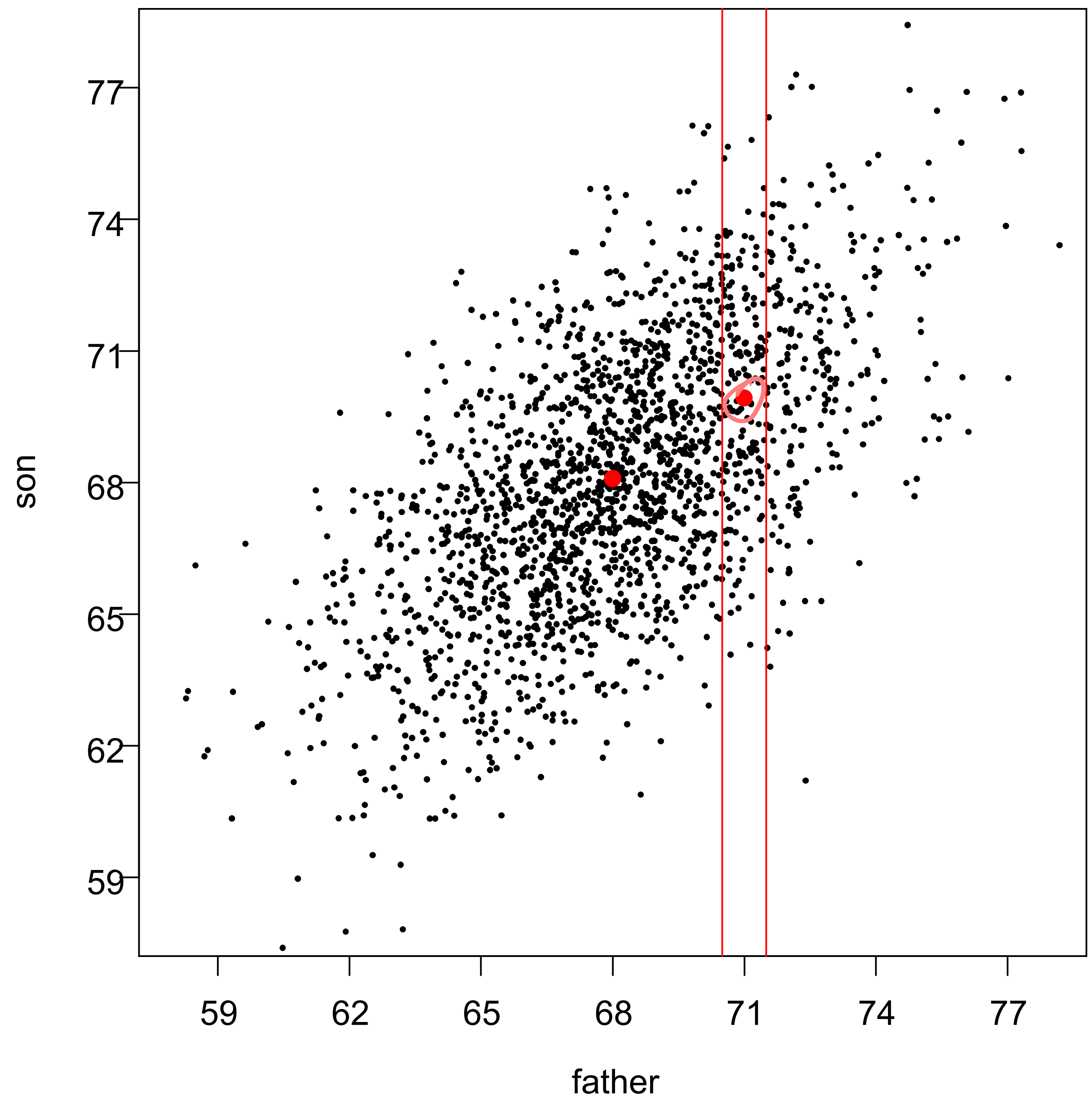
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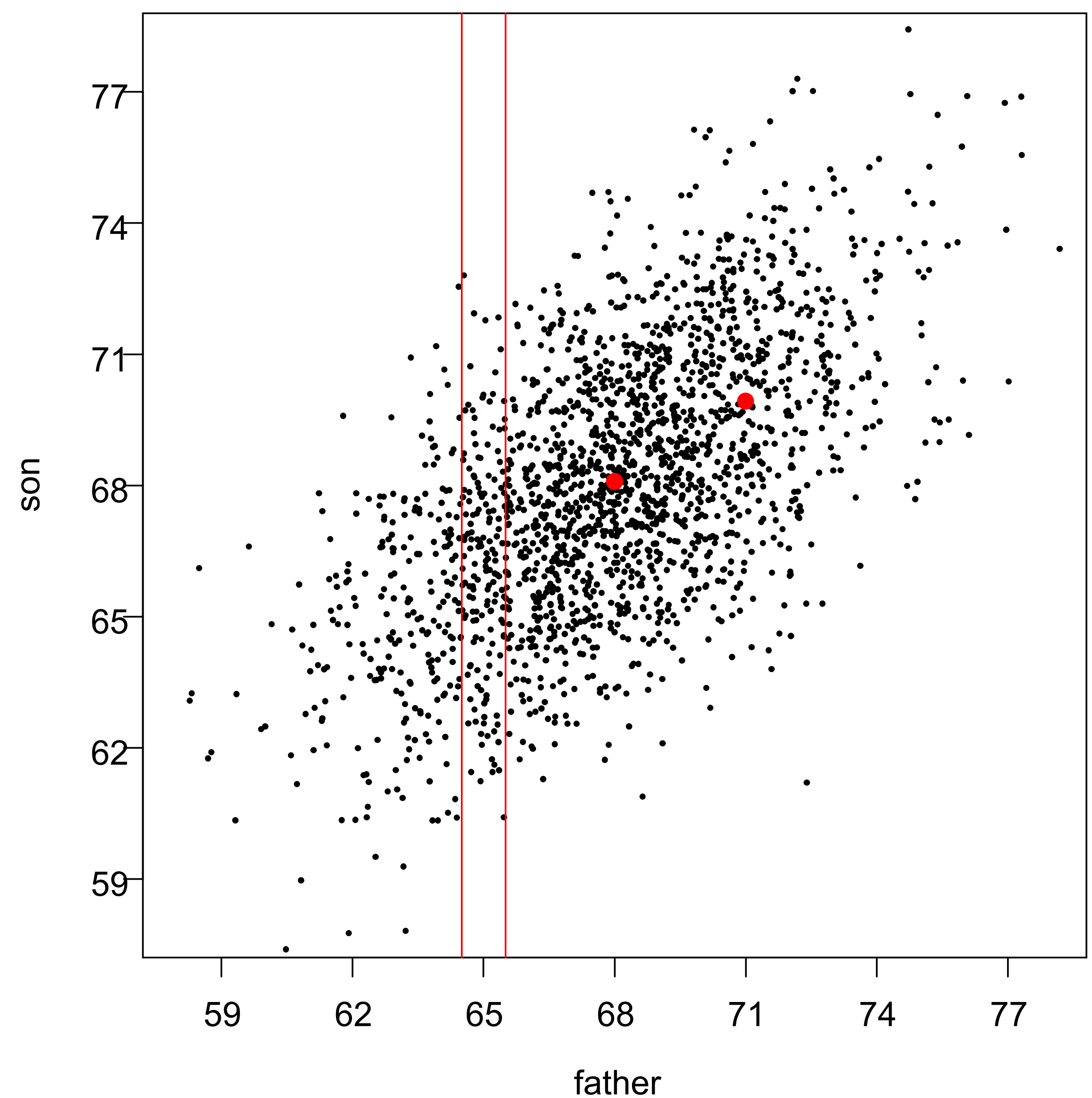
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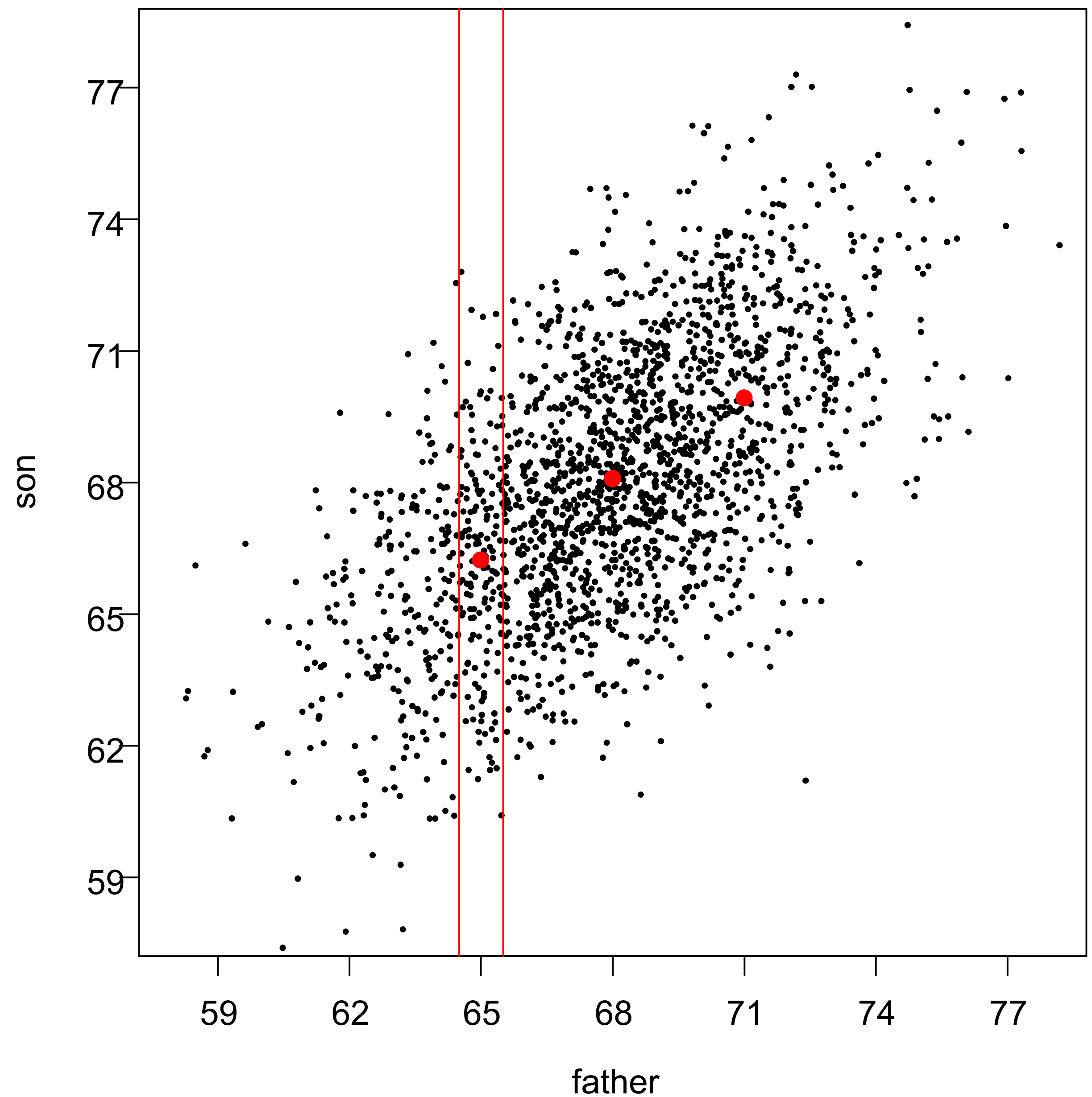
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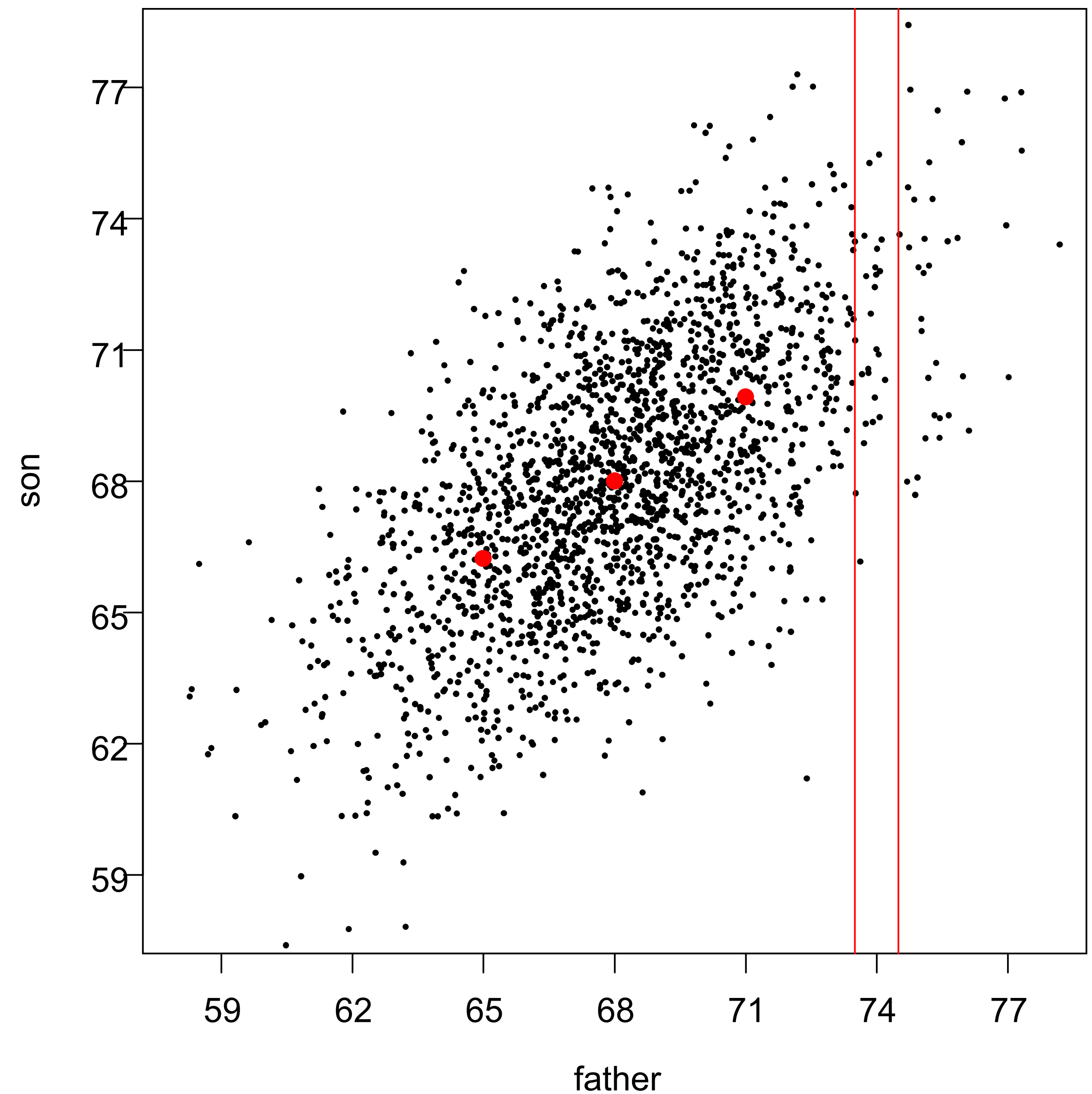
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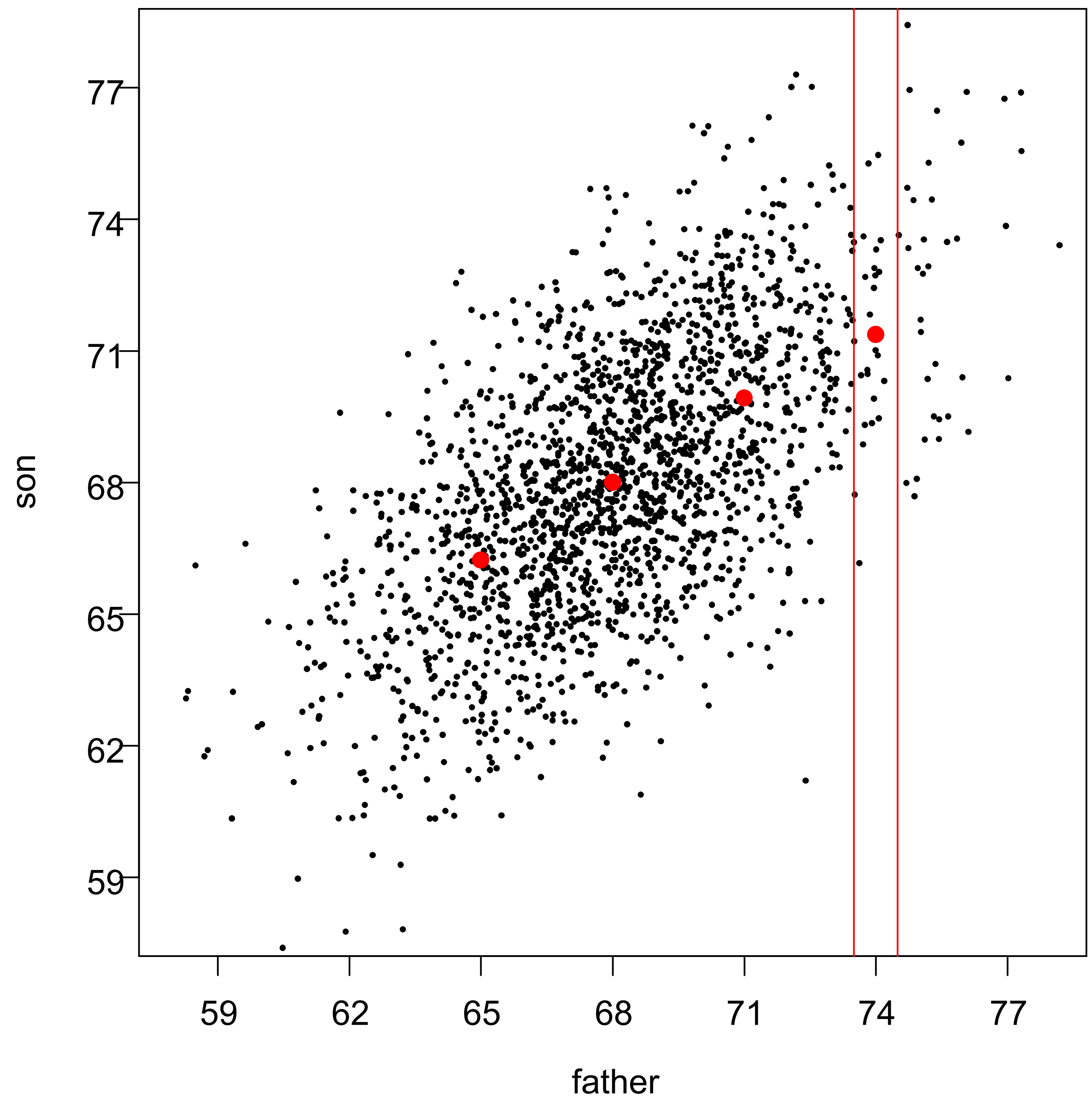

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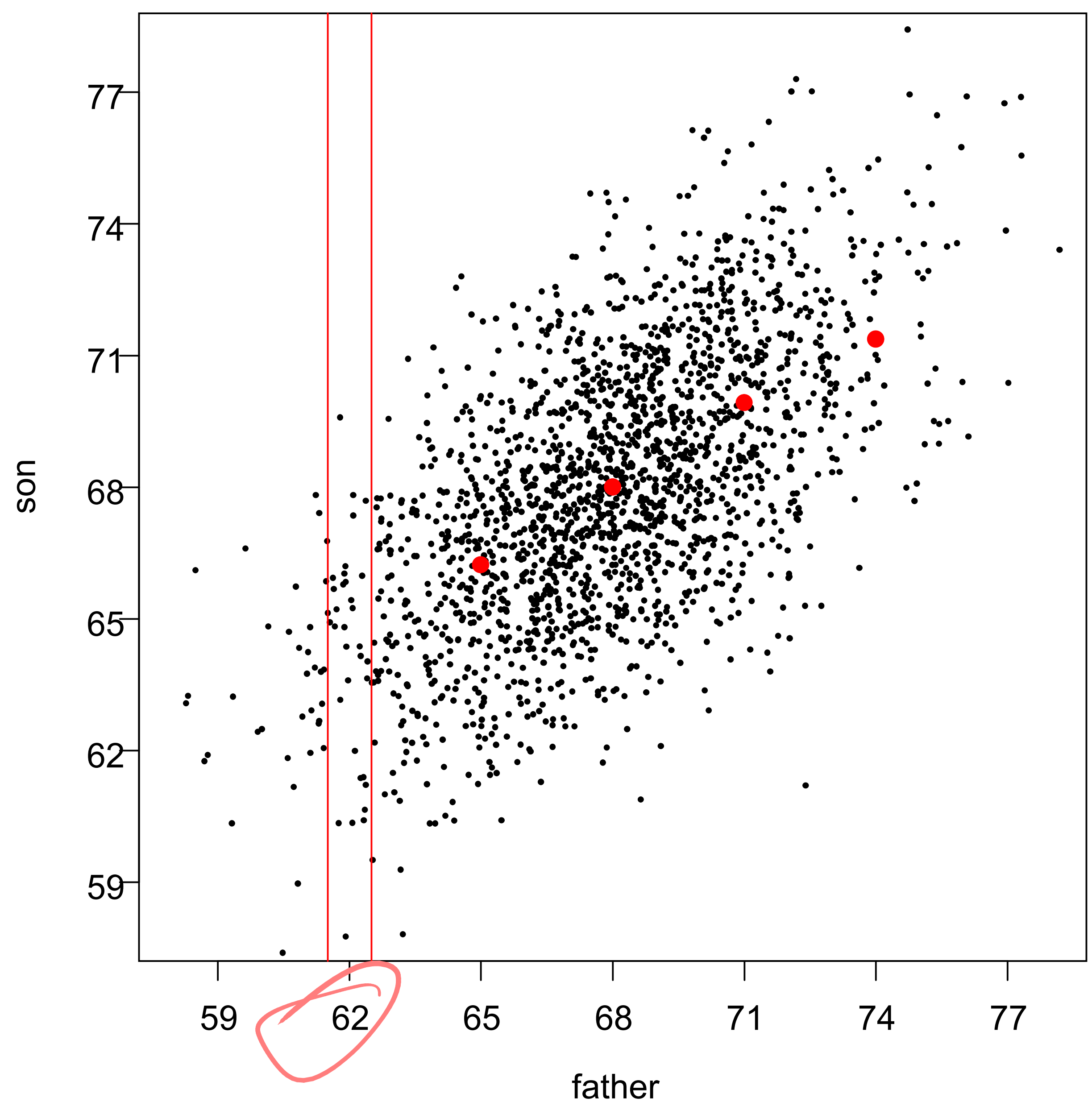
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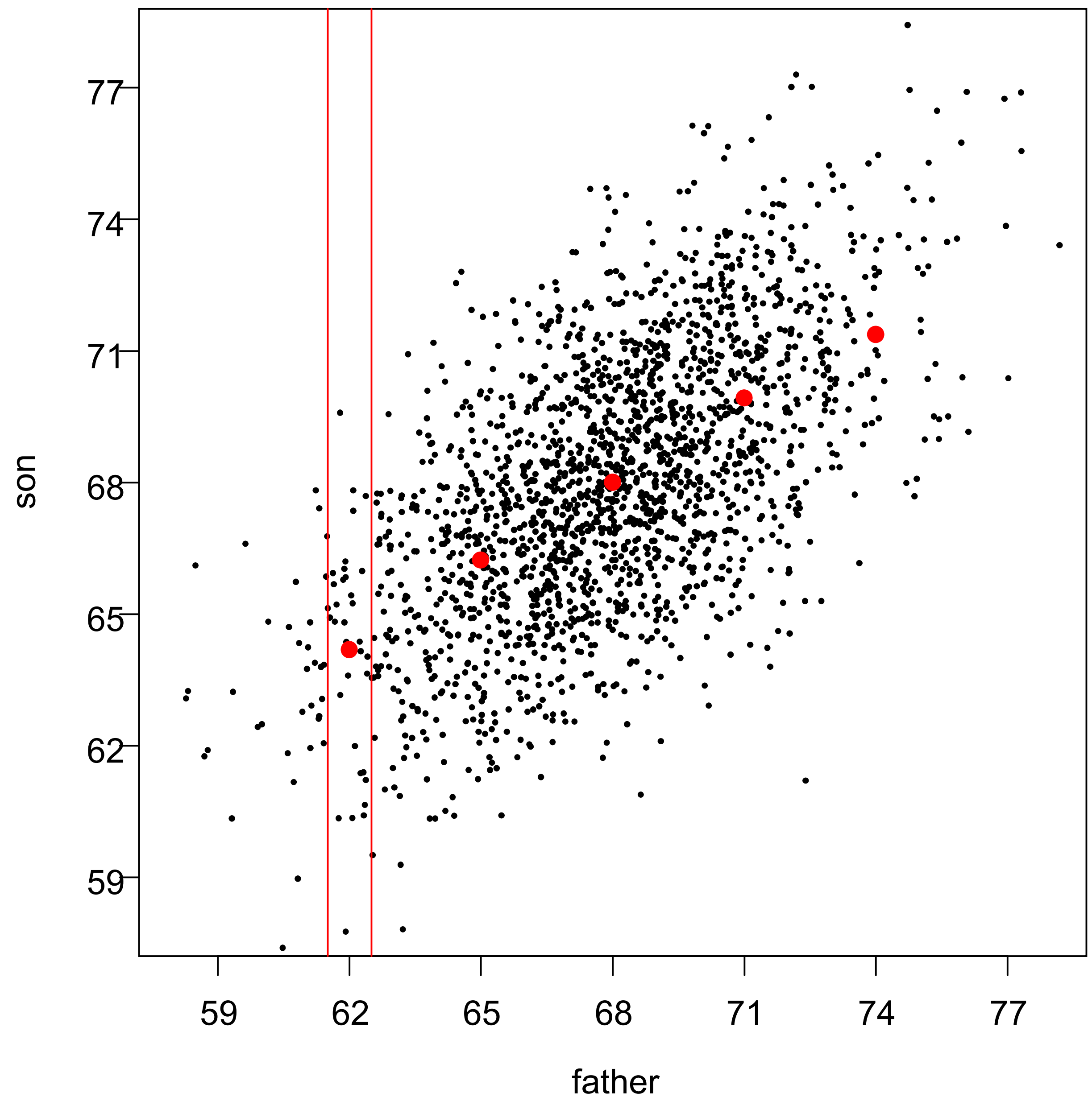
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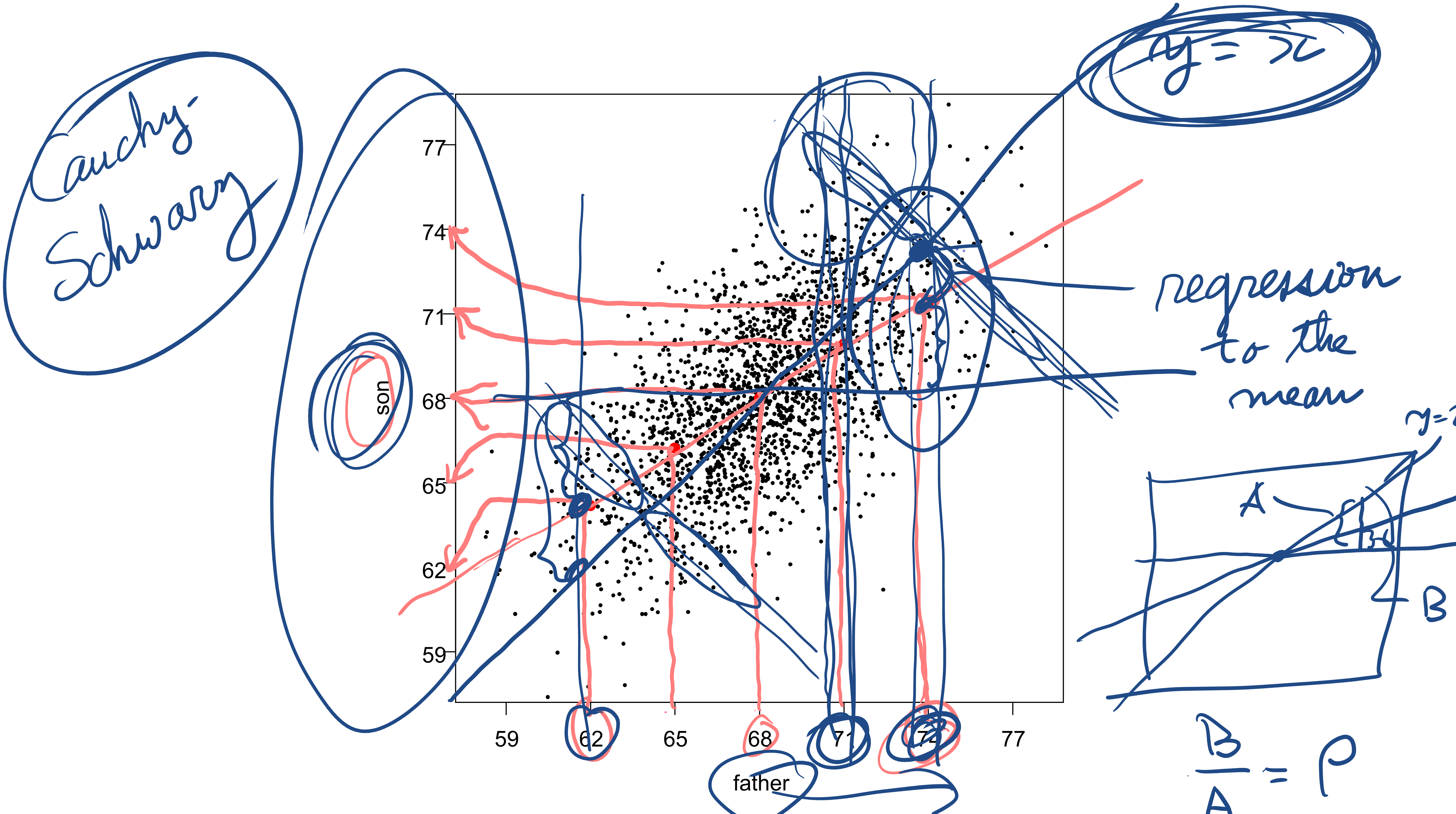
father


Cauchy-Schwarz
y = x
regression to the mean
son
father
77
74
71
68
65
62
59
59 62 65 68 71 74 77
y=x
A
B
B/A = ρ

$$Z_x = \frac{X - \mu_x}{\sigma_x} = \text{\# of sd's from the mean}$$

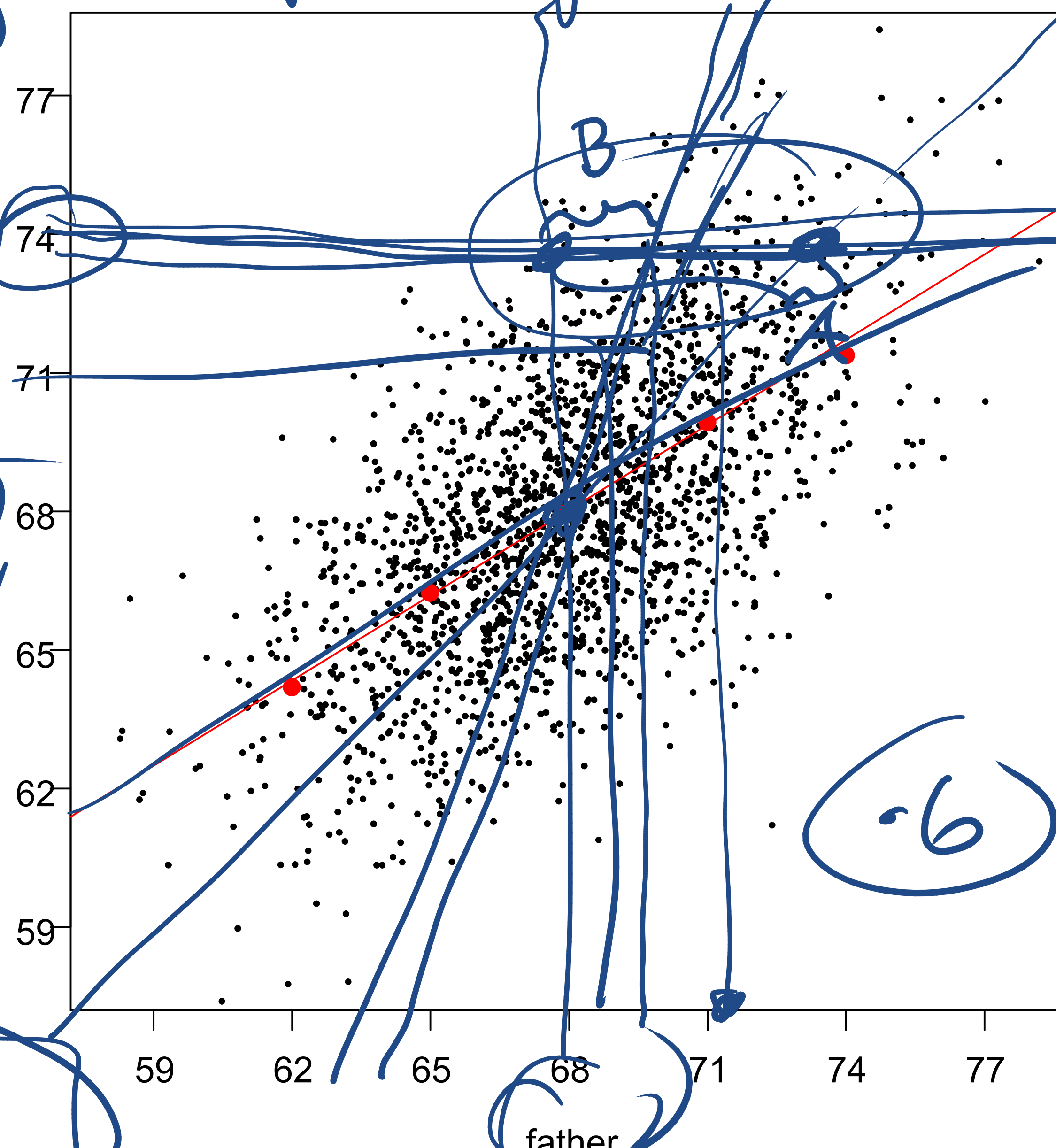
$$\frac{B}{A} = \rho$$

$$Z_y = \frac{Y - \mu_y}{\sigma_y}$$

$$\hat{Z}_y = a + b Z_x$$

$$= 0 + \rho_{xy} Z_x$$

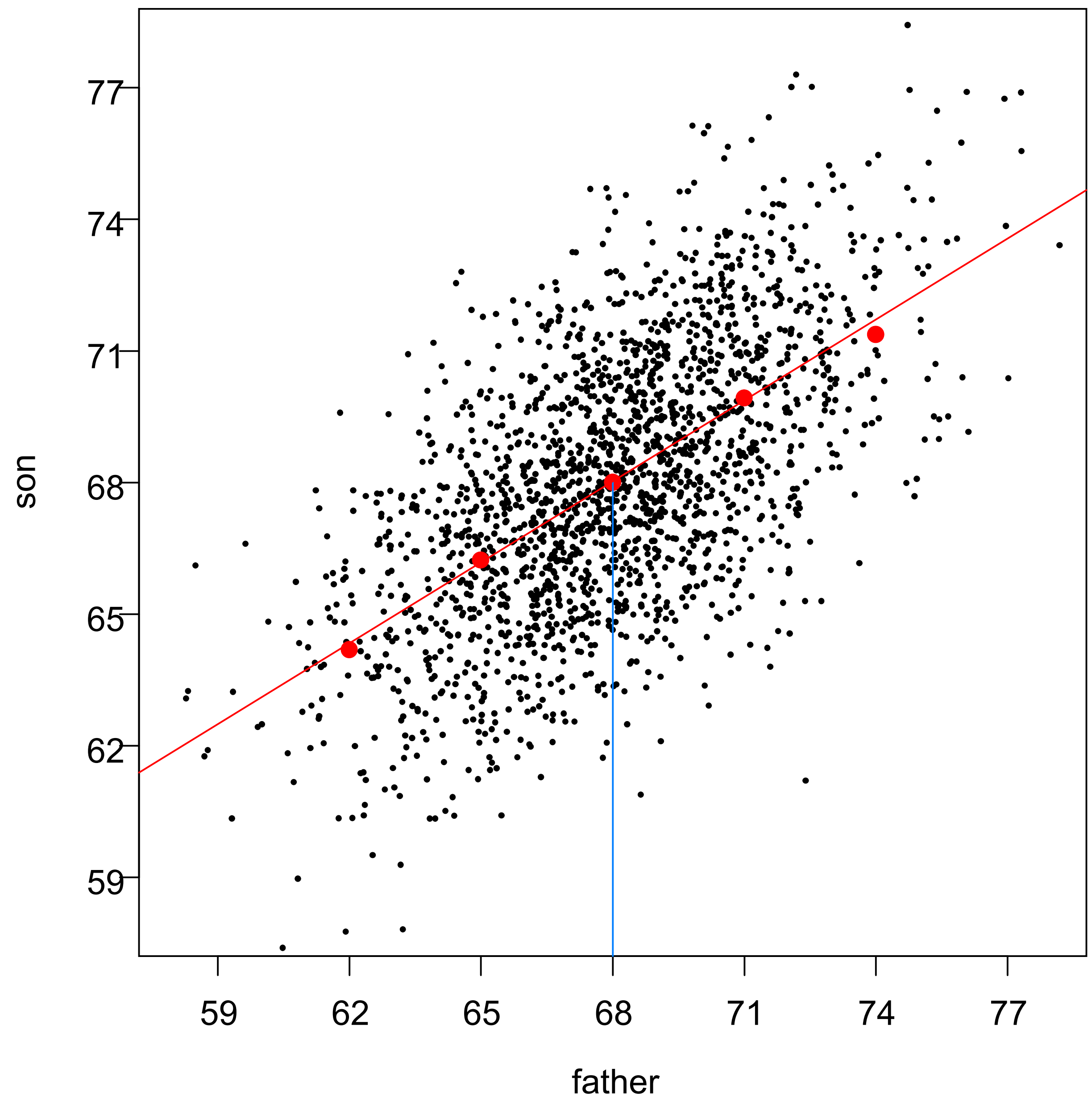
$$\hat{Z}_y = \rho_{xy} Z_x$$




.6

slope

$$\frac{Cov(Y, X)}{Var(X)} = \rho_{xy} \times \frac{\sigma_y}{\sigma_x}$$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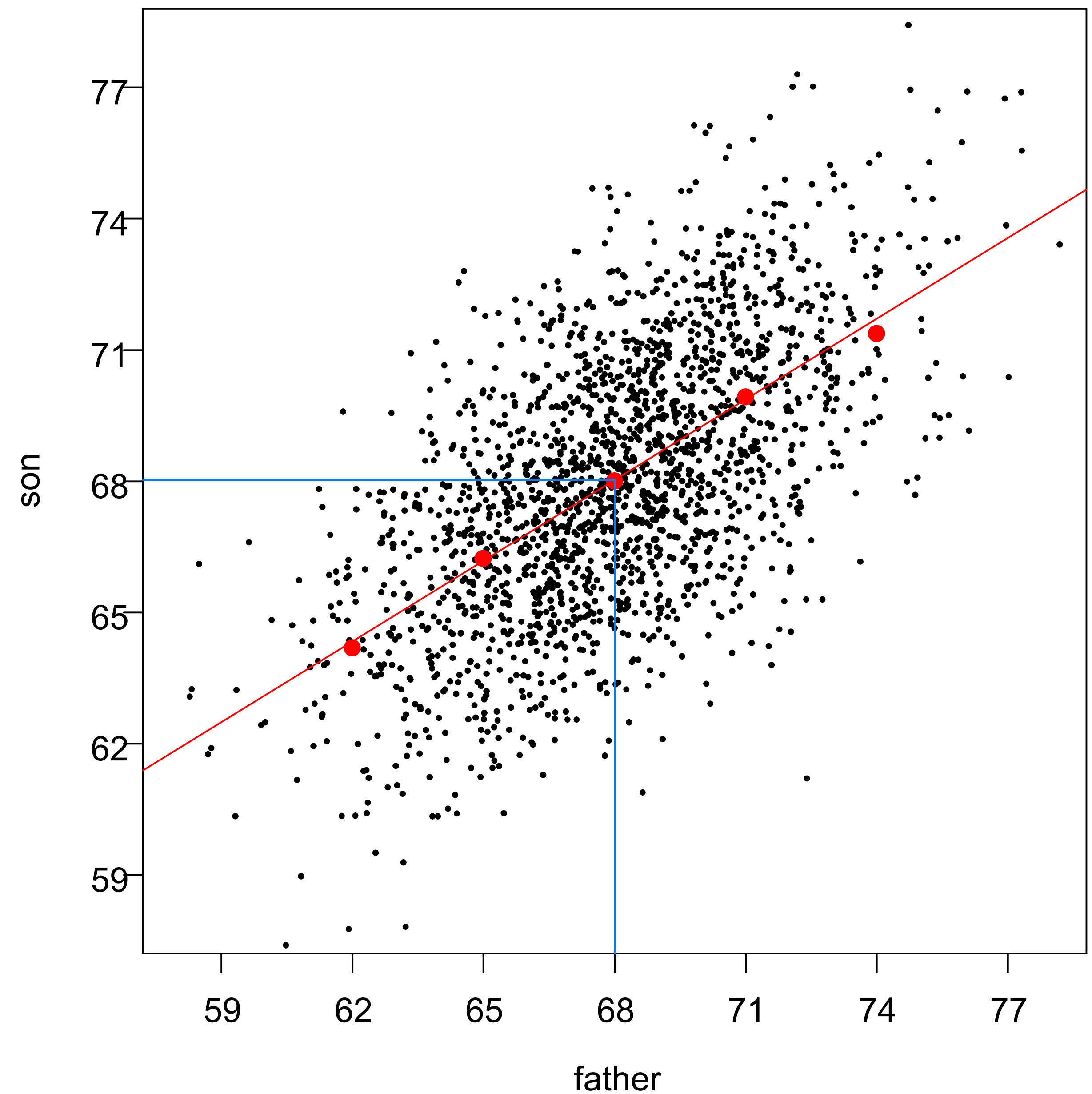
father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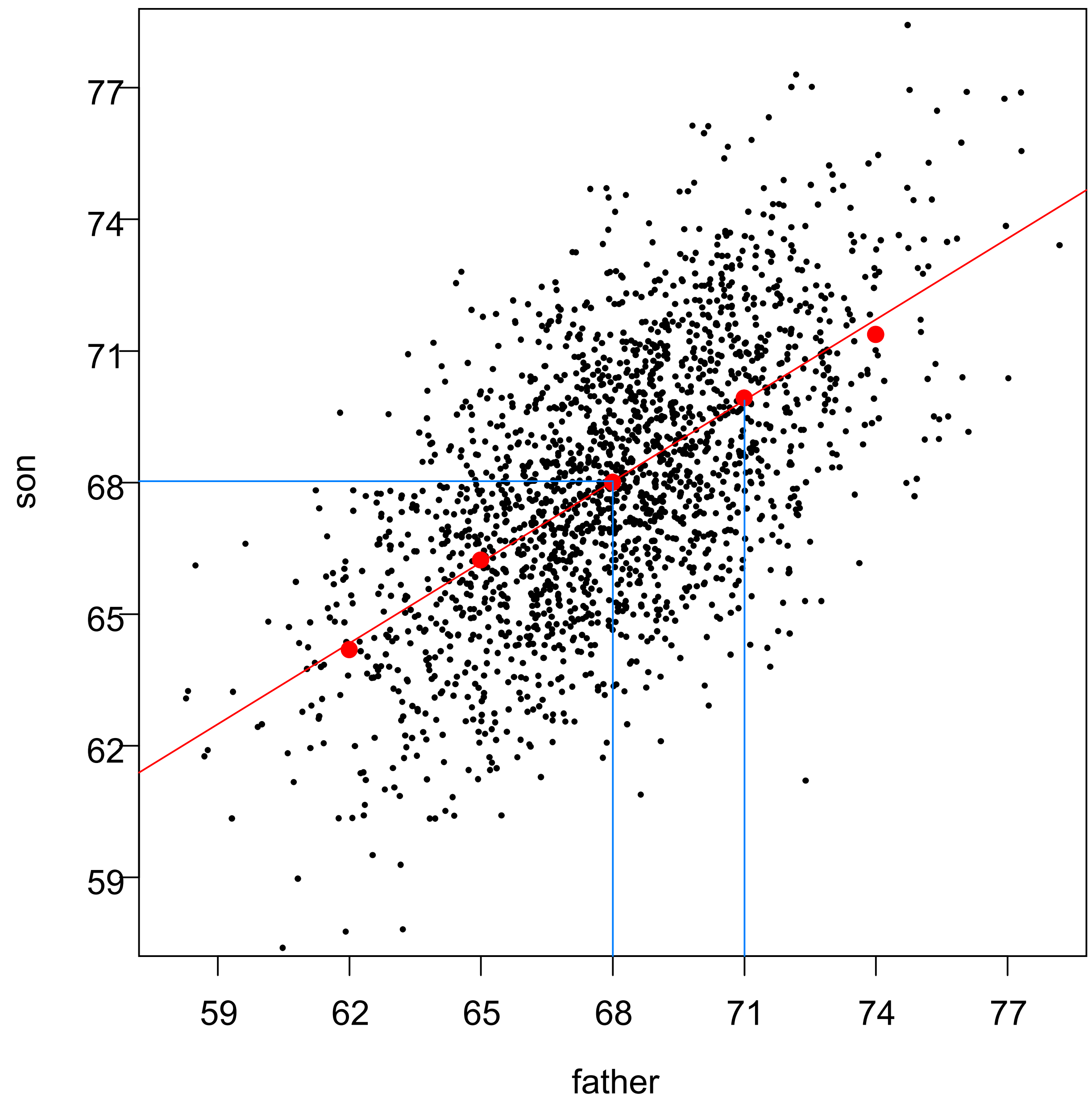
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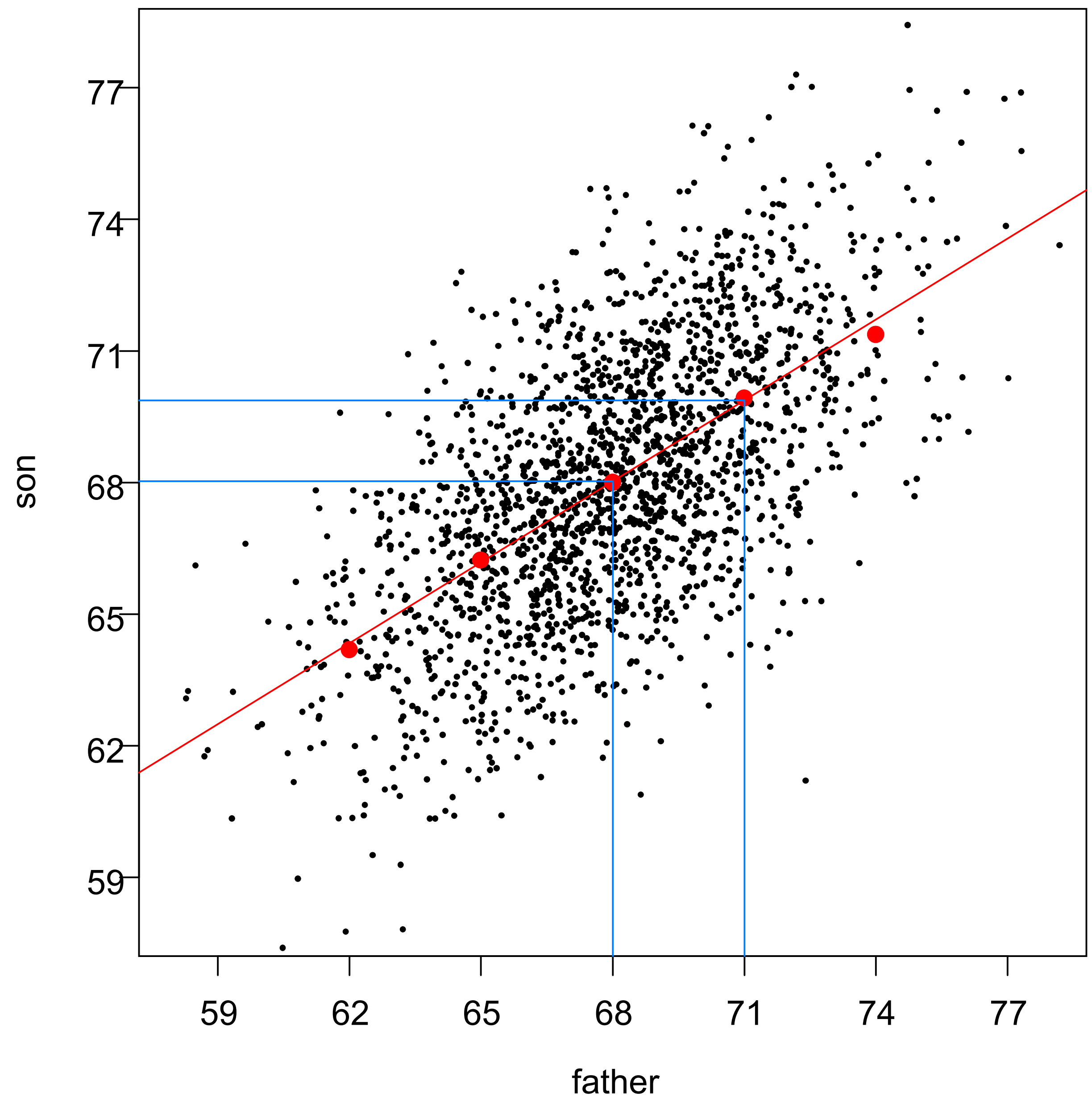

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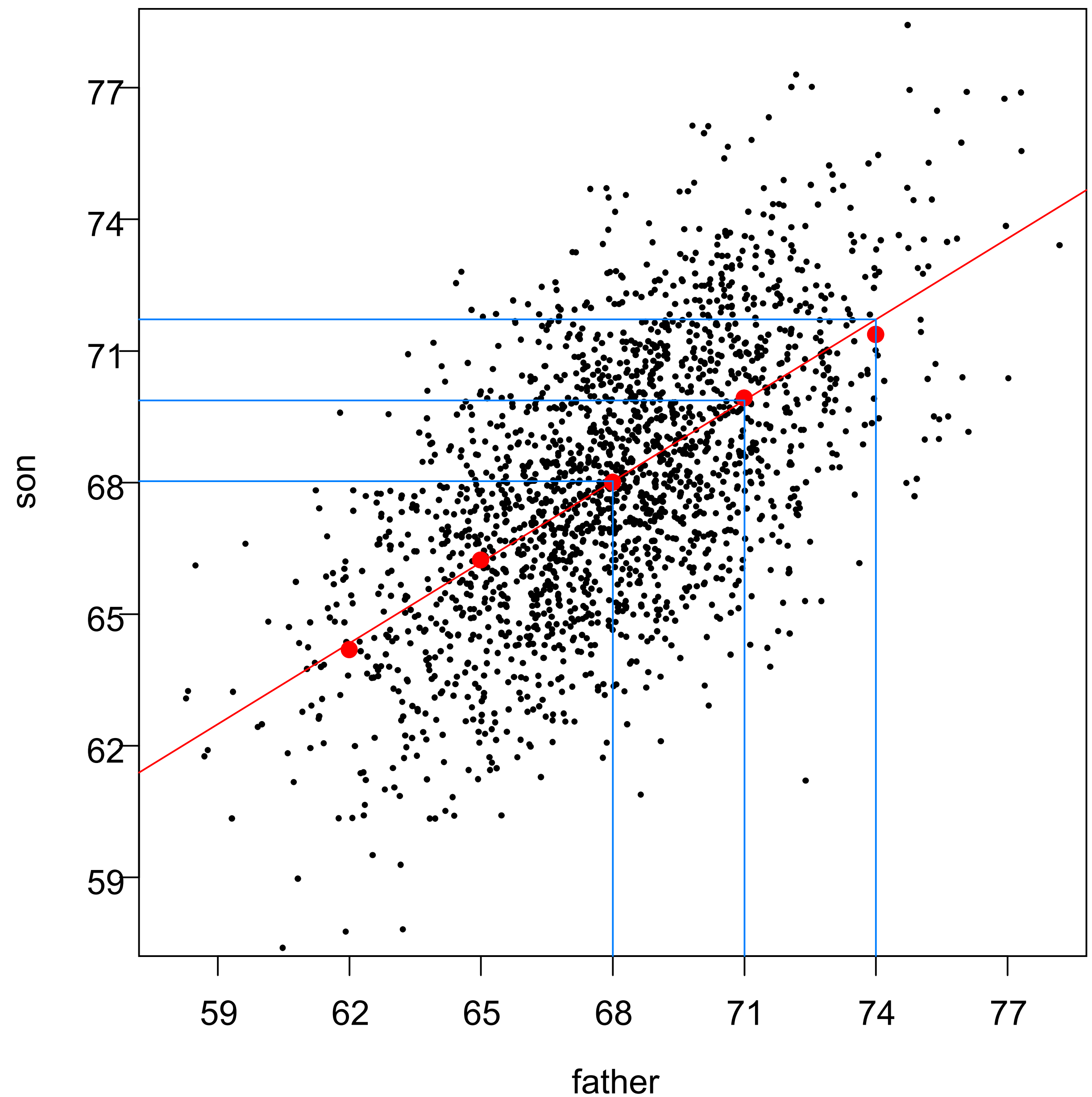
father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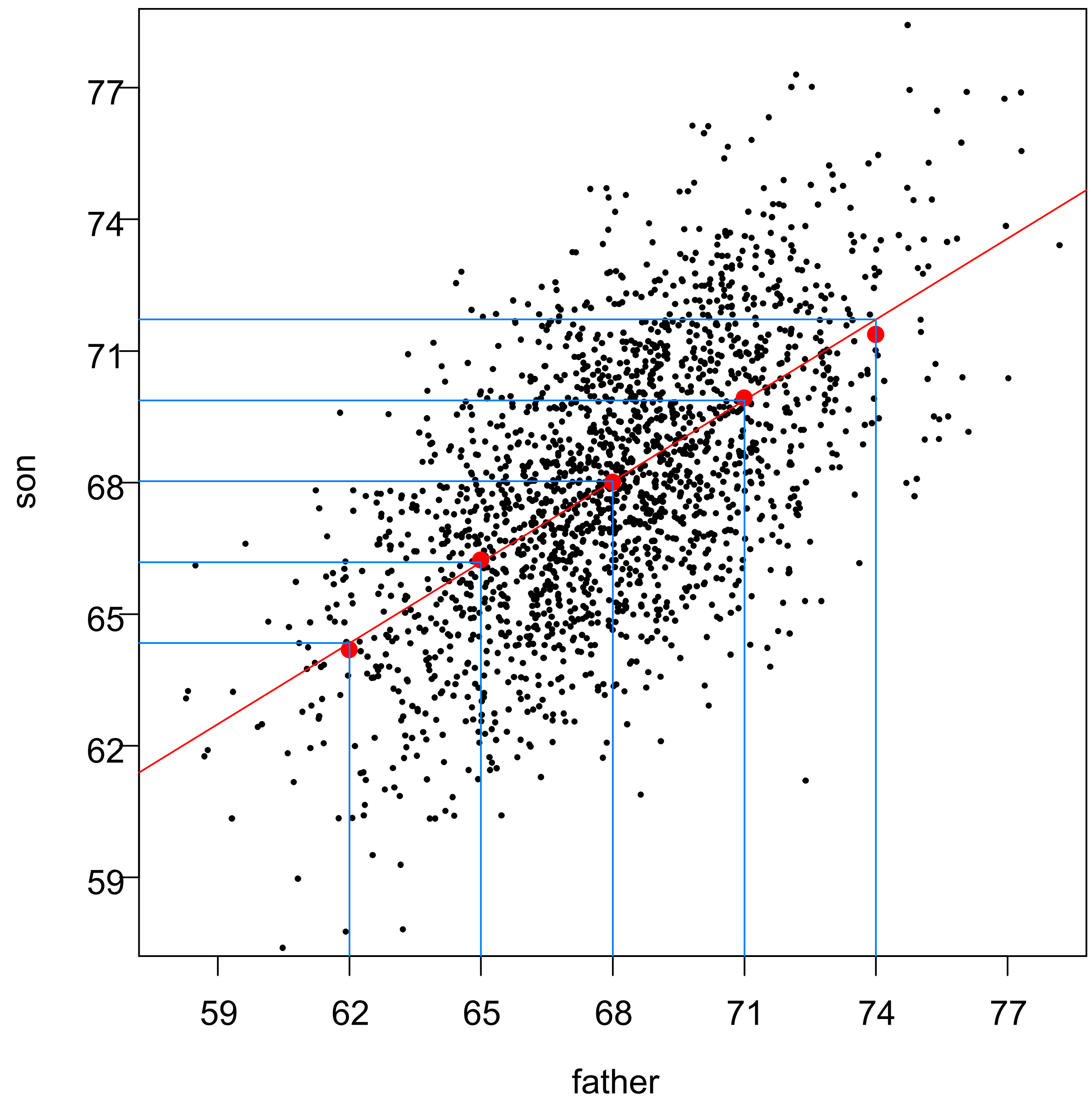
son
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77

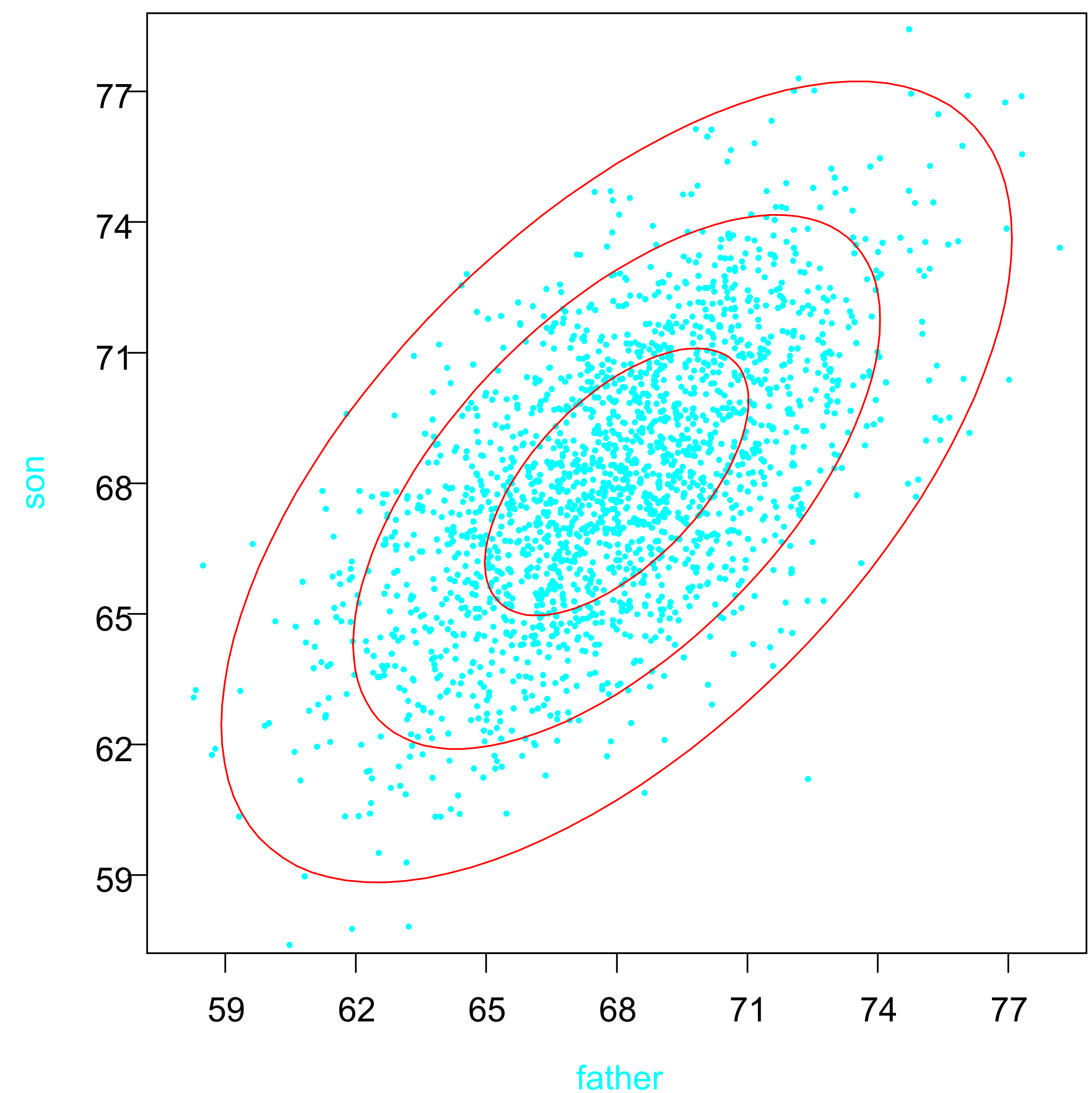
| Father's height | Son's expected height |
|---|---|
| 68 | |

| Father’s height | Son’s expected height |
| --- | --- |
| $68$ | $68$ |

| Father's height | Son's expected height |
|---|---|
| $68$ | $68$ |
| $71 = 68 + 3$ | |

| Father's height | Son's expected height |
|---|---|
| $68$ | $68$ |
| $71 = 68 + 3$ | $69.8 = 68 + r \times 3$ |

| Father's height | Son's expected height |
|---|---|
| $68$ | $68$ |
| $71 = 68 + 3$ | $69.8 = 68 + r \times 3$ |
| $74 = 68 + 6$ | $71.6 = 68 + r \times 6$ |
| $65 = 68 - 3$ | $66.2 = 68 - r \times 3$ |
| $62 = 68 - 6$ | $64.4 = 68 - r \times 6$ |



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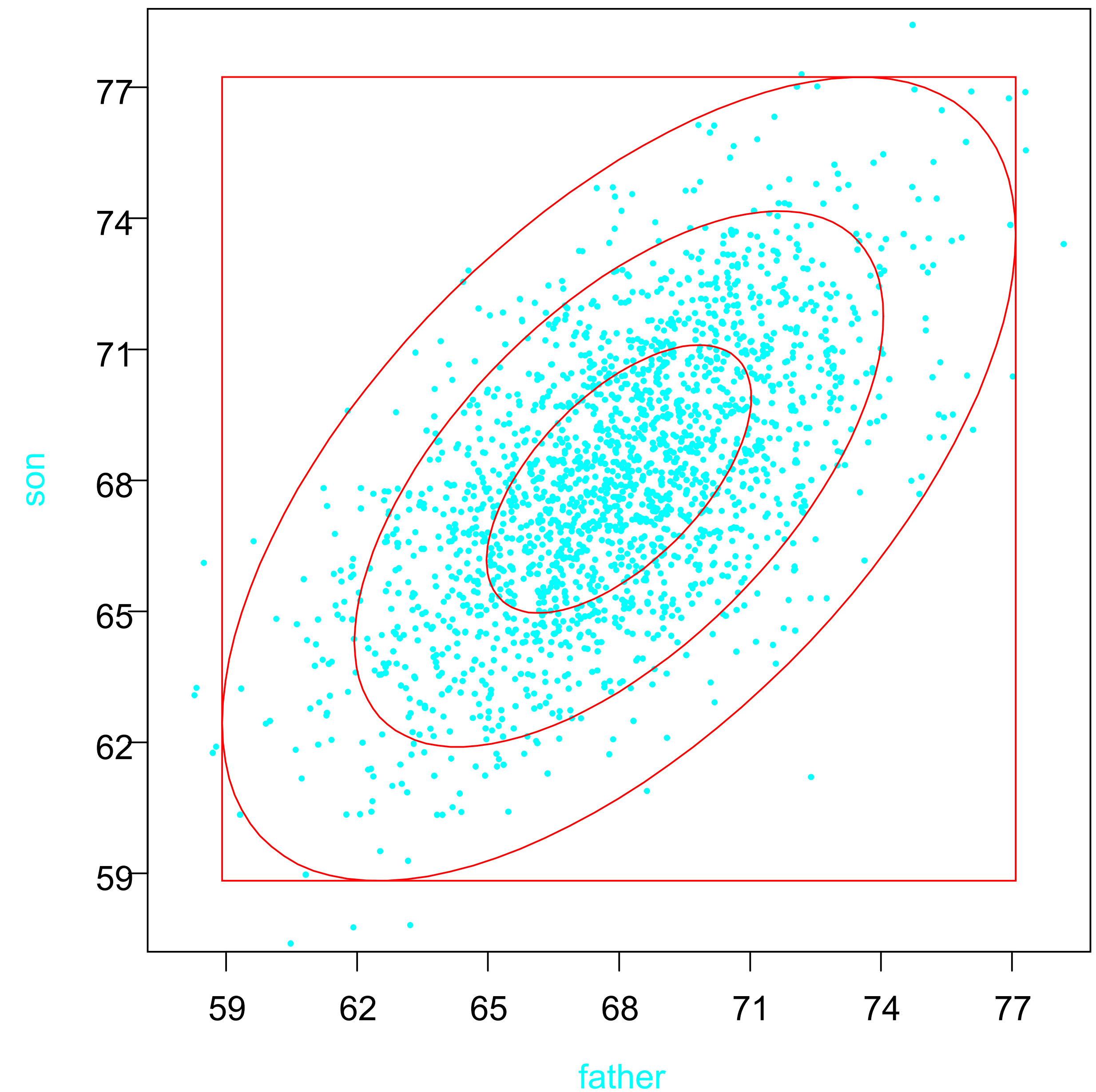
father



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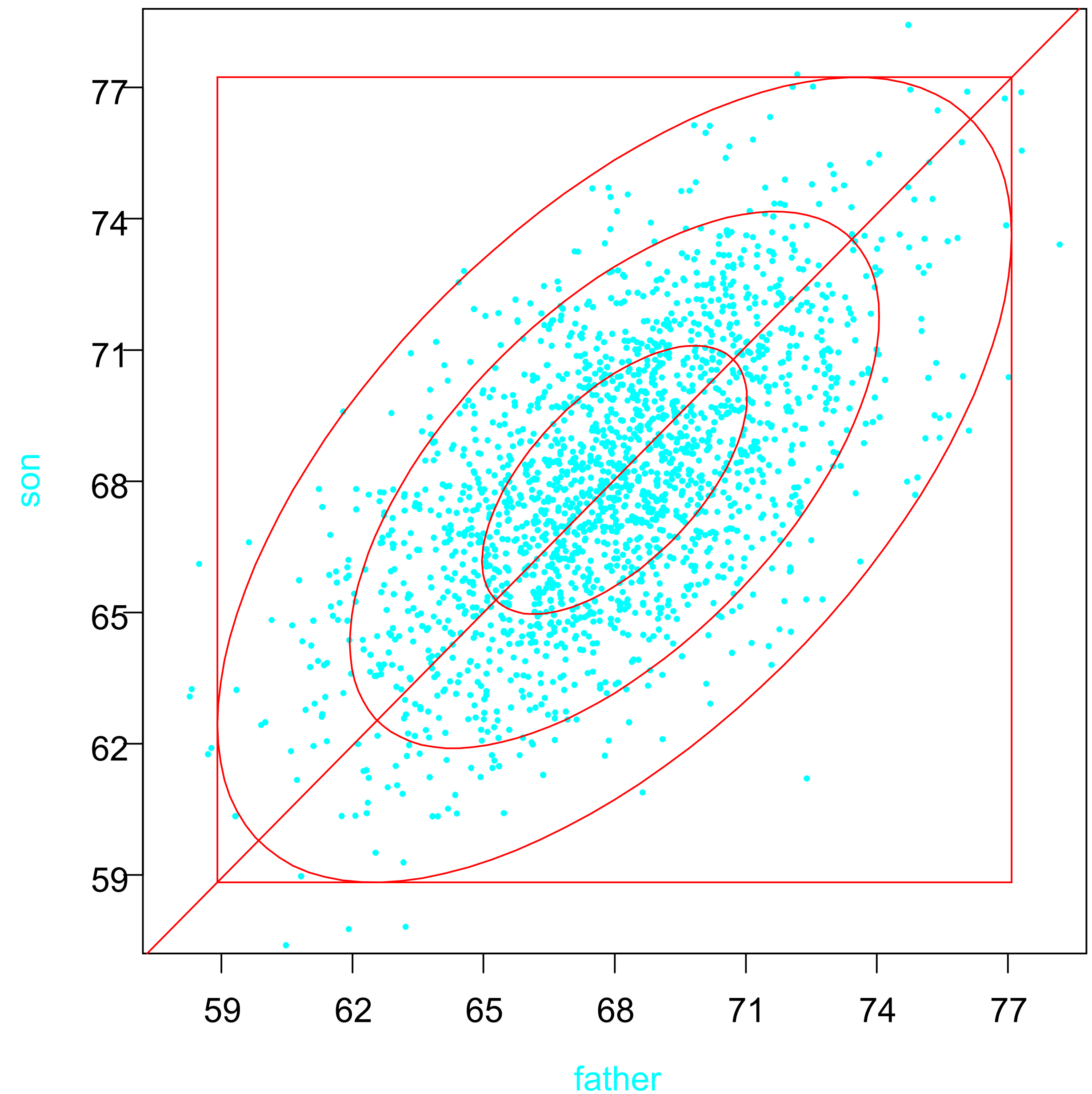

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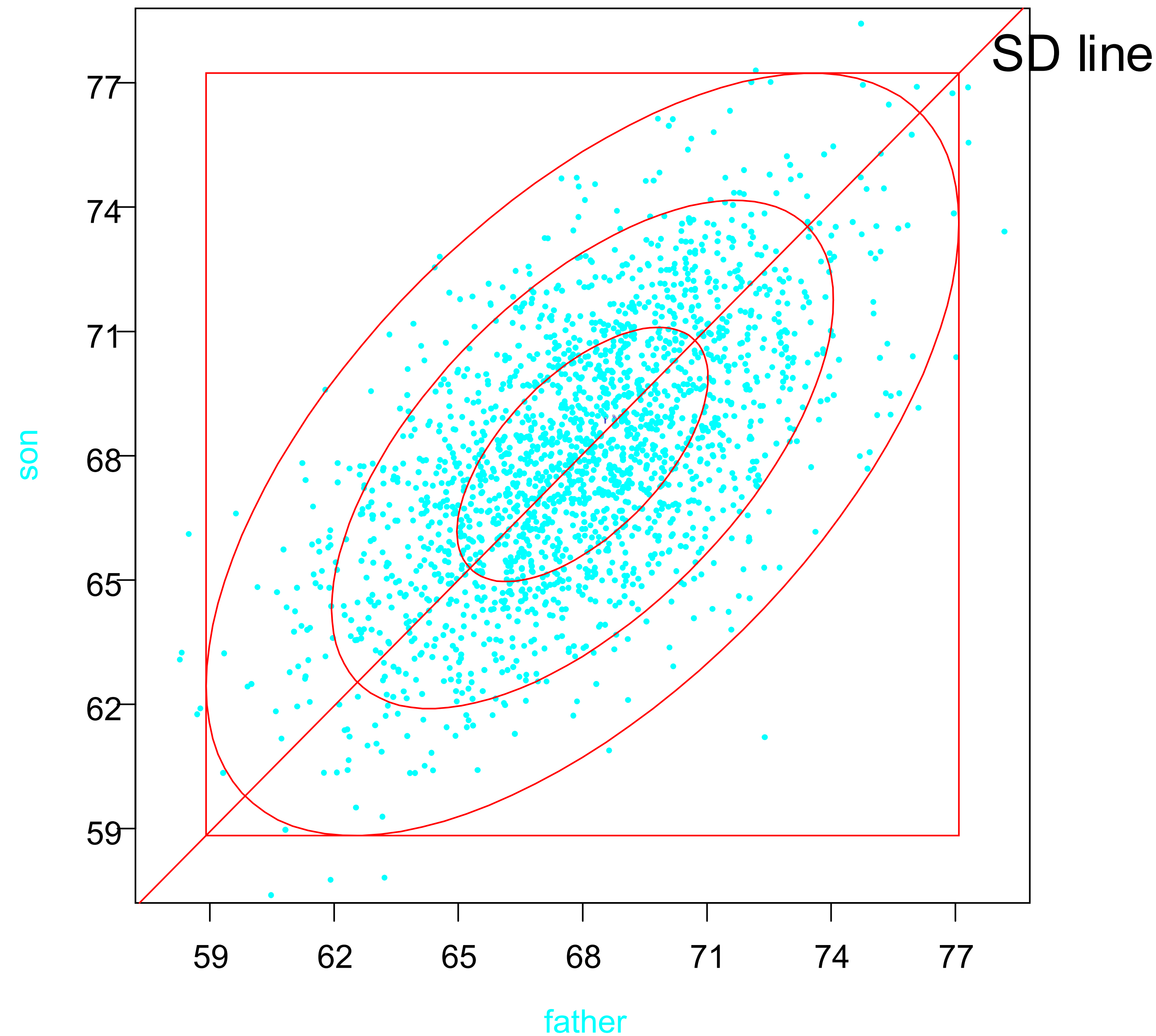
SD line
son
77
74
71
68
65
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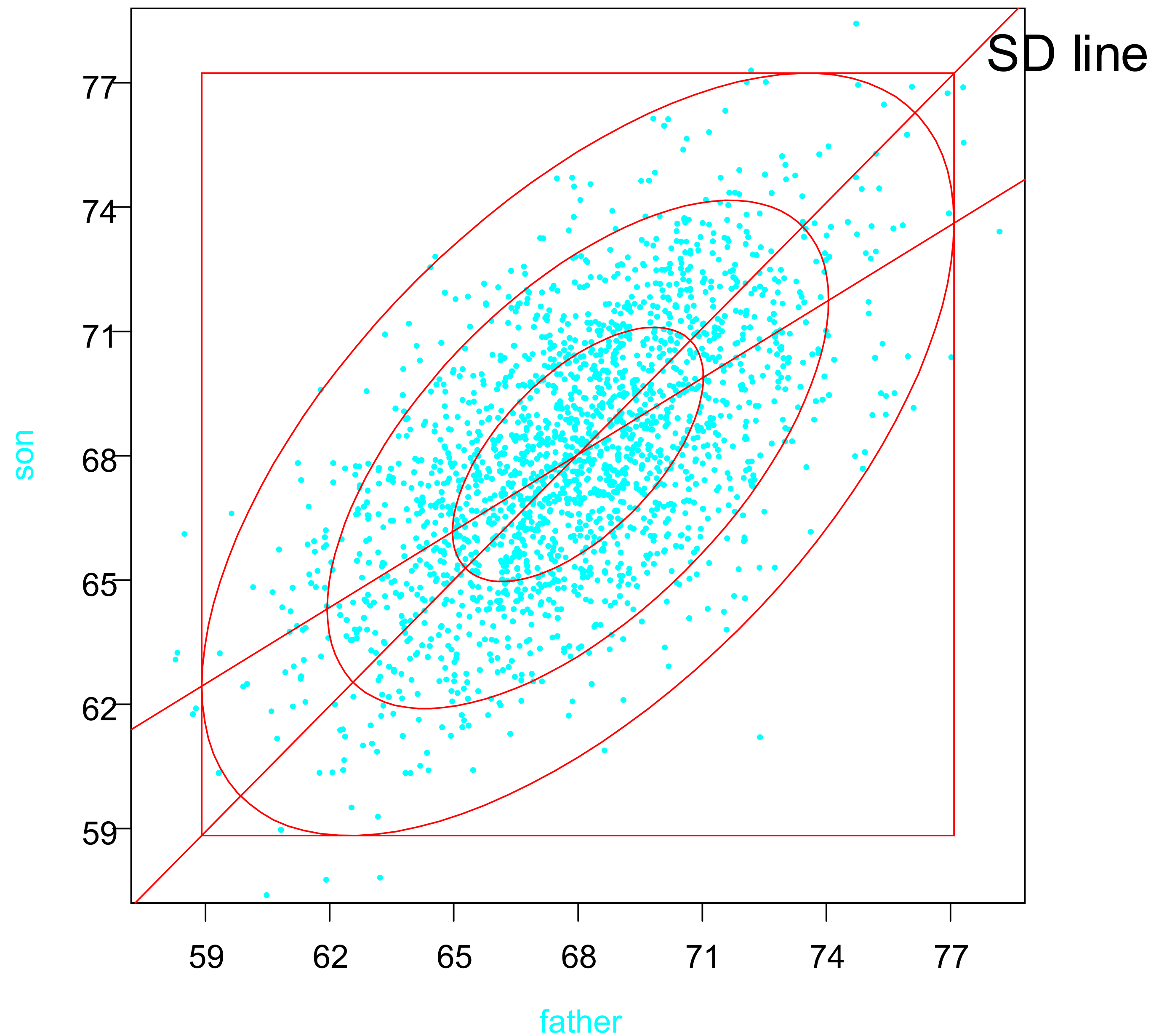
59
62
65
68
71
74
77
father


SD line
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son
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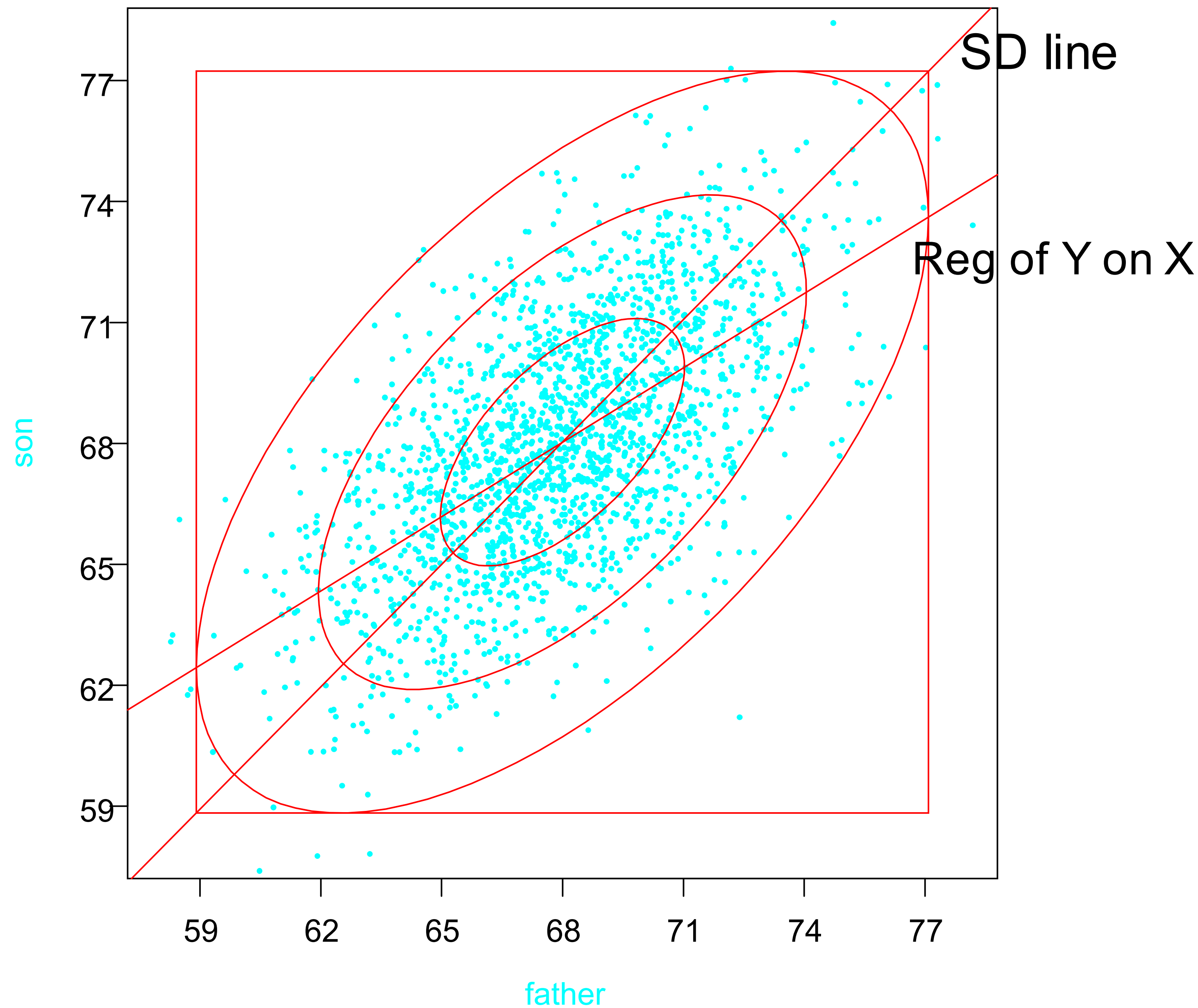
father


SD line
Reg of Y on X
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son
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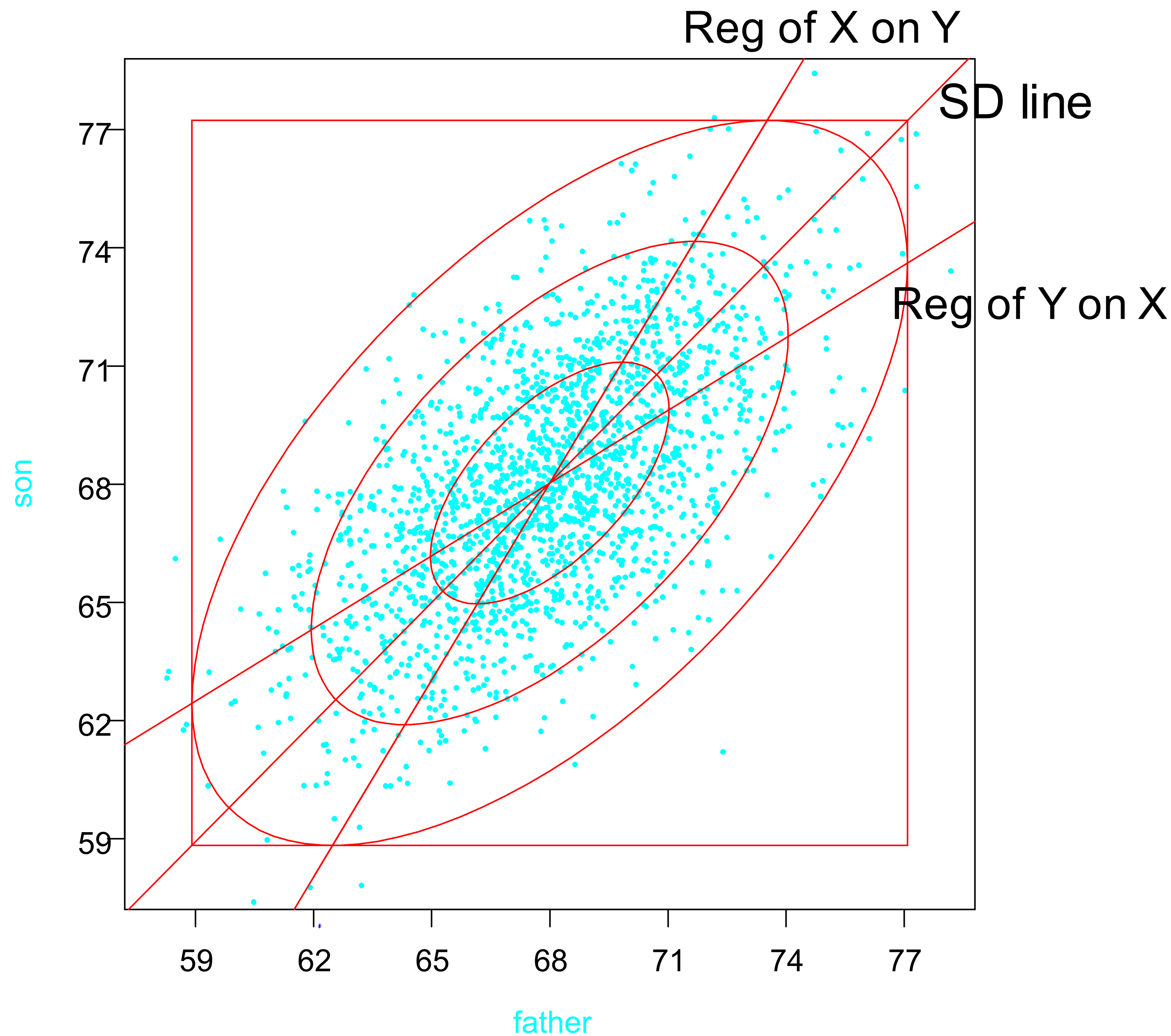
68
71
74
77
father


Reg of X on Y
SD line
Reg of Y on X
77
74
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68
65
62
59
son
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62
65
68
71
74
77
father

Regression Paradox:

## Mathematics:

$$f(X|Y) = f^{-1}(Y|X)$$

## Statistics:

$$E(X|Y) \neq E^{-1}(Y|X)$$

unless $X$ and $Y$ are exact functions of each other.

In this example, the overall distribution of heights remains the same from generation to generation.

But, following individuals, it seems that heights are *regressing* to the overall mean.

Moreover, this works both forward and backward in time!

# Regression Paradox

- The overall distribution of heights stays the same from generation to generation
- But when you follow individuals it looks like heights are being compressed towards the mean.
- Moreover, the same thing happens whether you go forward or backward in time.

Simply

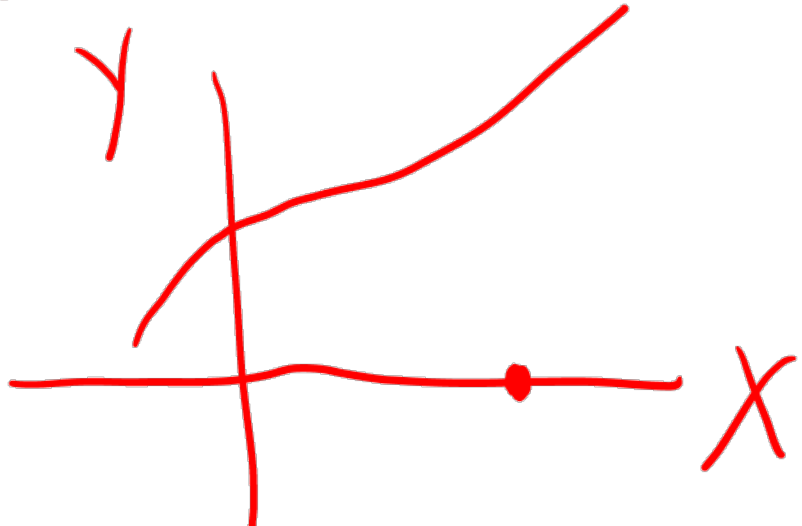
The regression of $Y$ on $X$, $E(Y|X)$ is not the mathematical inverse of the regression of $X$ on $Y$, $E(X|Y)$

Mathematical inverse

Simply

The regression of $Y$ on $X$, $E(Y|X)$ is not the mathematical inverse of the regression of $X$ on $Y$, $E(X|Y)$

Mathematical inverse



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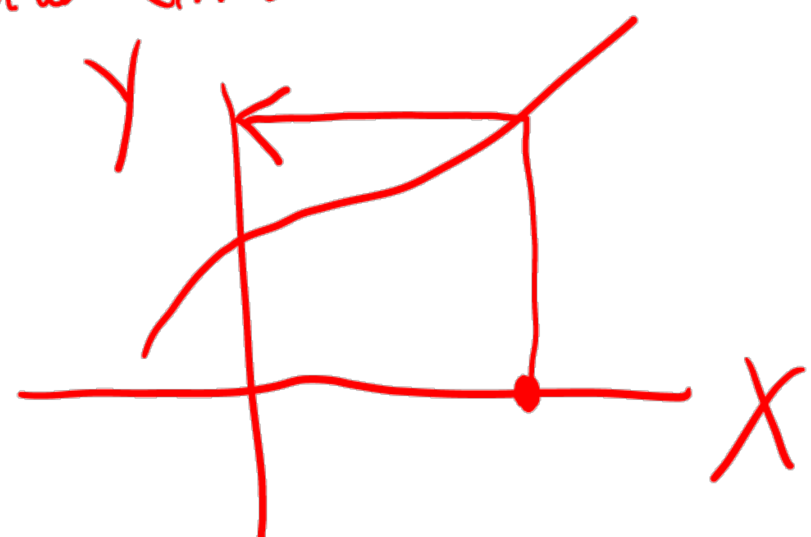
Mathematical inverse



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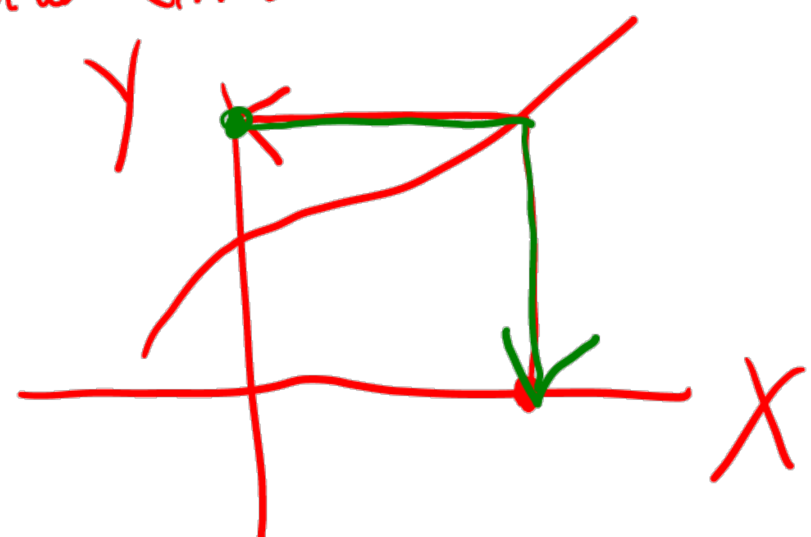
Mathematical inverse



**Simply**

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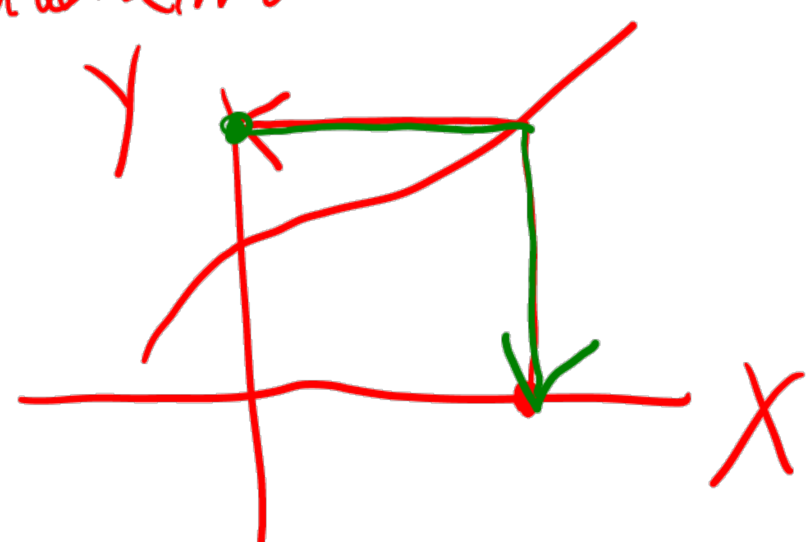
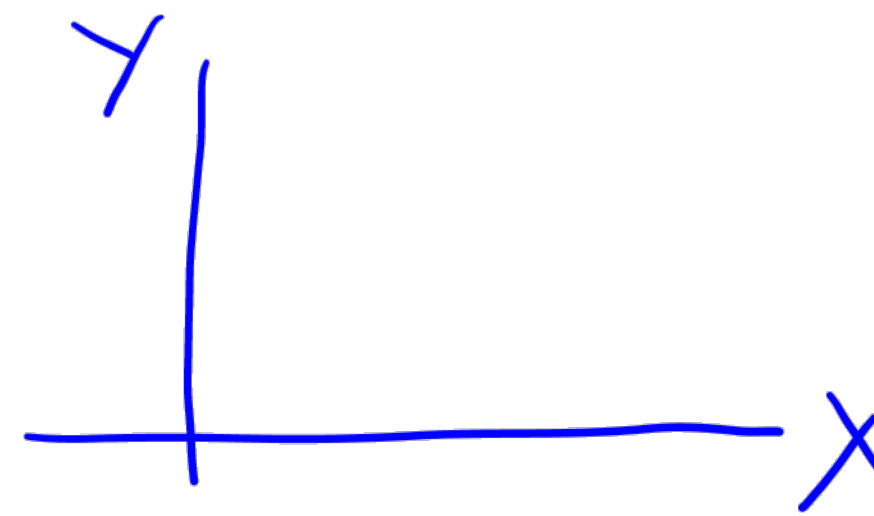
Mathematical inverse




Statistical inverse



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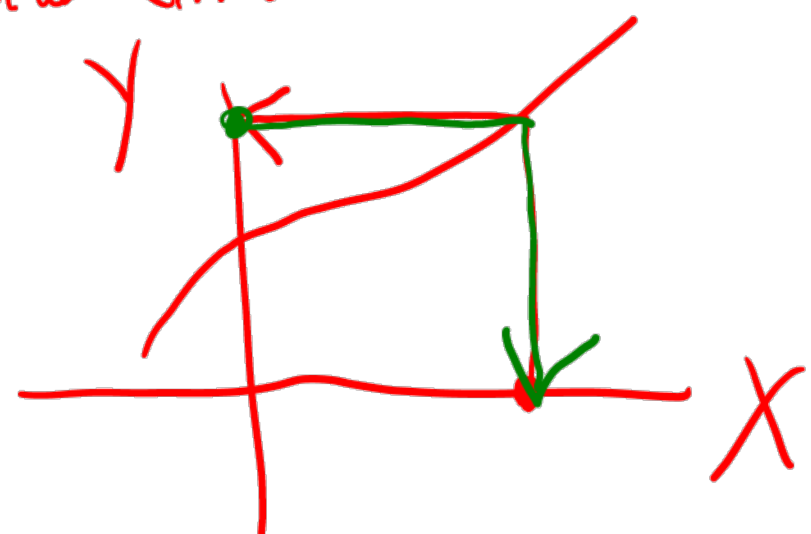
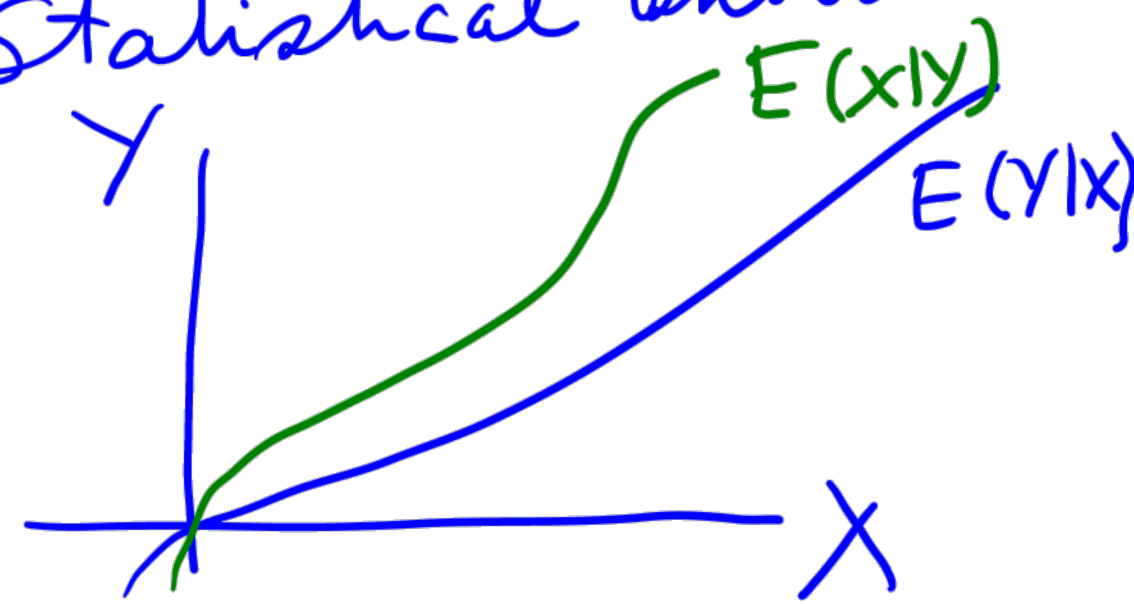
Mathematical inverse




Statistical Inverse



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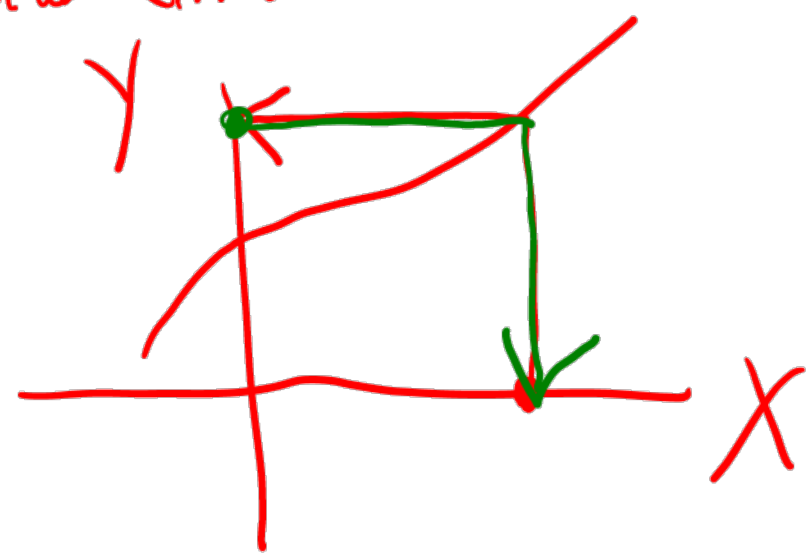
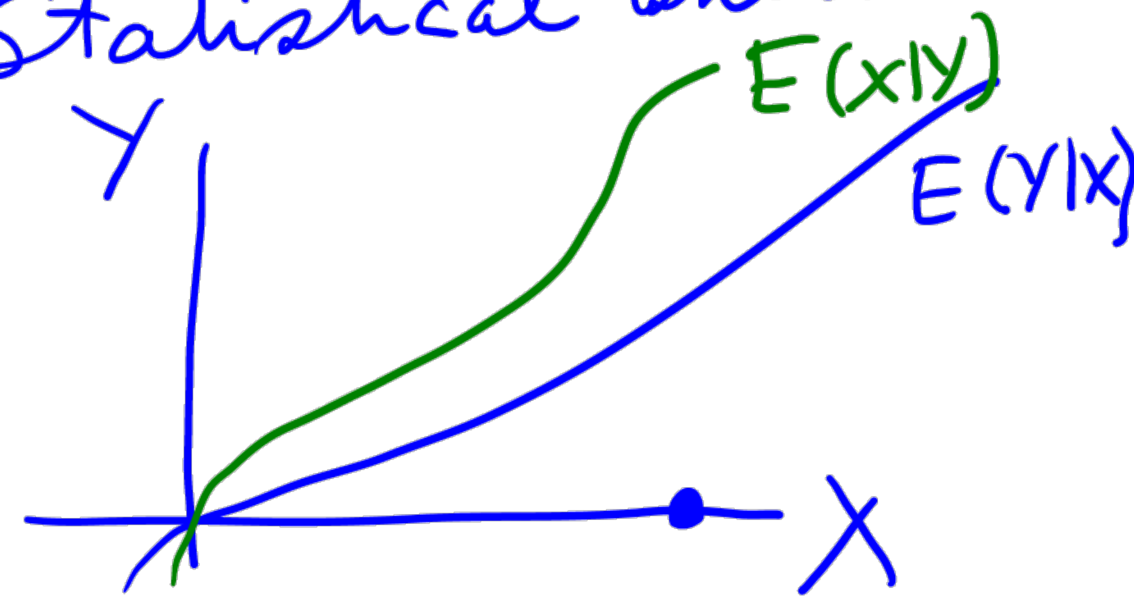
Mathematical inverse



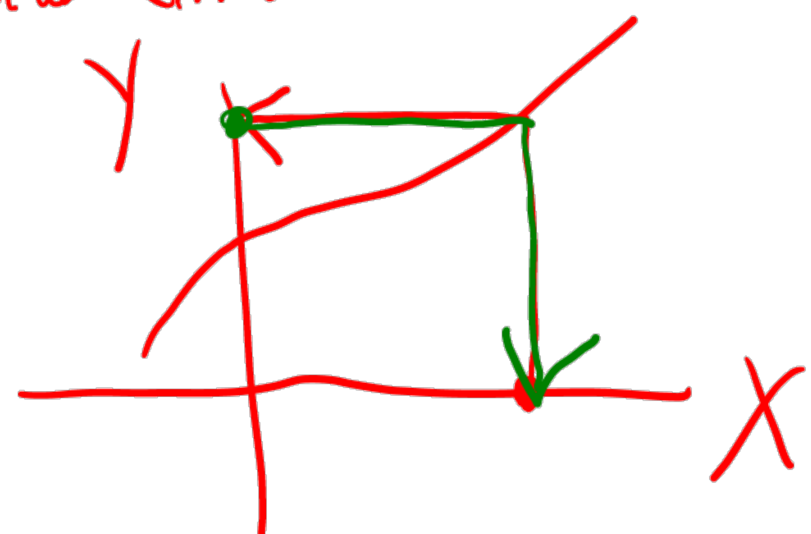

Statistical inverse



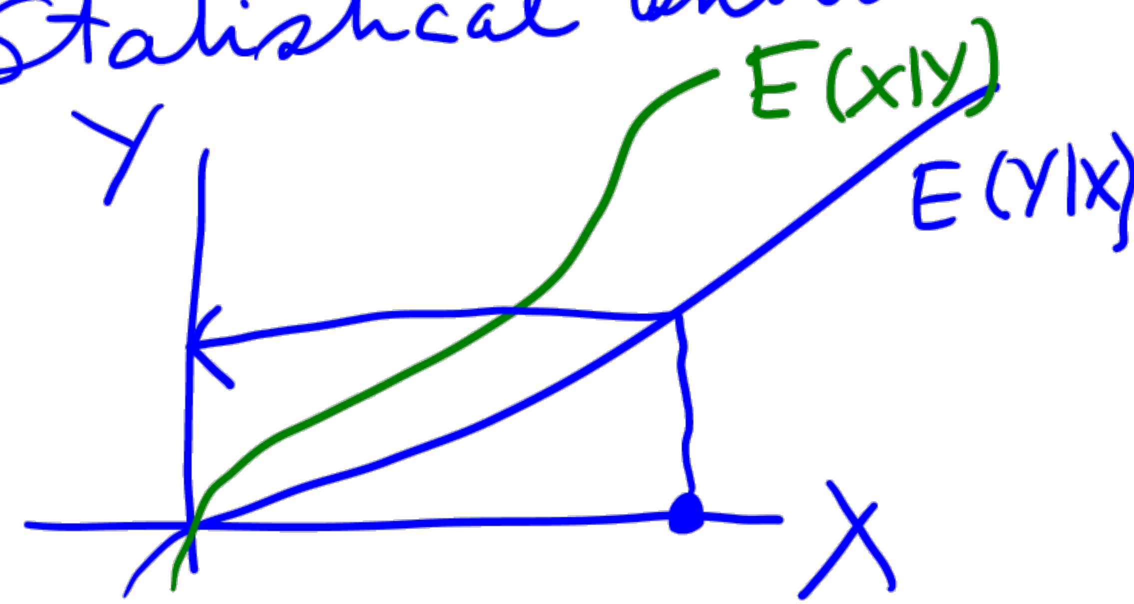
## Simply

The regression of Y on X, E(Y|X)
is not the mathematical inverse of
the regression of X on Y, E(X|Y)

Mathematical inverse




Statistical inverse



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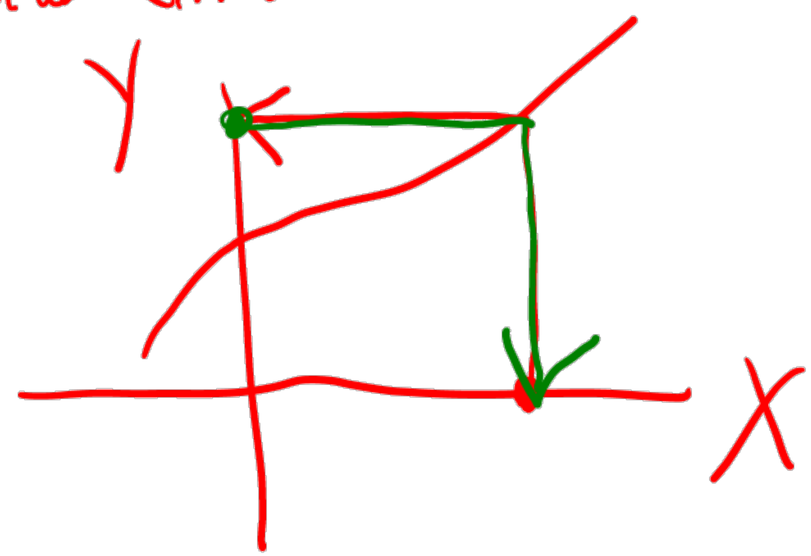
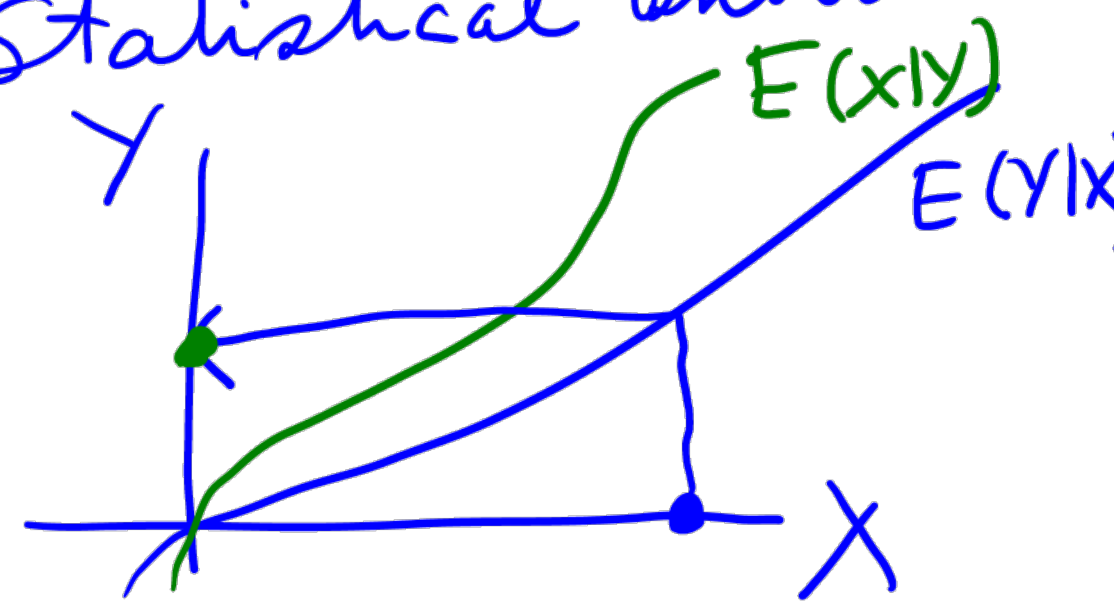
Mathematical inverse




Statistical inverse



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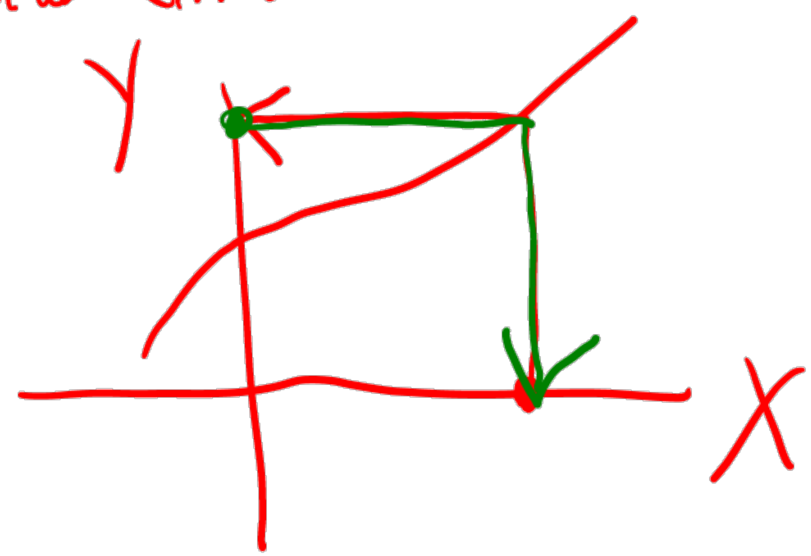
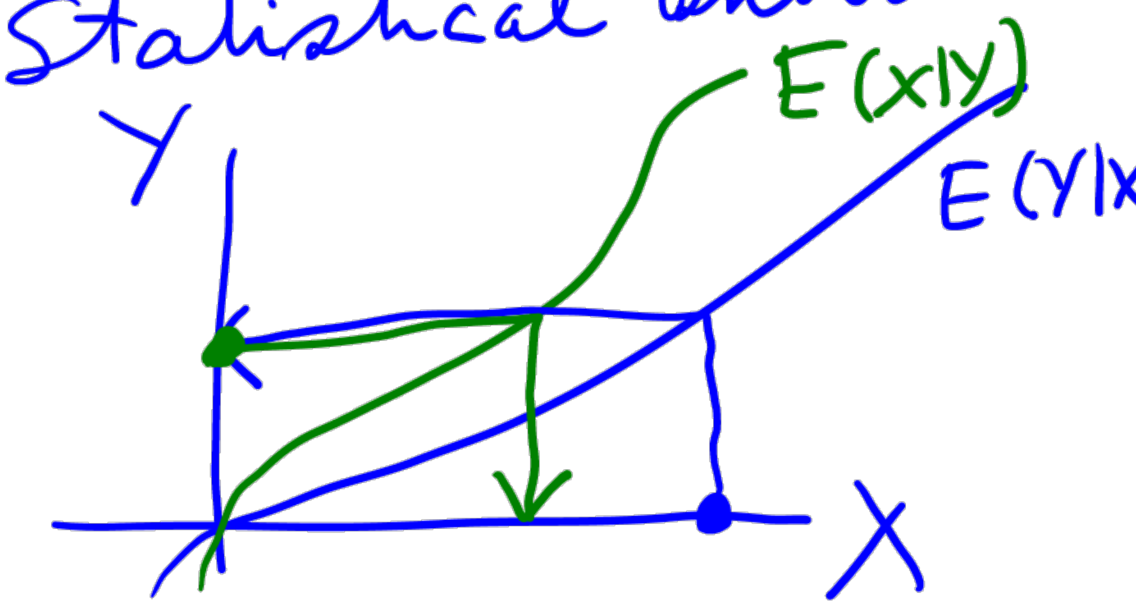
Mathematical inverse




Statistical inverse



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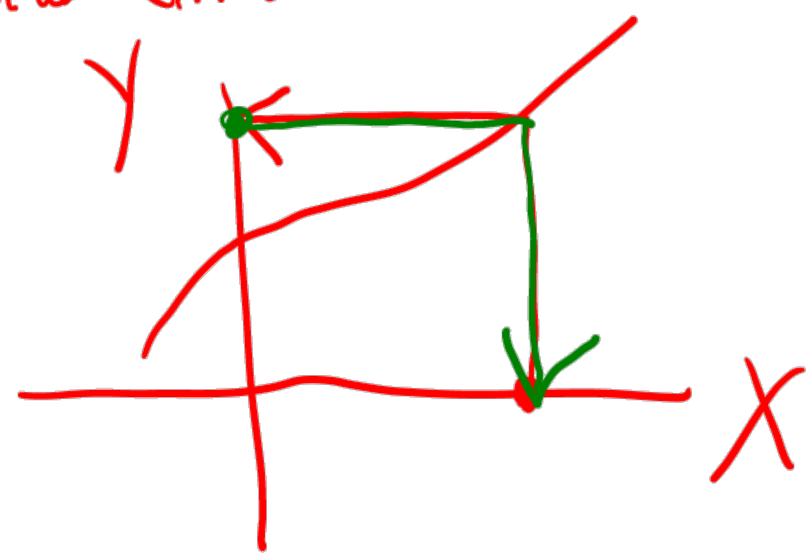
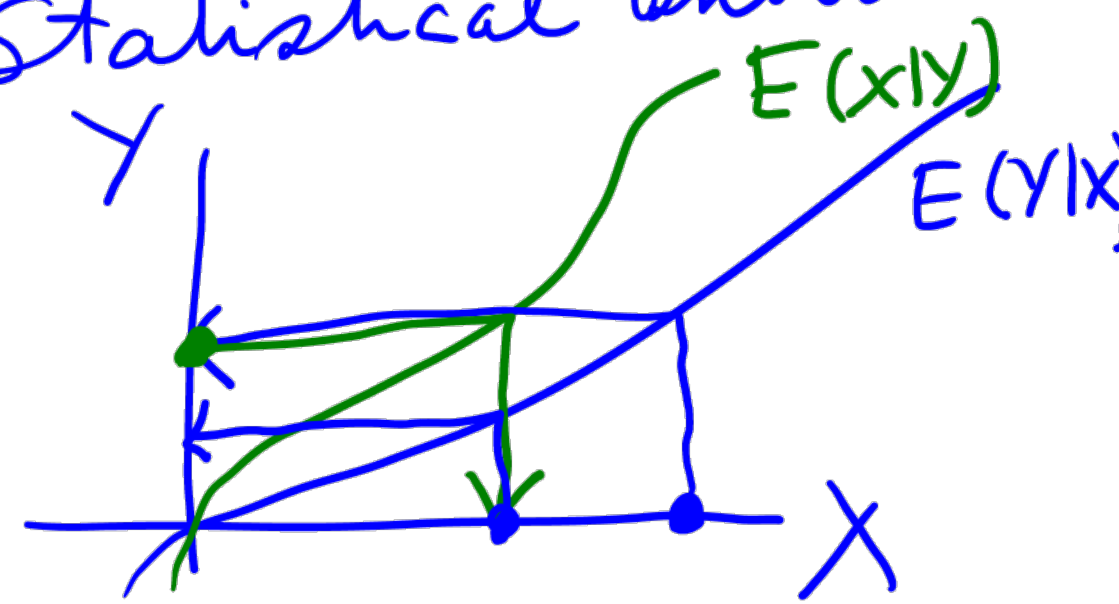
Mathematical inverse




Statistical inverse



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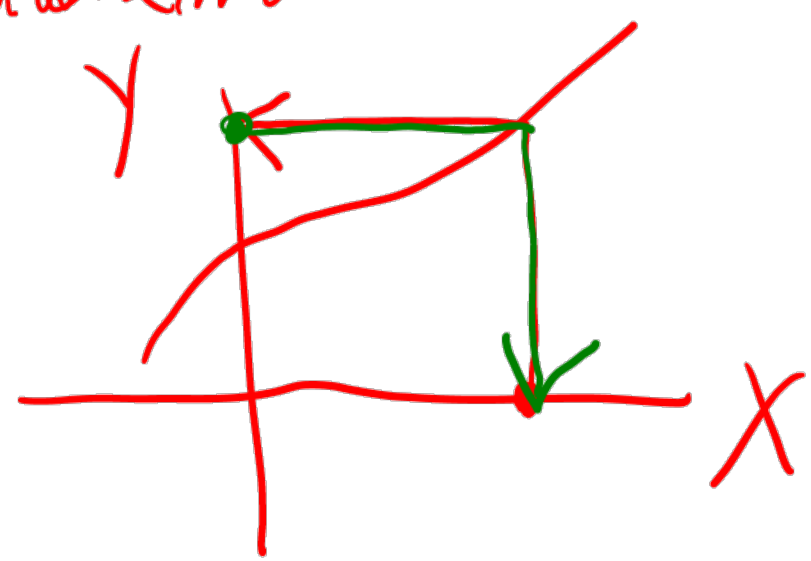
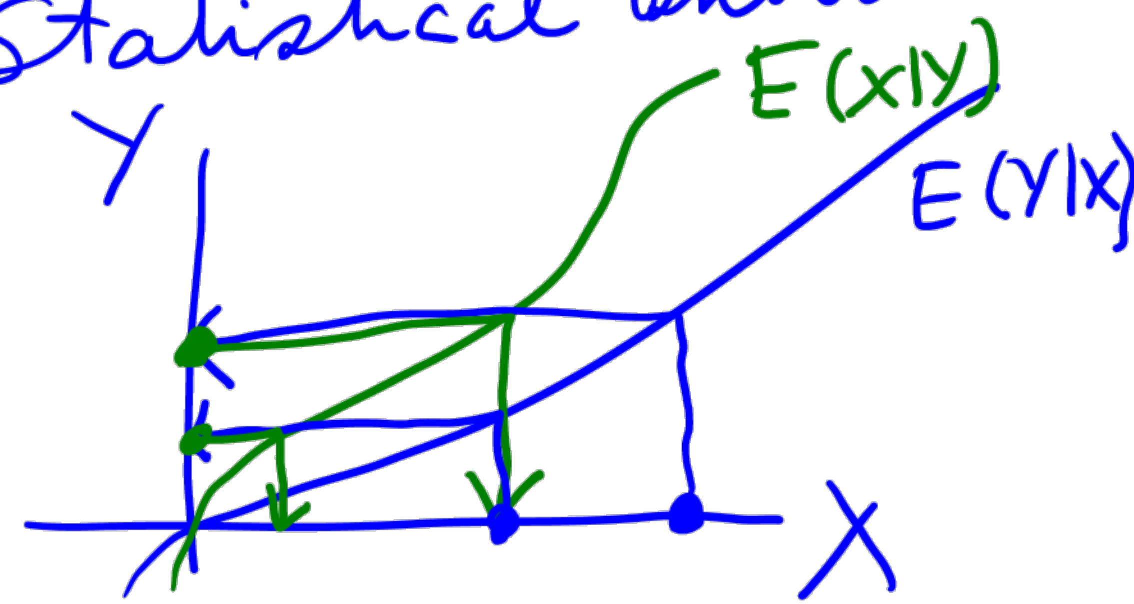
Mathematical inverse




Statistical inverse



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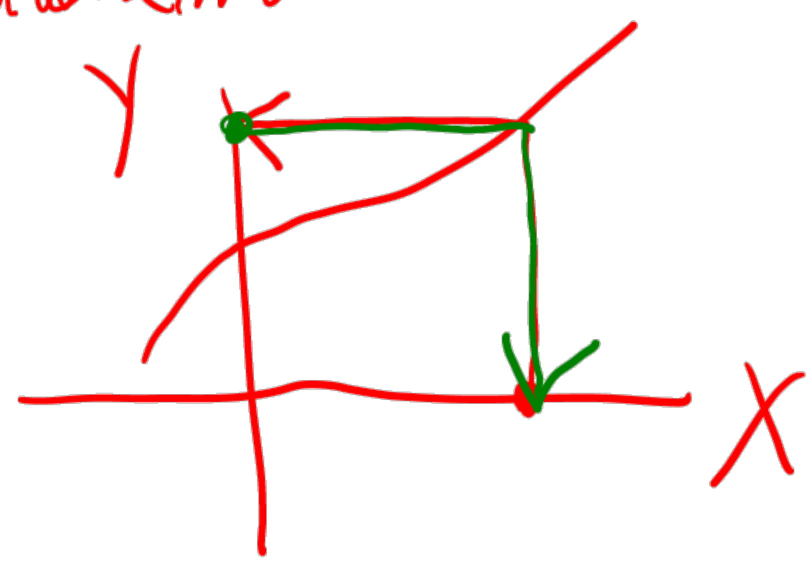
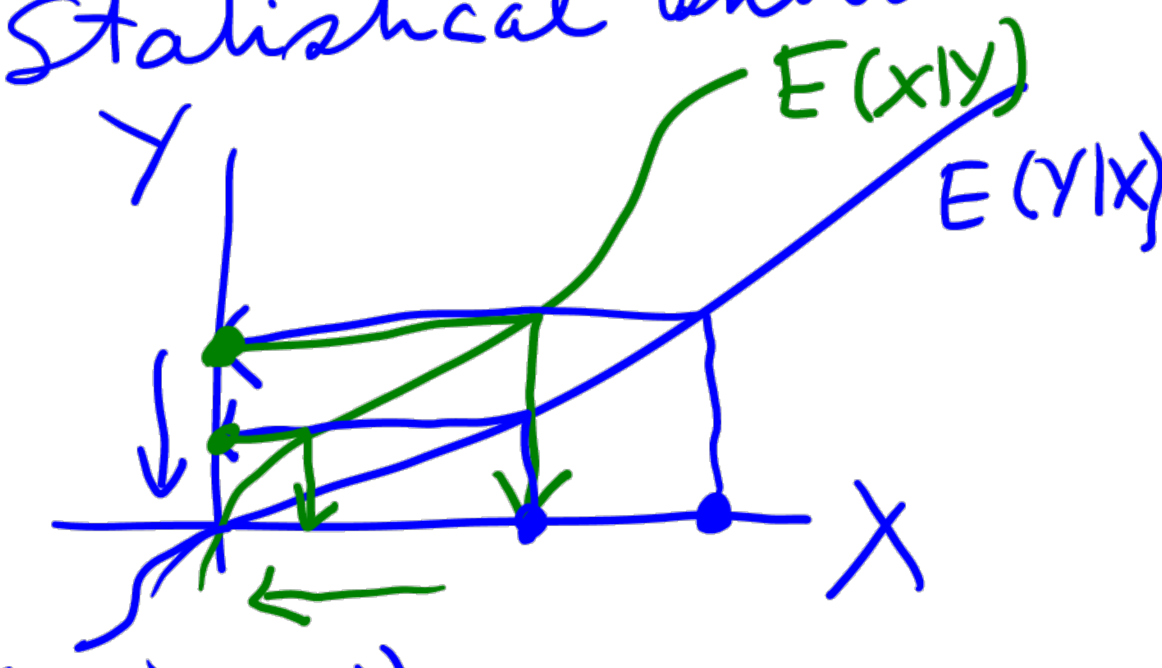
Mathematical inverse




Statistical inverse



# 2 Eyeballing a scatterplot

At a glance:

- What is the correlation?
- Is the relationship statistically significant?
- Approximate 95% confidence interval?

# Some data:

```
1971 Canadian Occupational Prestige Data
[1] Occupational title
[2] Average education of incumbents, years
[3] Average income of incumbents, dollars
[4] Percent of incumbents who are women
[5] Pineo-Porter prestige score for occupation
[6] Canadian Census occupational code
[7] Type of occupation
        prof =  professional and technical
        wc   =  white collar
        bc   =  blue collar
        ?    =  missing (not classified)
Source: Census of Canada, 1971, in Fox (1997)
```

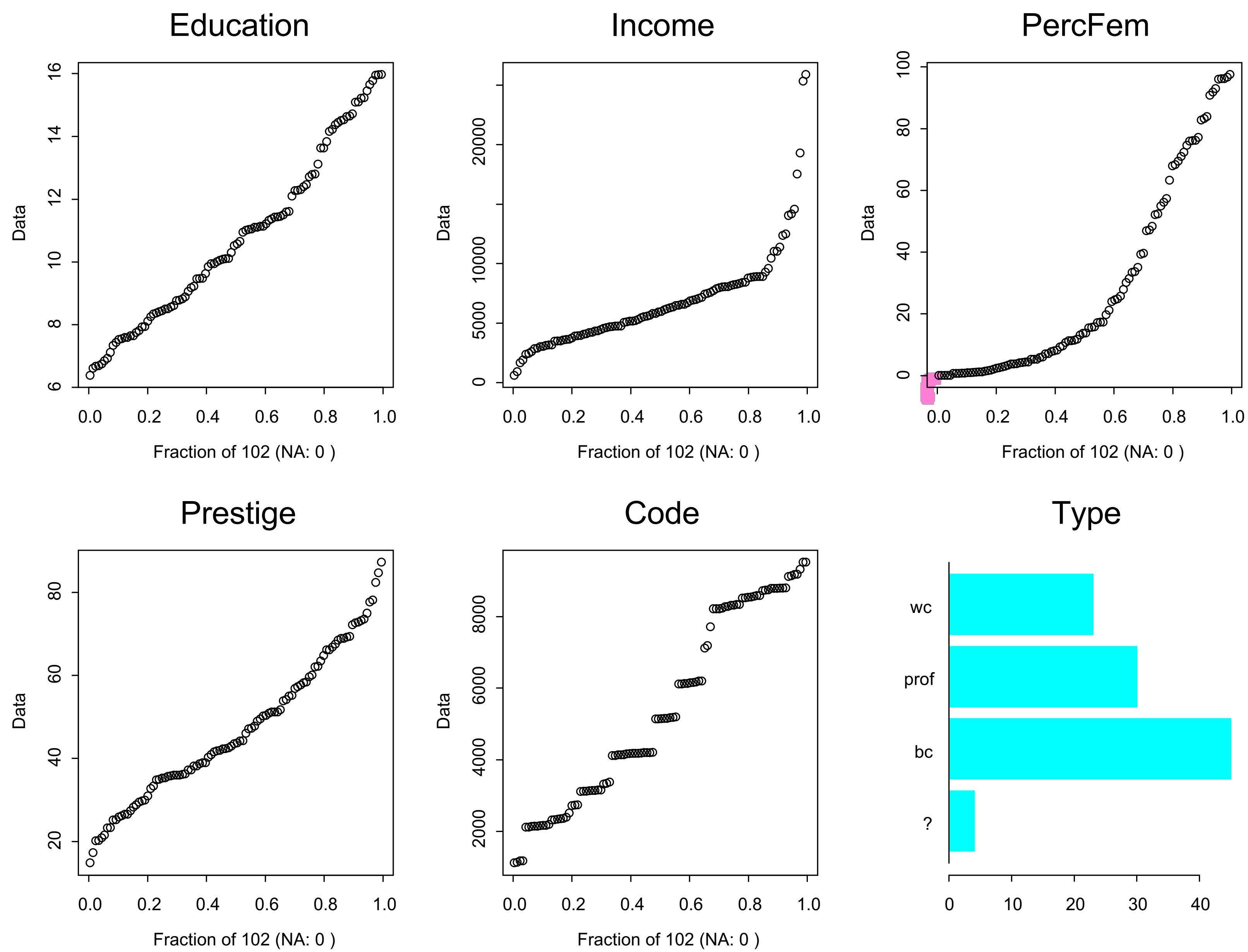
| | Education | Income | PercFem | Prestige | Code | Type |
|---|---|---|---|---|---|---|
| GOV_ADMINISTRATORS | 13.11 | 12351 | 11.16 | 68.8 | 1113 | prof |
| GENERAL_MANAGERS | 12.26 | 25879 | 4.02 | 69.1 | 1130 | prof |
| ACCOUNTANTS | 12.77 | 9271 | 15.70 | 63.4 | 1171 | prof |
| PURCHASING_OFFICERS | 11.42 | 8865 | 9.11 | 56.8 | 1175 | prof |
| CHEMISTS | 14.62 | 8403 | 11.68 | 73.5 | 2111 | prof |
| PHYSICISTS | 15.64 | 11030 | 5.13 | 77.6 | 2113 | prof |
| BIOLOGISTS | 15.09 | 8258 | 25.65 | 72.6 | 2133 | prof |
| ARCHITECTS | 15.44 | 14163 | 2.69 | 78.1 | 2141 | prof |
| CIVIL_ENGINEERS | 14.52 | 11377 | 1.03 | 73.1 | 2143 | prof |
| MINING_ENGINEERS | 14.64 | 11023 | 0.94 | 68.8 | 2153 | prof |

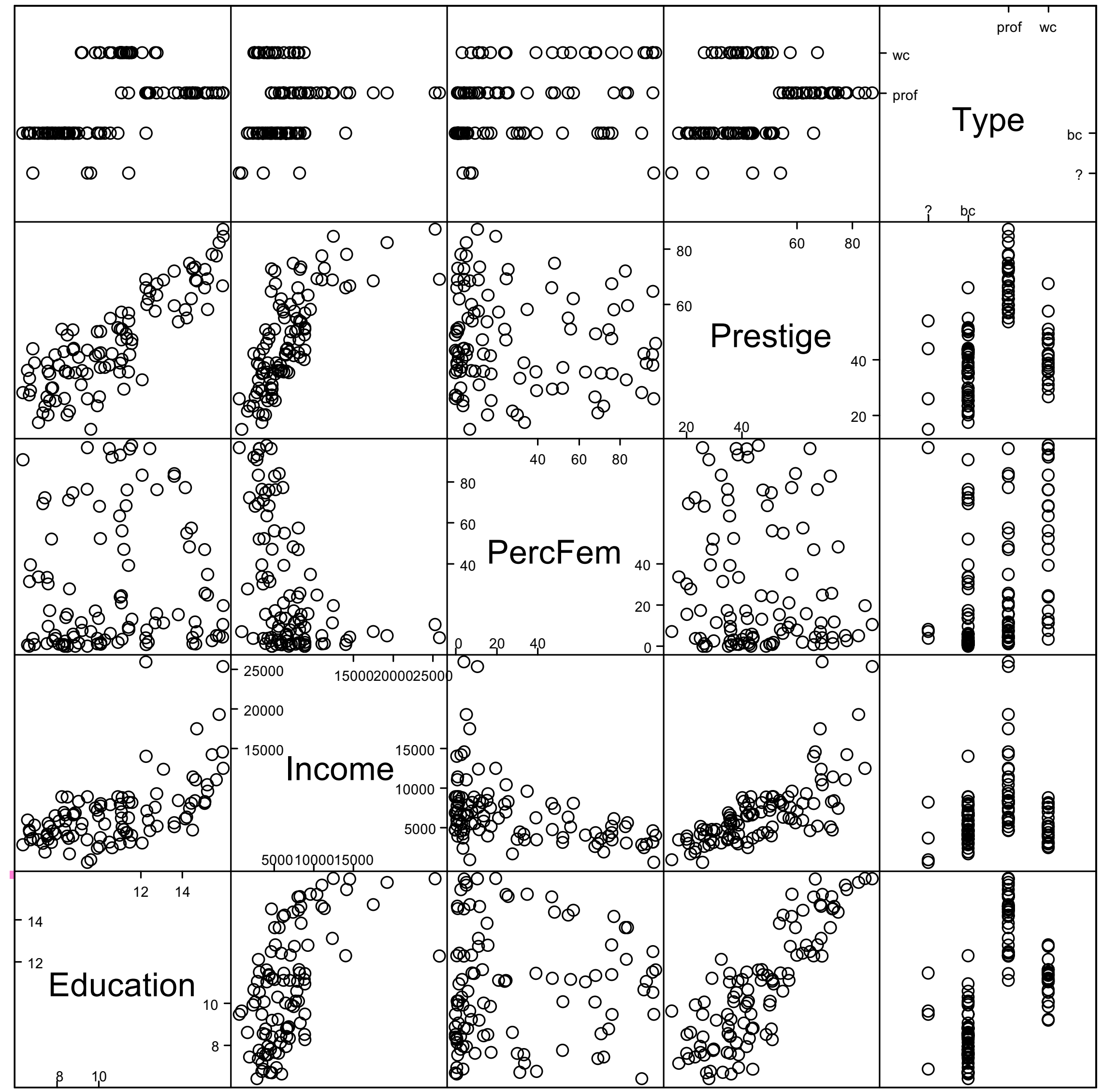
...

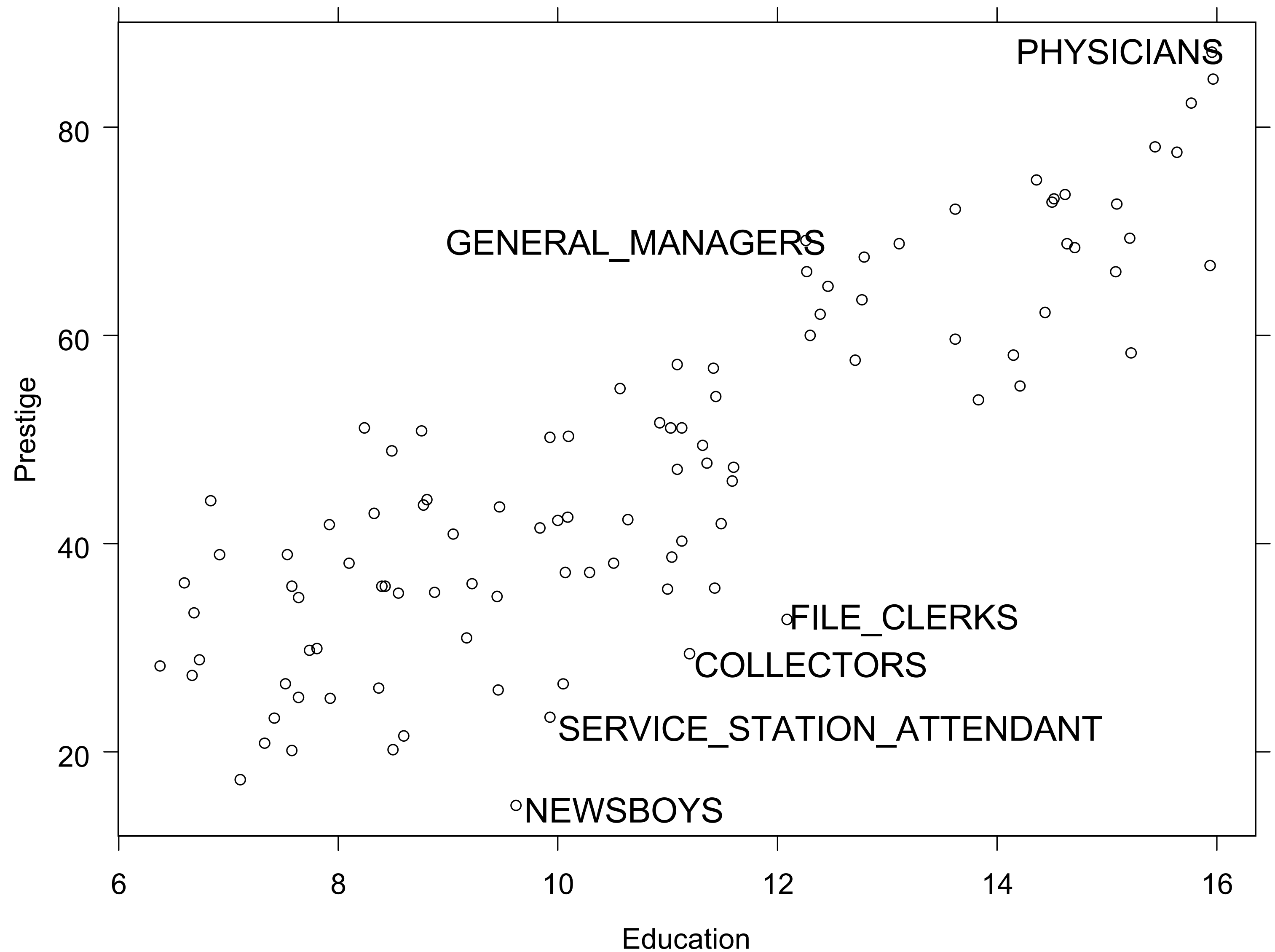
102 occupations

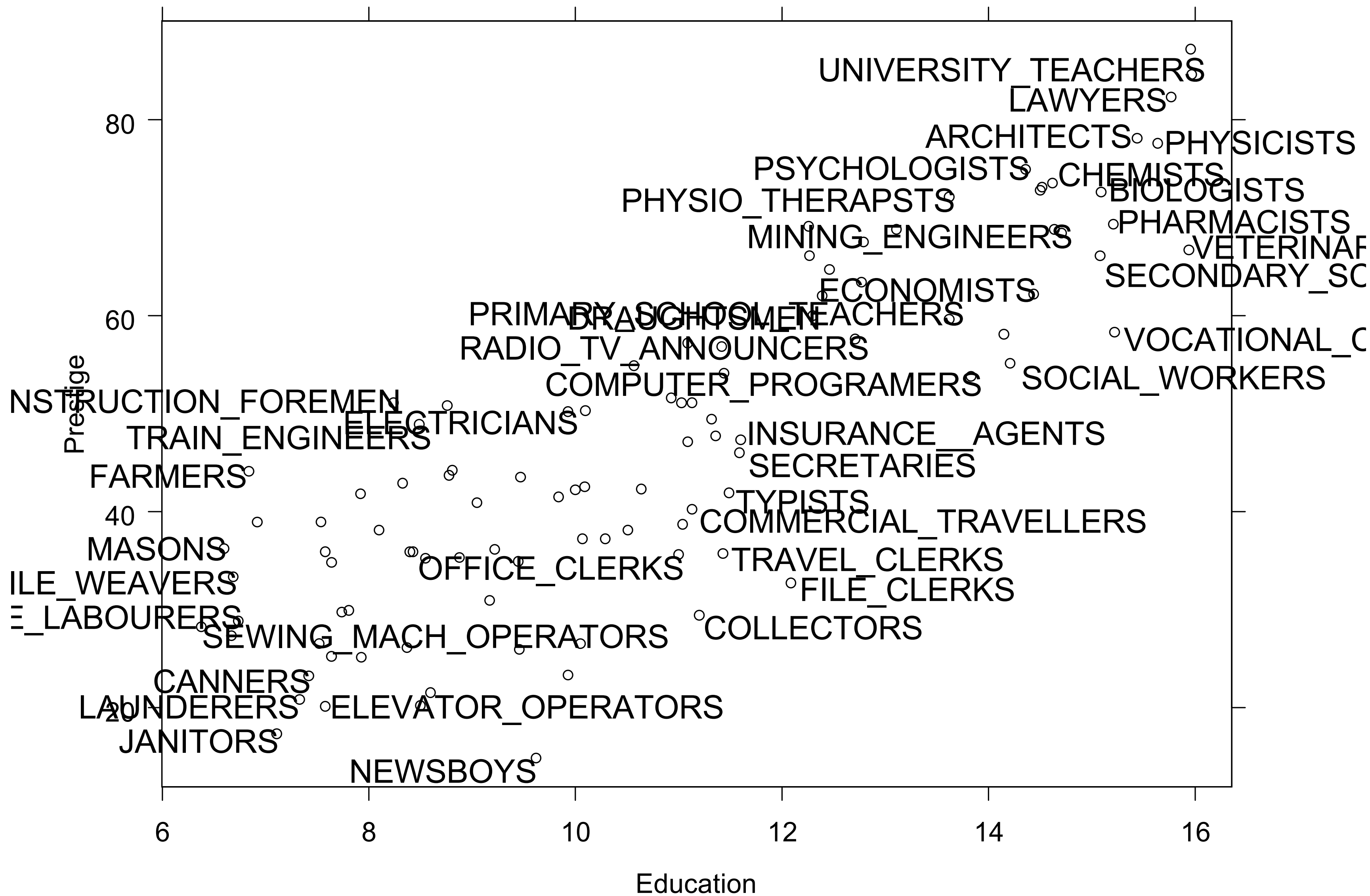
# Summary:

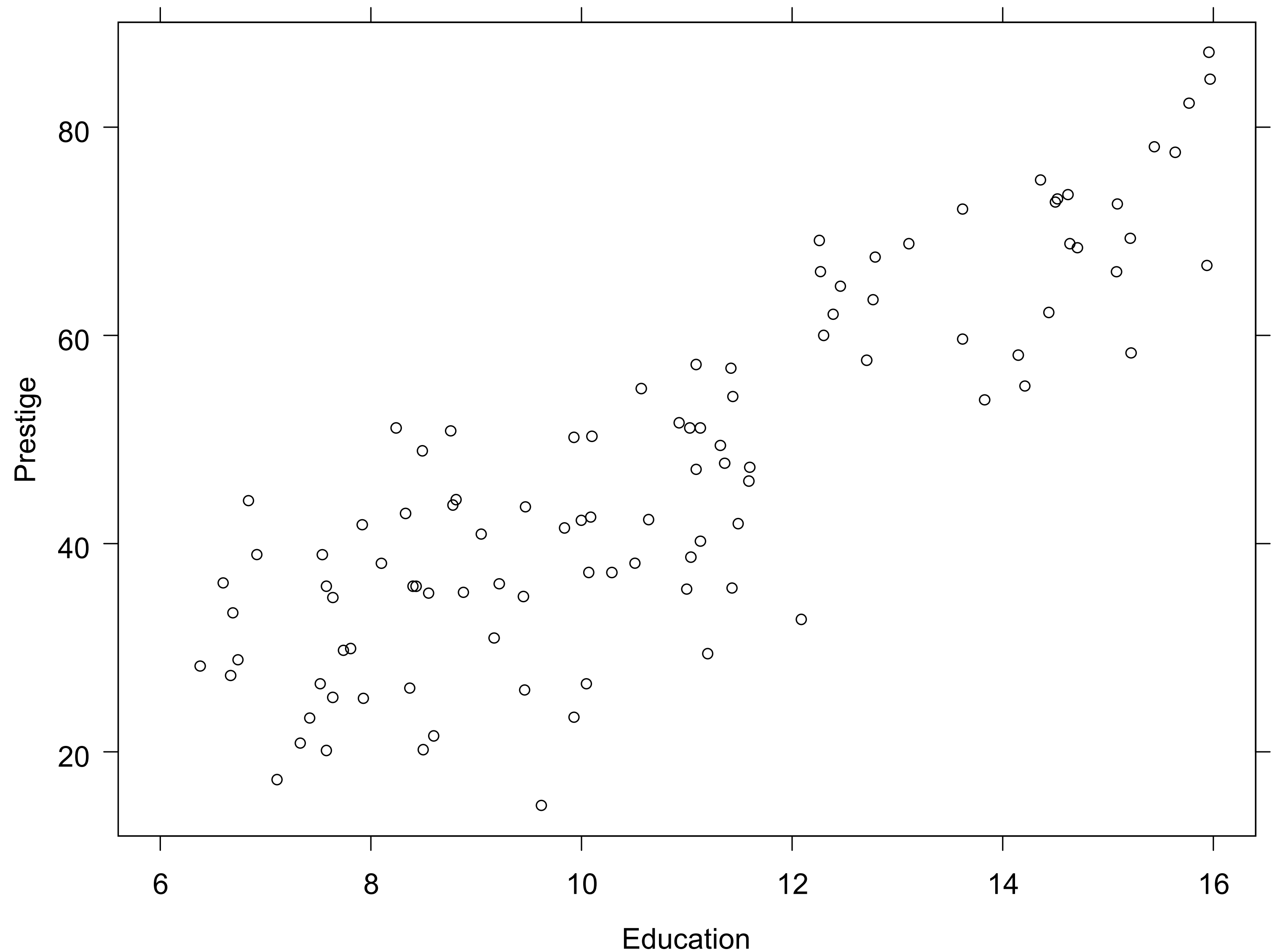
```
   Education            Income           PercFem
   Min.: 6.380       Min.:   611       Min.: 0.000
1st Qu.: 8.445    1st Qu.: 4106    1st Qu.: 3.592
 Median:10.540     Median: 5930     Median:13.600
   Mean:10.740       Mean: 6798       Mean:28.980
3rd Qu.:12.650    3rd Qu.: 8187    3rd Qu.:52.200
   Max.:15.970       Max.:25880       Max.:97.510


    Prestige           Code          Type
   Min.:14.80       Min.:1113        ?: 4
1st Qu.:35.22    1st Qu.:3120       bc:45
 Median:43.60     Median:5135     prof:30
   Mean:46.83       Mean:5402       wc:23
3rd Qu.:59.28    3rd Qu.:8312
   Max.:87.20       Max.:9517
```


Education
Income
PercFem
Prestige
Code
Type
Data
Fraction of 102 (NA: 0 )
wc
prof
bc
?


Type
prof
wc
bc
?
Prestige
PercFem
Income
Education


PHYSICIANS
GENERAL_MANAGERS
FILE_CLERKS
COLLECTORS
SERVICE_STATION_ATTENDANT
NEWSBOYS
80
60
40
20
Prestige
6
8
10
12
14
16
Education


UNIVERSITY_TEACHERS
LAWYERS
ARCHITECTS
PHYSICISTS
PSYCHOLOGISTS
CHEMISTS
PHYSIO_THERAPSTS
BIOLOGISTS
PHARMACISTS
MINING_ENGINEERS
VETERINAR
SECONDARY_SC
ECONOMISTS
PRIMARY_SCHOOL_TEACHERS
DRAUGHTSMEN
RADIO_TV_ANNOUNCERS
VOCATIONAL_C
SOCIAL_WORKERS
COMPUTER_PROGRAMERS
NSTRUCTION_FOREMEN
ELECTRICIANS
TRAIN_ENGINEERS
INSURANCE__AGENTS
SECRETARIES
FARMERS
TYPISTS
COMMERCIAL_TRAVELLERS
MASONS
TRAVEL_CLERKS
OFFICE_CLERKS
ILE_WEAVERS
FILE_CLERKS
E_LABOURERS
COLLECTORS
SEWING_MACH_OPERATORS
CANNERS
LAUNDERERS
ELEVATOR_OPERATORS
JANITORS
NEWSBOYS
80
60
40
20
Prestige
6
8
10
12
14
16
Education


Prestige
Education
80
60
40
20
6
8
10
12
14
16

# Typical regression output:

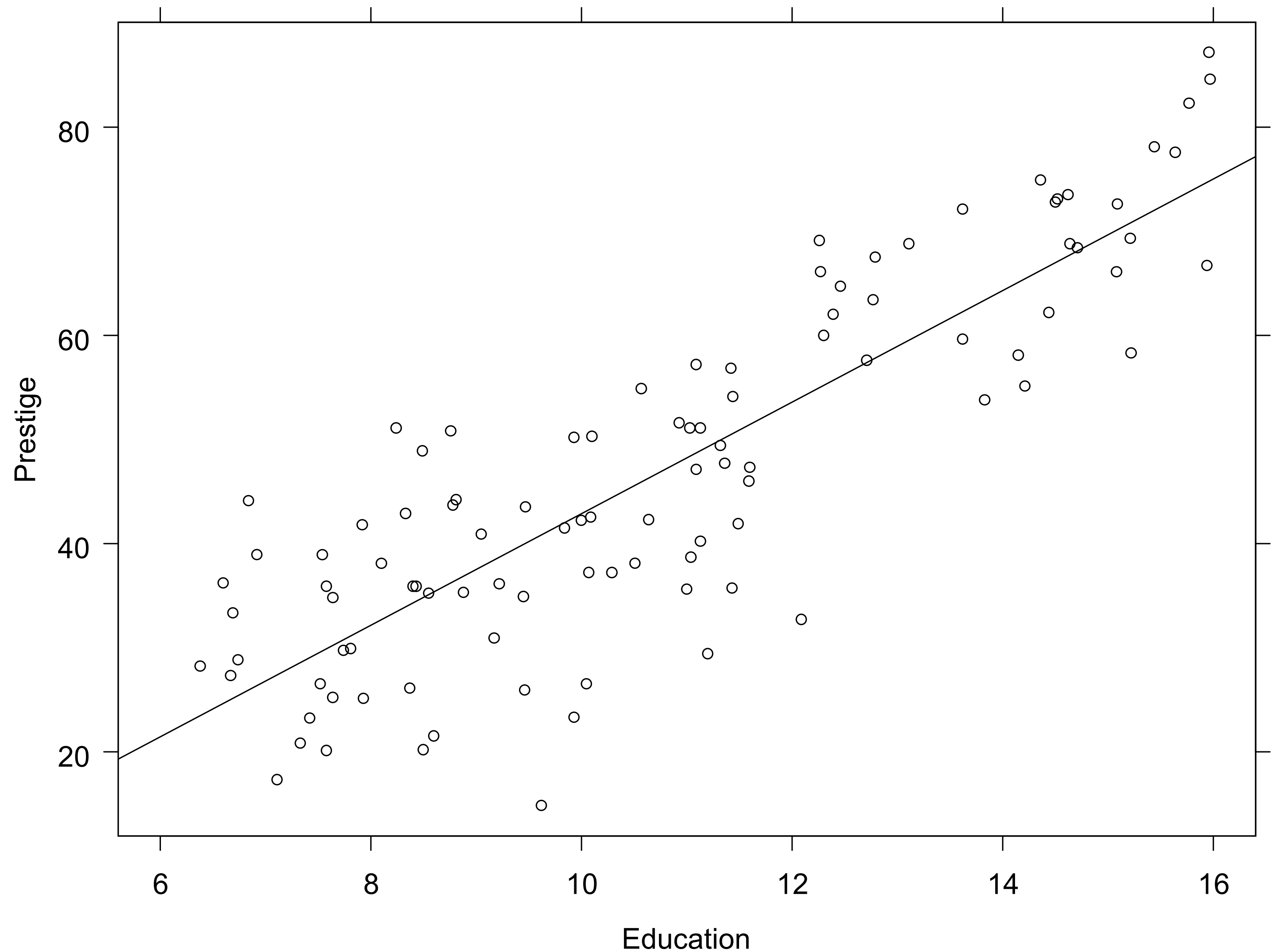
```
> summary(fit <- lm(Prestige ~ Education, pdat))

Call: lm(formula = Prestige ~ Education, data = pdat)
Residuals:
    Min     1Q Median    3Q   Max
 -26.04 -6.523 0.6611 6.743 18.16

Coefficients:
               Value Std. Error  t value Pr(>|t|)
(Intercept) -10.7320   3.6771    -2.9186   0.0043
  Education   5.3609   0.3320    16.1478   0.0000

Residual standard error: 9.103 on 100 degrees of freedom
Multiple R-Squared: 0.7228
F-statistic: 260.8 on 1 and 100 degrees of freedom,
          the p-value is 0
```

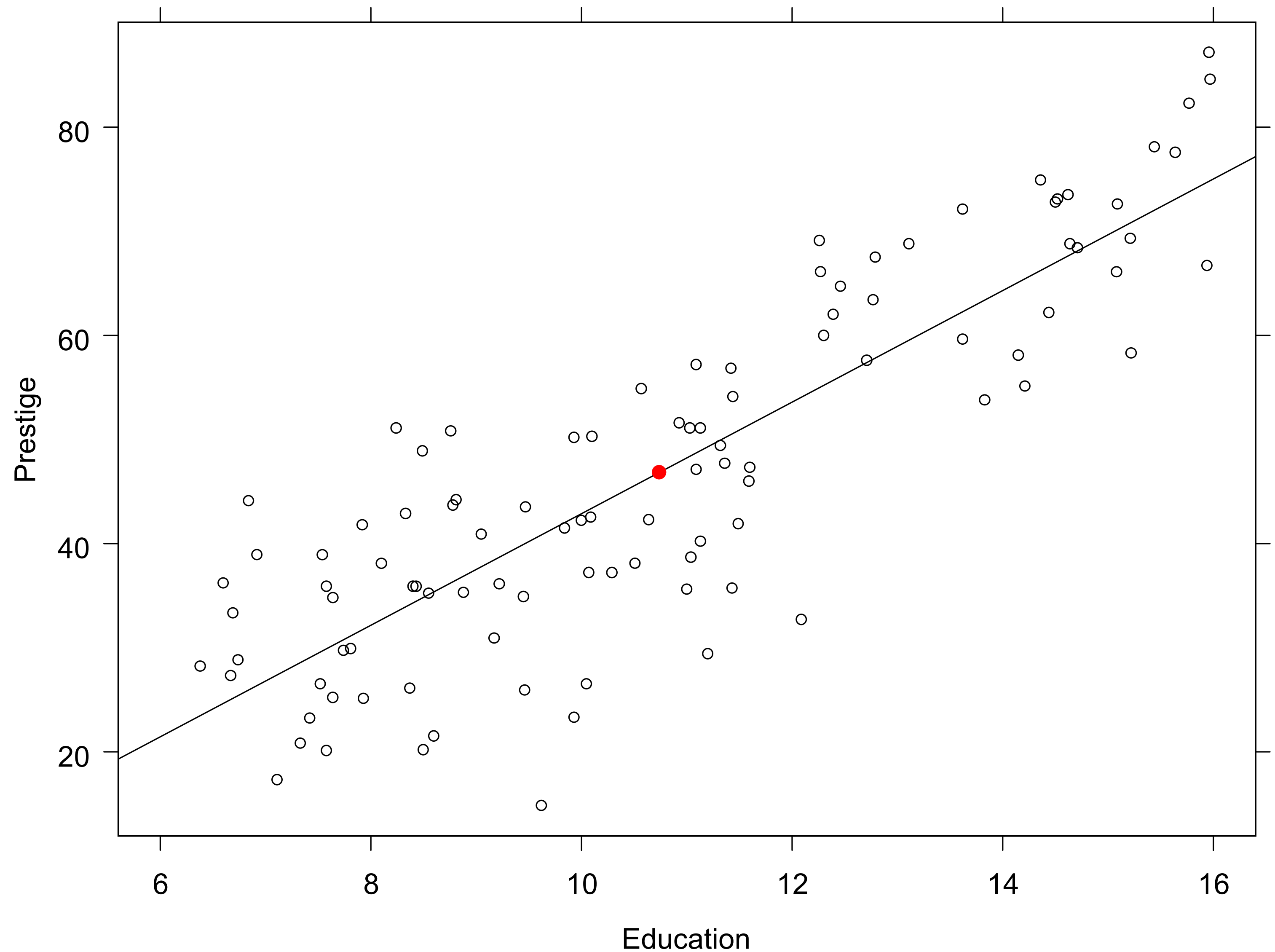
# Slope of the least-squares line:

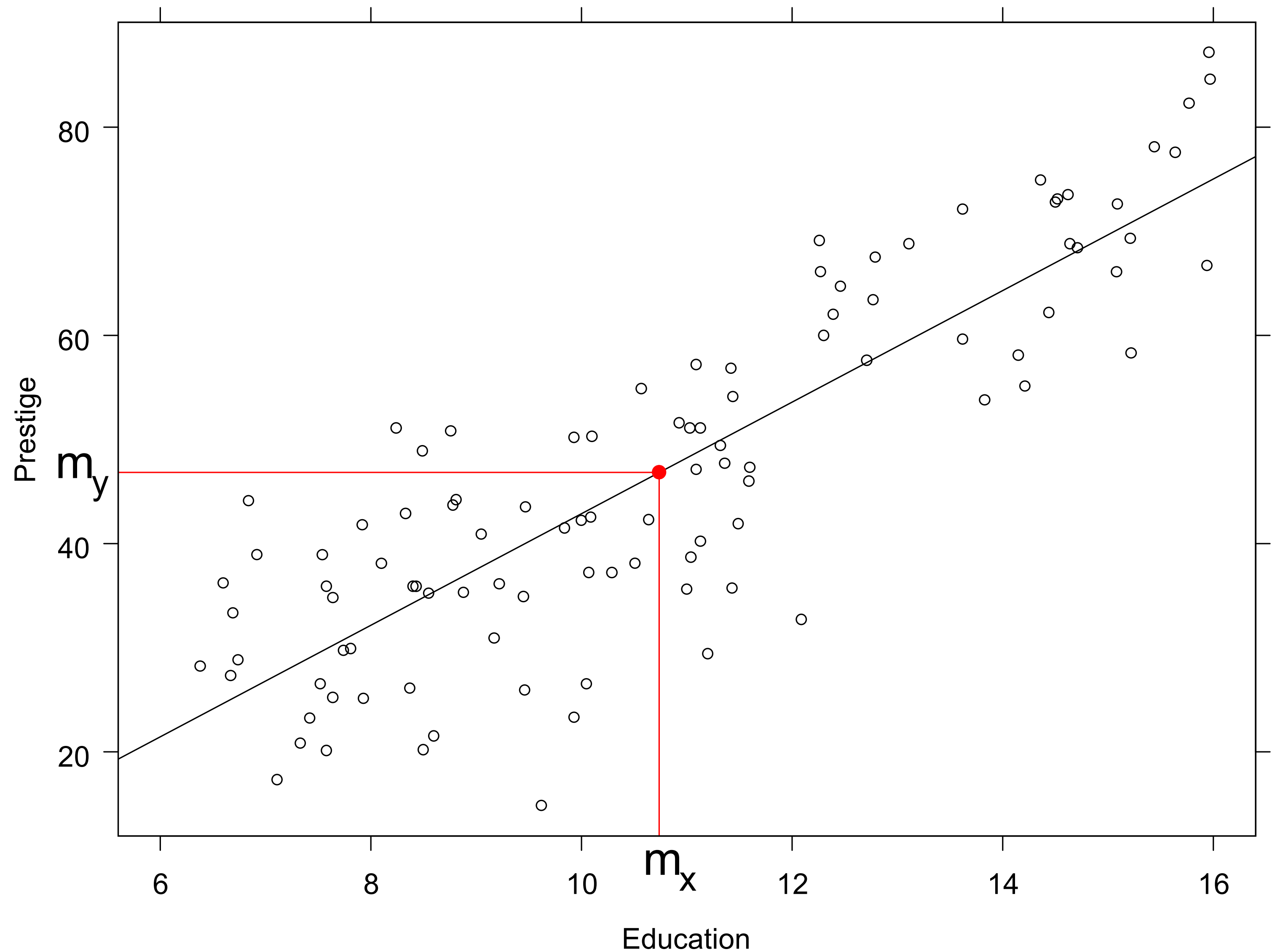
$$\begin{aligned} b &= \frac{\sum(X_i - \overline{X})(Y_i - \overline{Y})}{\sum(X_i - \overline{X})^2} \\ &= \frac{\left[\sum(X_i - \overline{X})(Y_i - \overline{Y})\right]/[n-1]}{\left[\sum(X_i - \overline{X})^2\right]/[n-1]} \\ &= \frac{\widehat{\mathrm{Cov}}(X,Y)}{\widehat{\mathrm{Var}}(X)} \\ &= \frac{s_{XY}}{s_X^2} \end{aligned}$$

# Intercept:

$$\overline{Y} = a + b\overline{X}$$


80
60
40
20
Prestige
6
8
10
12
14
16
Education


Prestige
80
60
40
20
6
8
10
12
14
16
Education


80
60
40
20
Prestige
$m_y$
6
8
10
$m_x$
12
14
16
Education

# Data ellipse:

Data ellipse of radius $r$ :

$$\mathcal{E}_r = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} : \left[ \begin{pmatrix} X \\ Y \end{pmatrix} - \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix} \right]' \begin{bmatrix} s_X^2 & s_{XY} \\ s_{XY} & s_Y^2 \end{bmatrix}^{-1} \left[ \begin{pmatrix} X \\ Y \end{pmatrix} - \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix} \right] = r^2 \right\}$$

As a transformation of the unit circle:

Let

$$\mathbf{S} = \begin{bmatrix} s_X^2 & s_{XY} \\ s_{XY} & s_Y^2 \end{bmatrix}$$

Let $\mathbf{S}^{1/2}$ be any 'square root' of $\mathbf{S}$ in the sense that:

$$\mathbf{S} = \mathbf{S}^{1/2} \left( \mathbf{S}^{1/2} \right)'$$

A good one is the Choleski square root:

$$\mathbf{S}^{1/2} = \begin{bmatrix} s_X & 0 \\ s_{XY}/s_X & s_{Y \cdot X} \end{bmatrix}$$

where $s_{Y \cdot X}$ denotes the 'partial standard deviation' of $Y$ adjusted for $X$ :

$$s_{Y \cdot X} = \sqrt{s_Y^2 - s_{XY}^2/s_X^2}$$

Then

$$\mathcal{E}_r = \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix} + r\mathbf{S}^{1/2}\mathcal{U}$$

where $\mathcal{U}$ is the unit circle.

If we let $SSE = \sum(Y_i - \widehat{Y}_i)^2$, then:

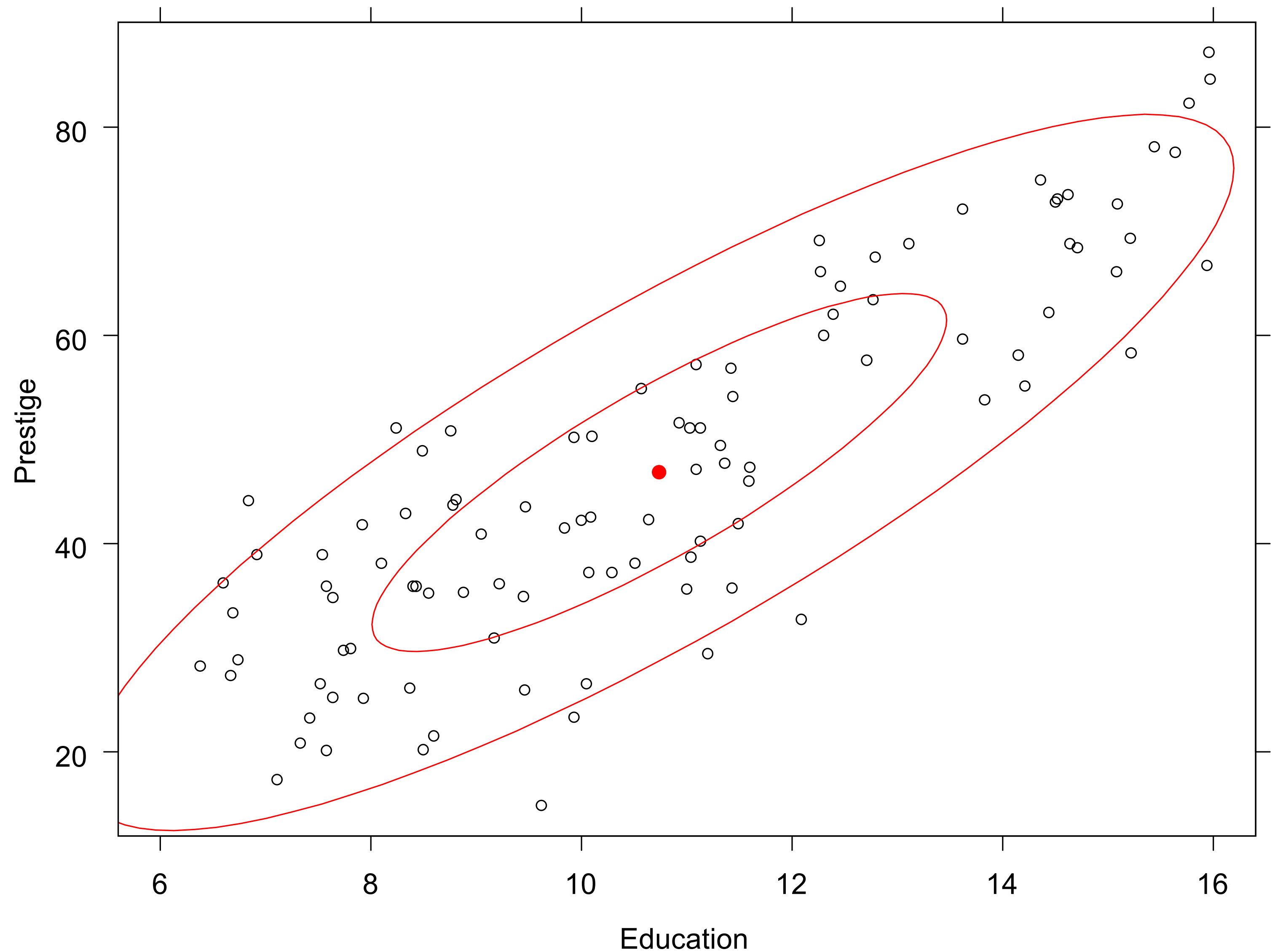
$$s_{Y \cdot X} = \sqrt{\frac{SSE}{n-1}}$$

# The usual standard error of regression is:

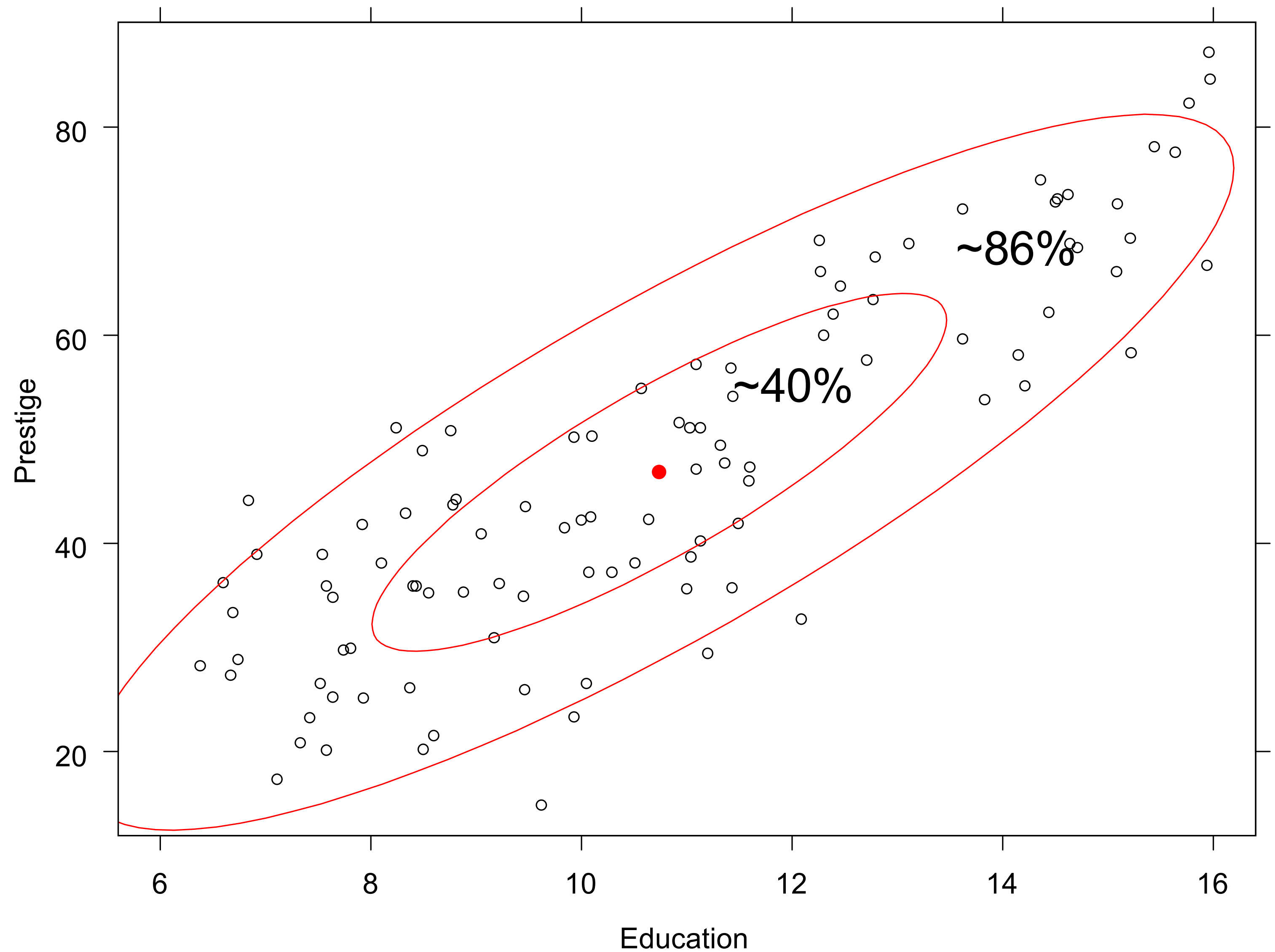
$$s_e = \sqrt{\frac{SSE}{n-2}}$$

So:

$$s_{Y \cdot X} \approx s_e$$



80
60
40
20
Prestige
6
8
10
12
14
16
Education

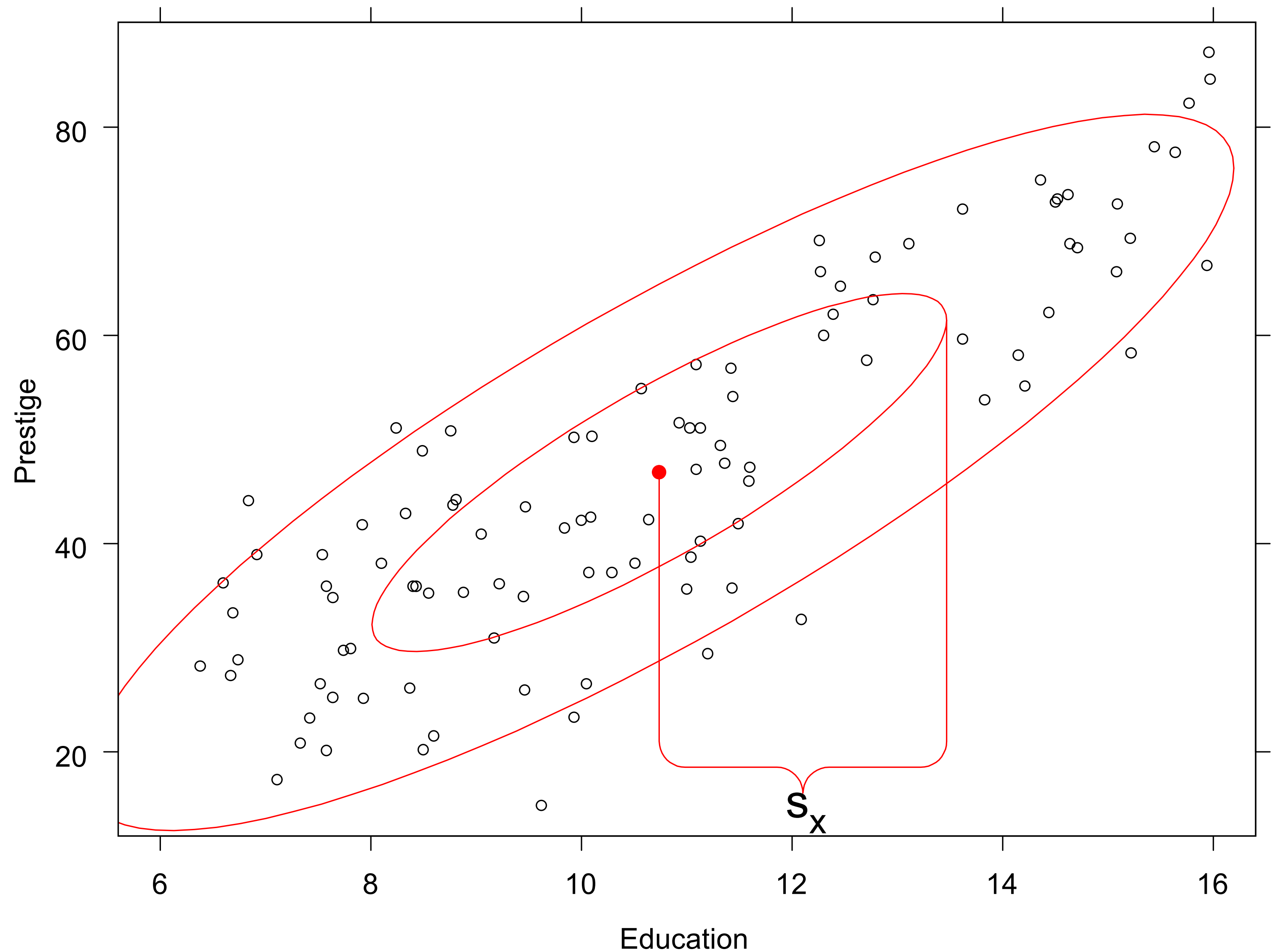


Prestige
Education
80
60
40
20
6
8
10
12
14
16
~86%
~40%

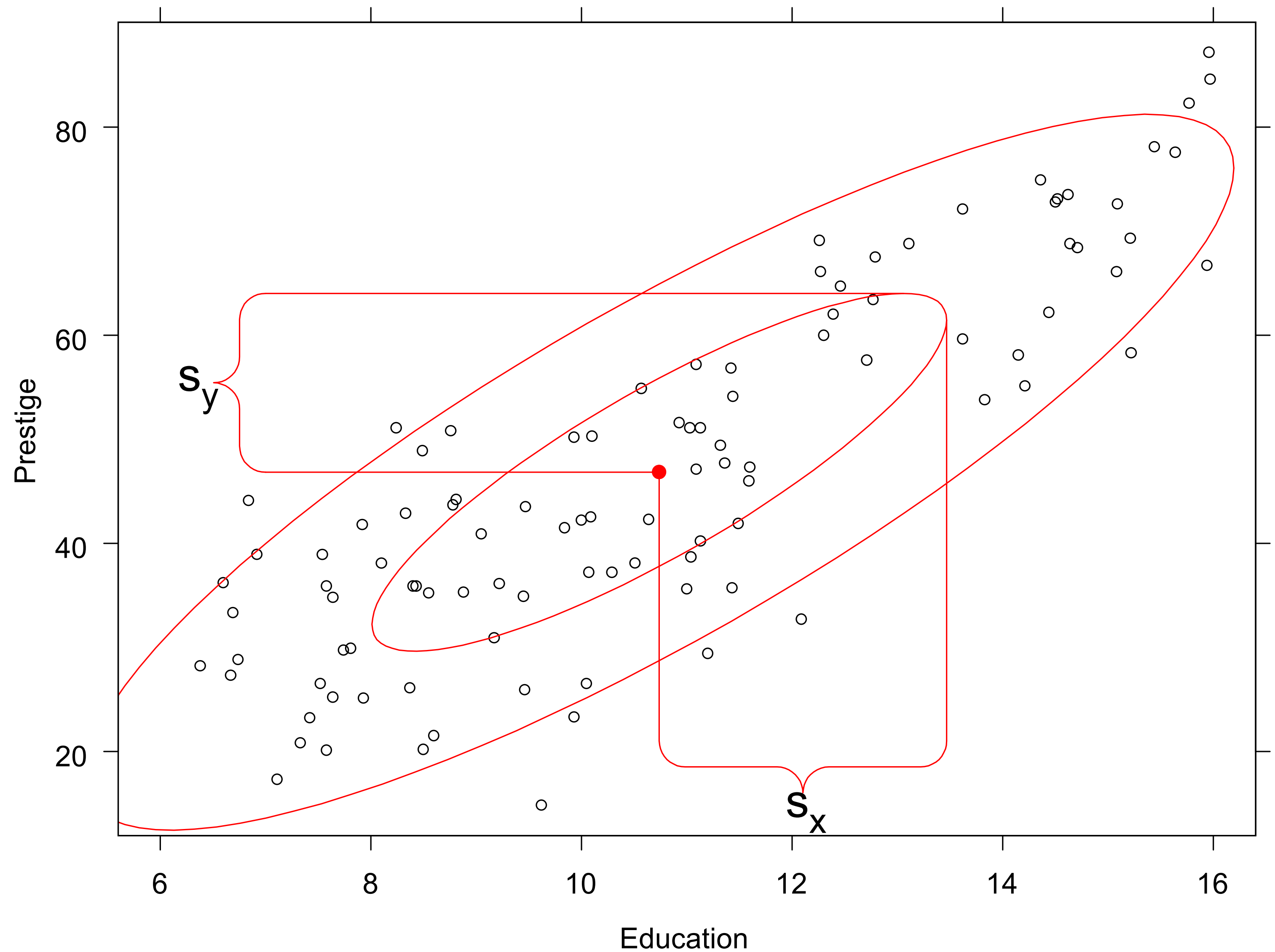
If the data cloud is approx. bivariate normal, then the proportion of points in $\mathcal{E}_r$ is approx:

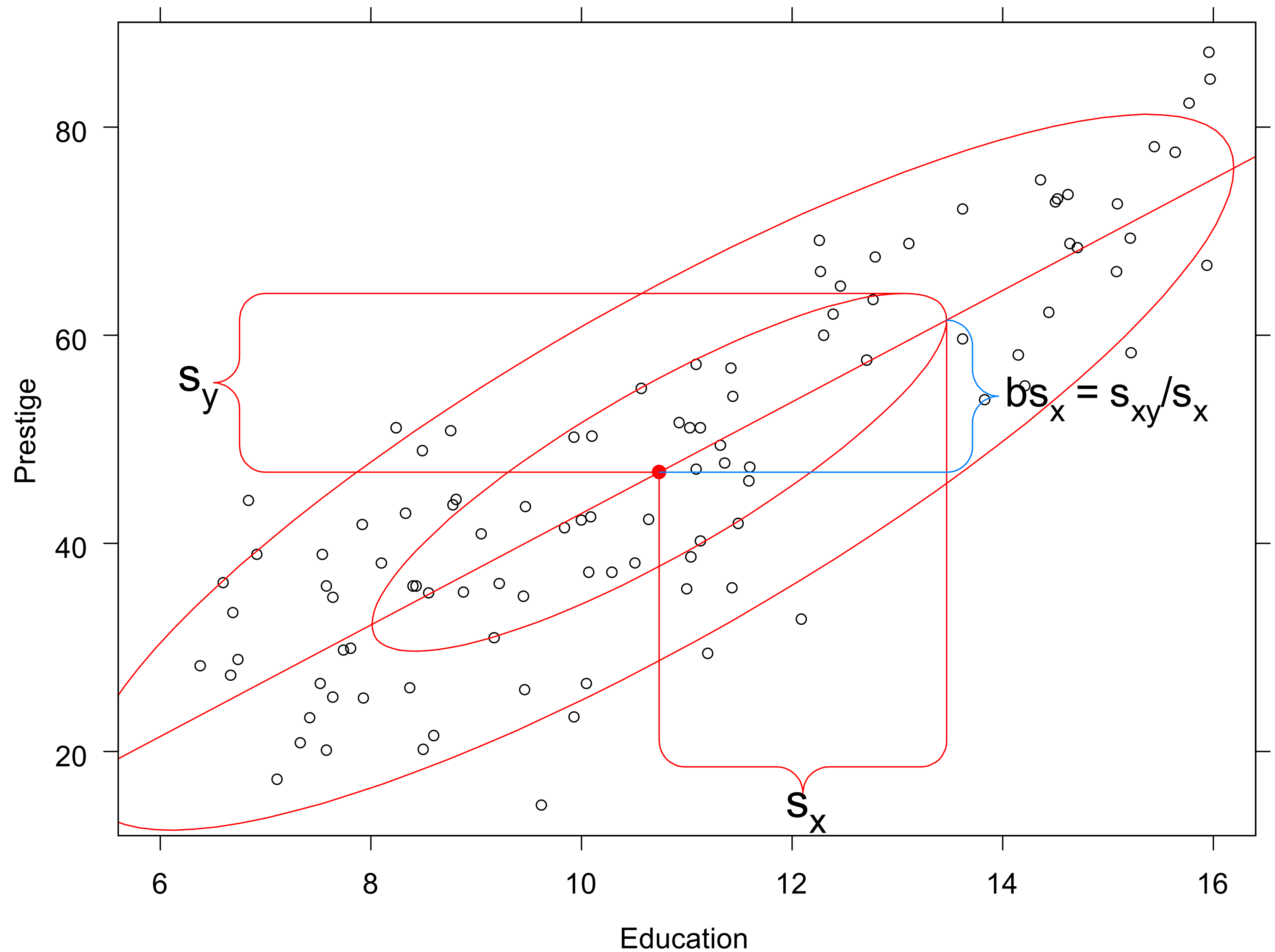
$$\Pr(\chi^2_2 \leq r^2)$$

where $\chi^2_2$ has a $\chi^2$ distribution with 2 degrees of freedom.

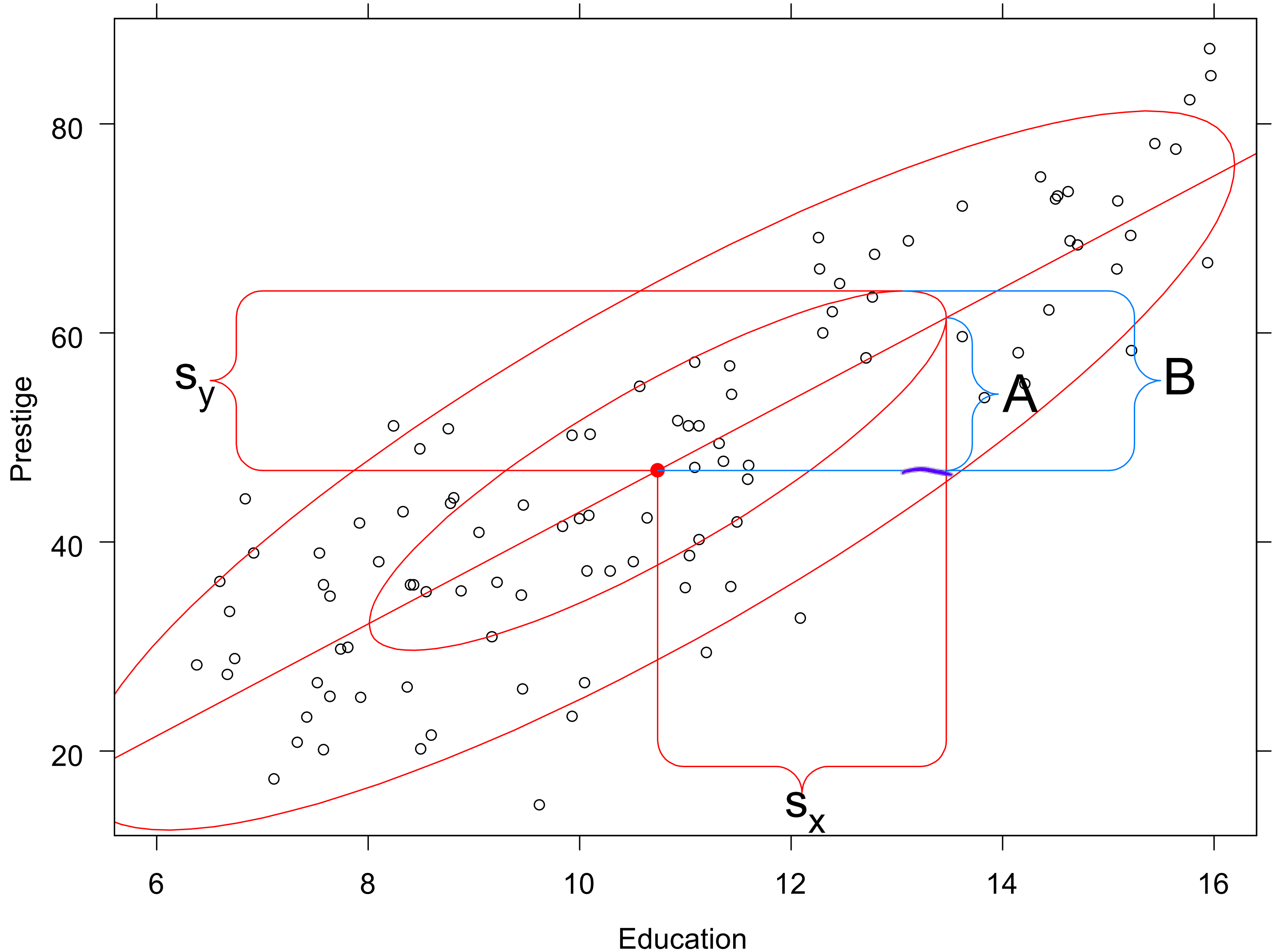


Prestige
80
60
40
20
6
8
10
12
14
16
Education
$S_X$

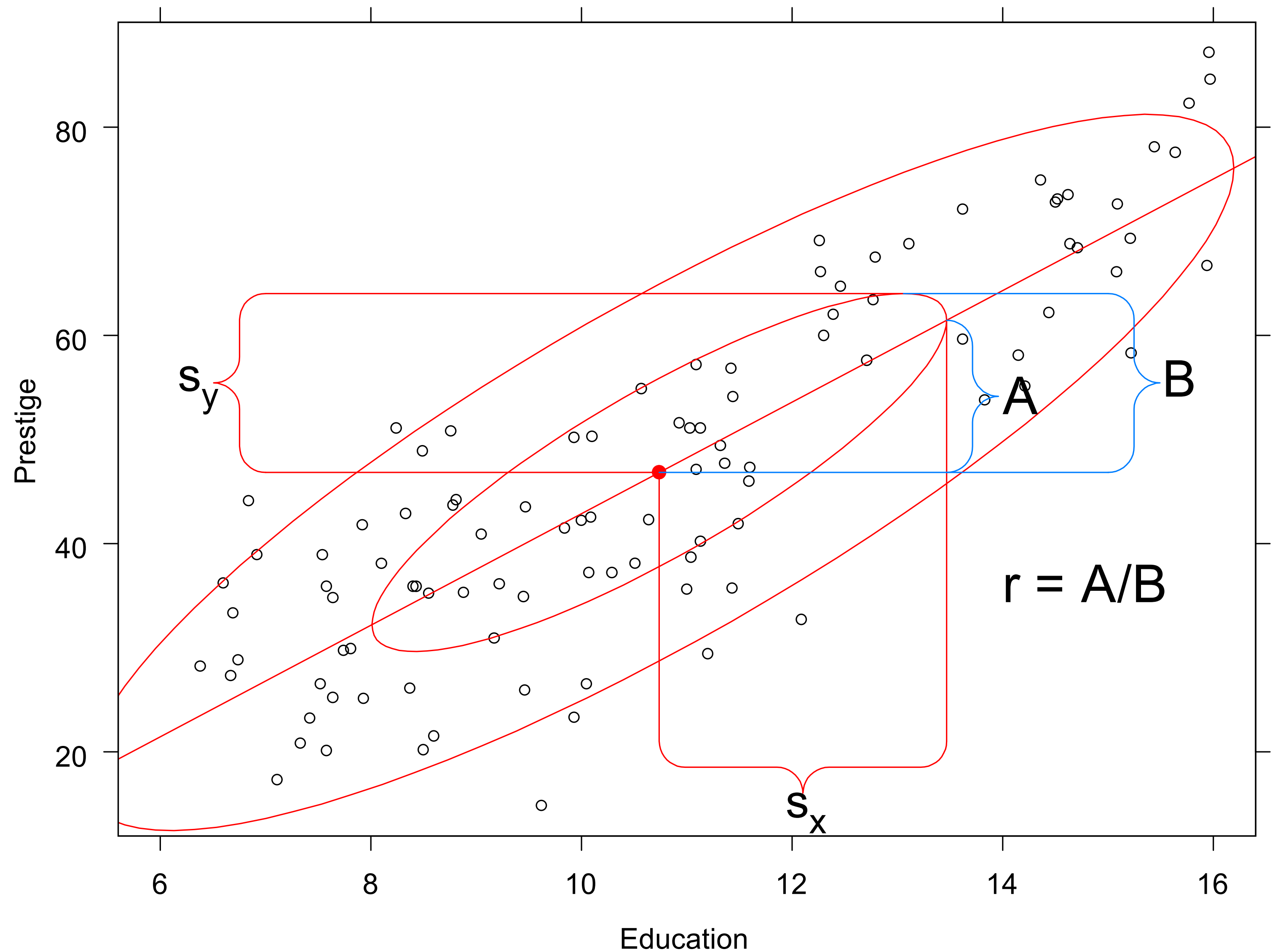

Prestige
Education
80
60
40
20
6
8
10
12
14
16
$s_y$
$s_x$



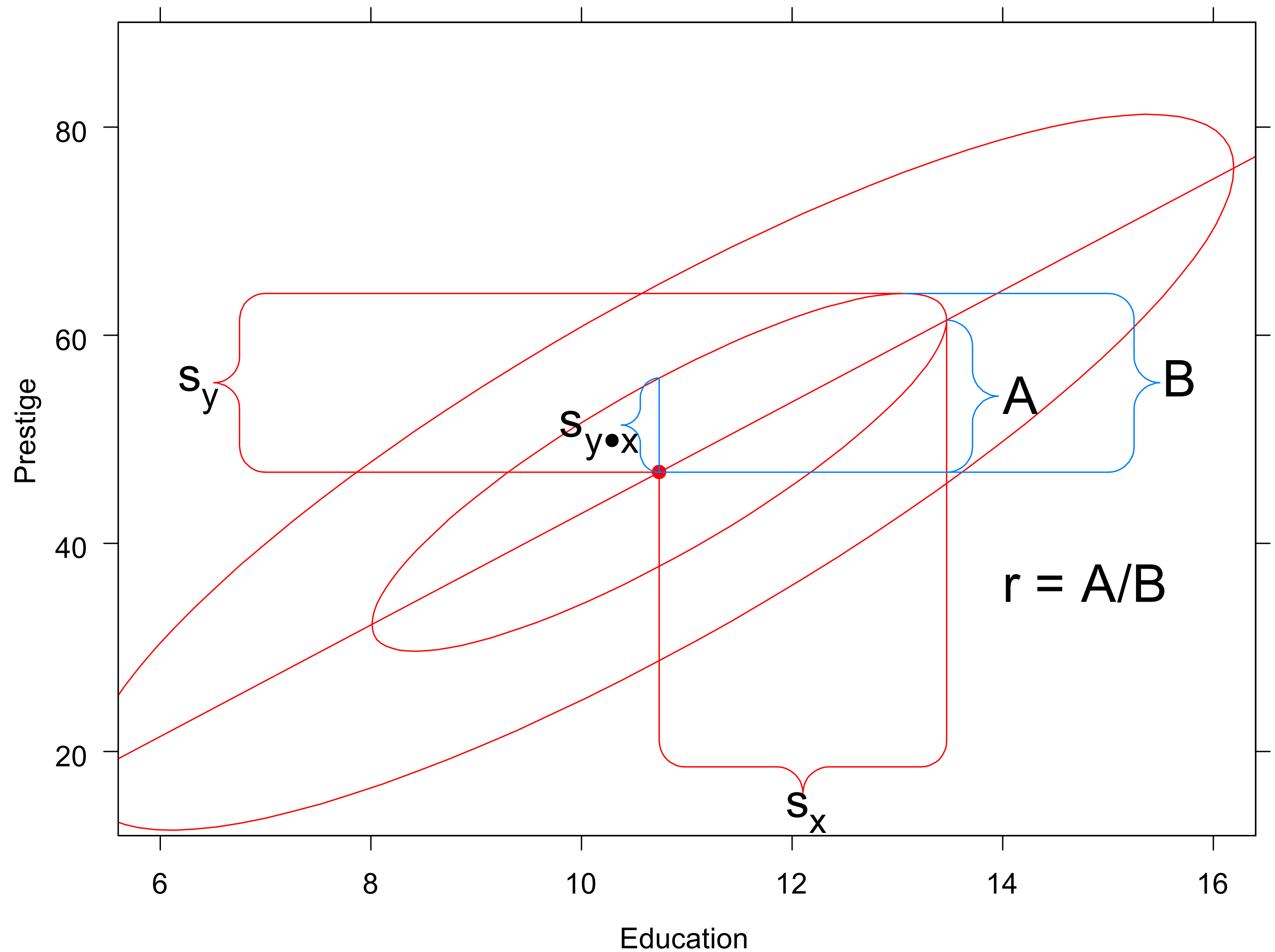
80
60
40
20
Prestige
6
8
10
12
14
16
Education
$s_y$
$s_x$
$bs_x = s_{xy}/s_x$

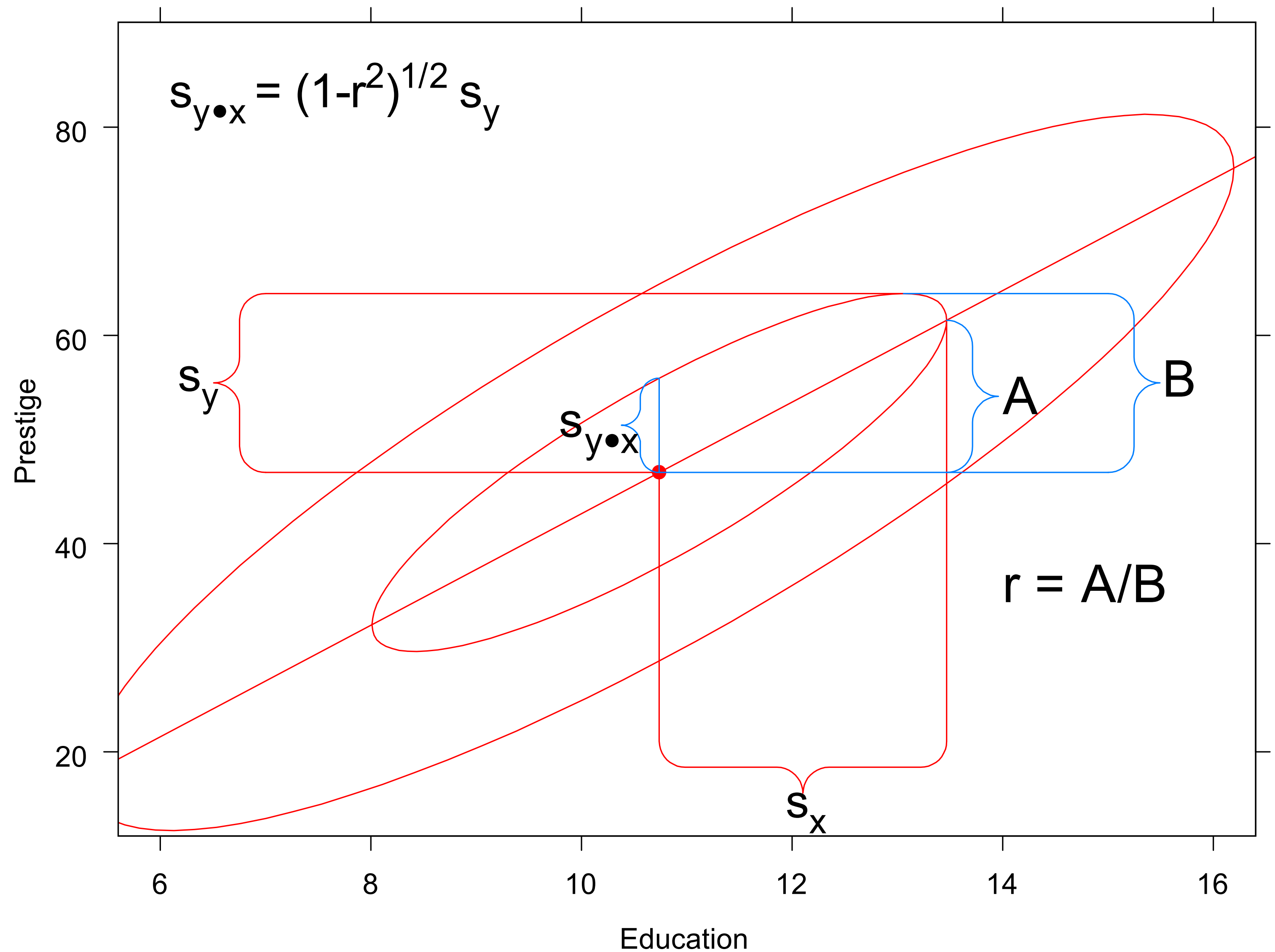


Prestige
Education
80
60
40
20
6
8
10
12
14
16
$s_y$
$s_x$
A
B

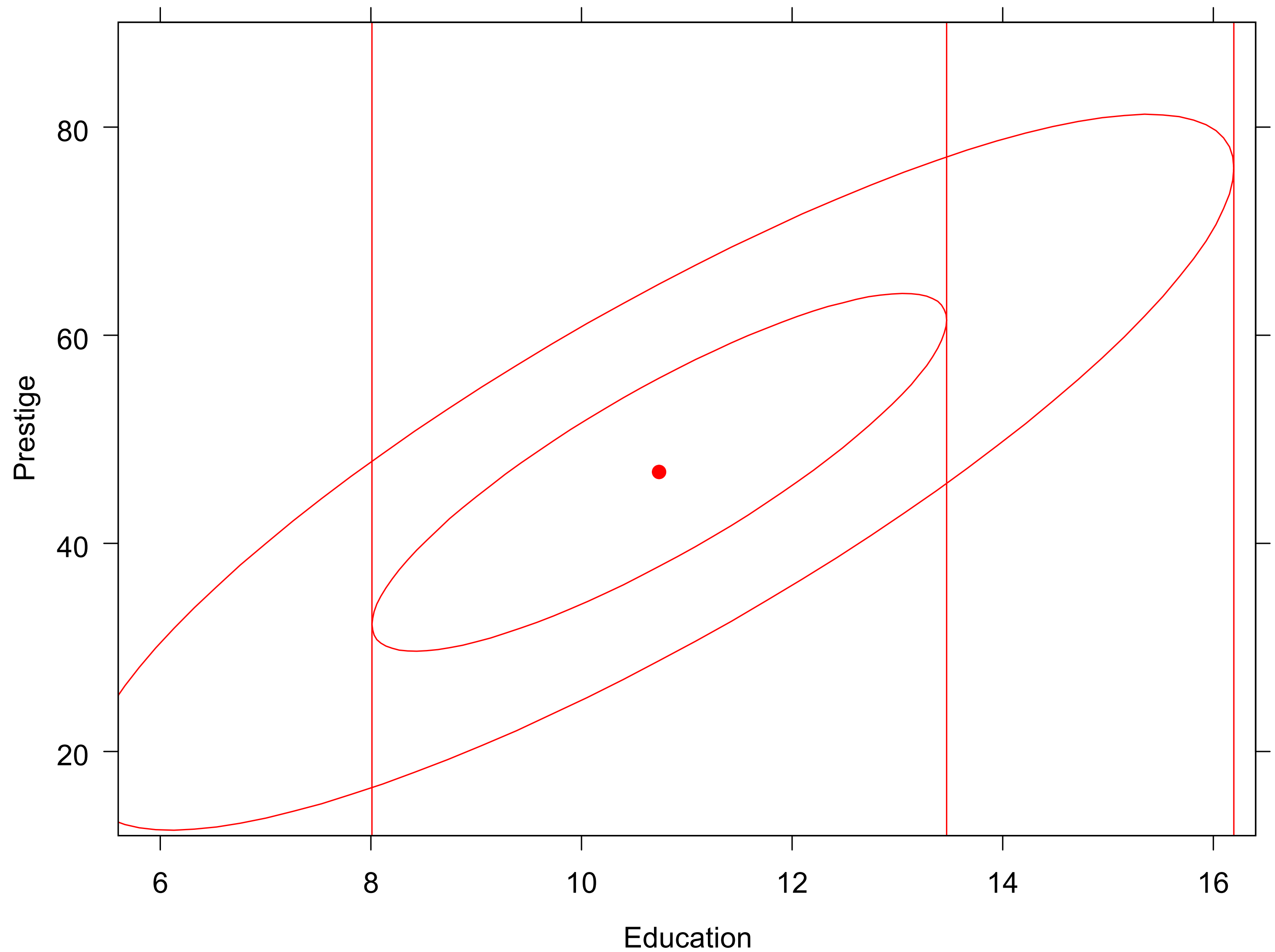


Prestige
Education
80
60
40
20
6
8
10
12
14
16
$s_y$
$s_x$
A
B
r = A/B

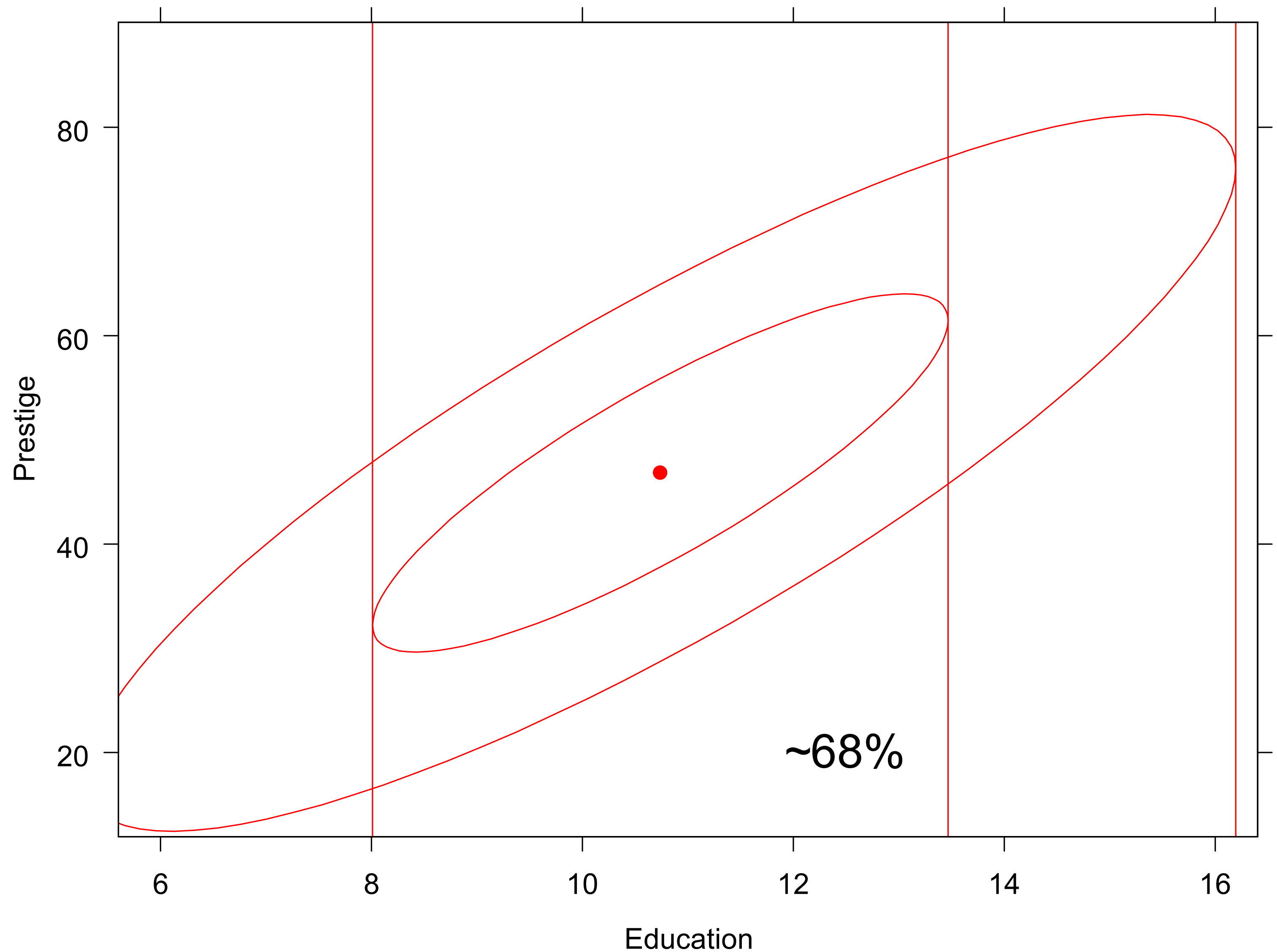

80
60
40
20
6
8
10
12
14
16
Prestige
Education
$s_y$
$s_{y \bullet x}$
$s_x$
A
B
r = A/B

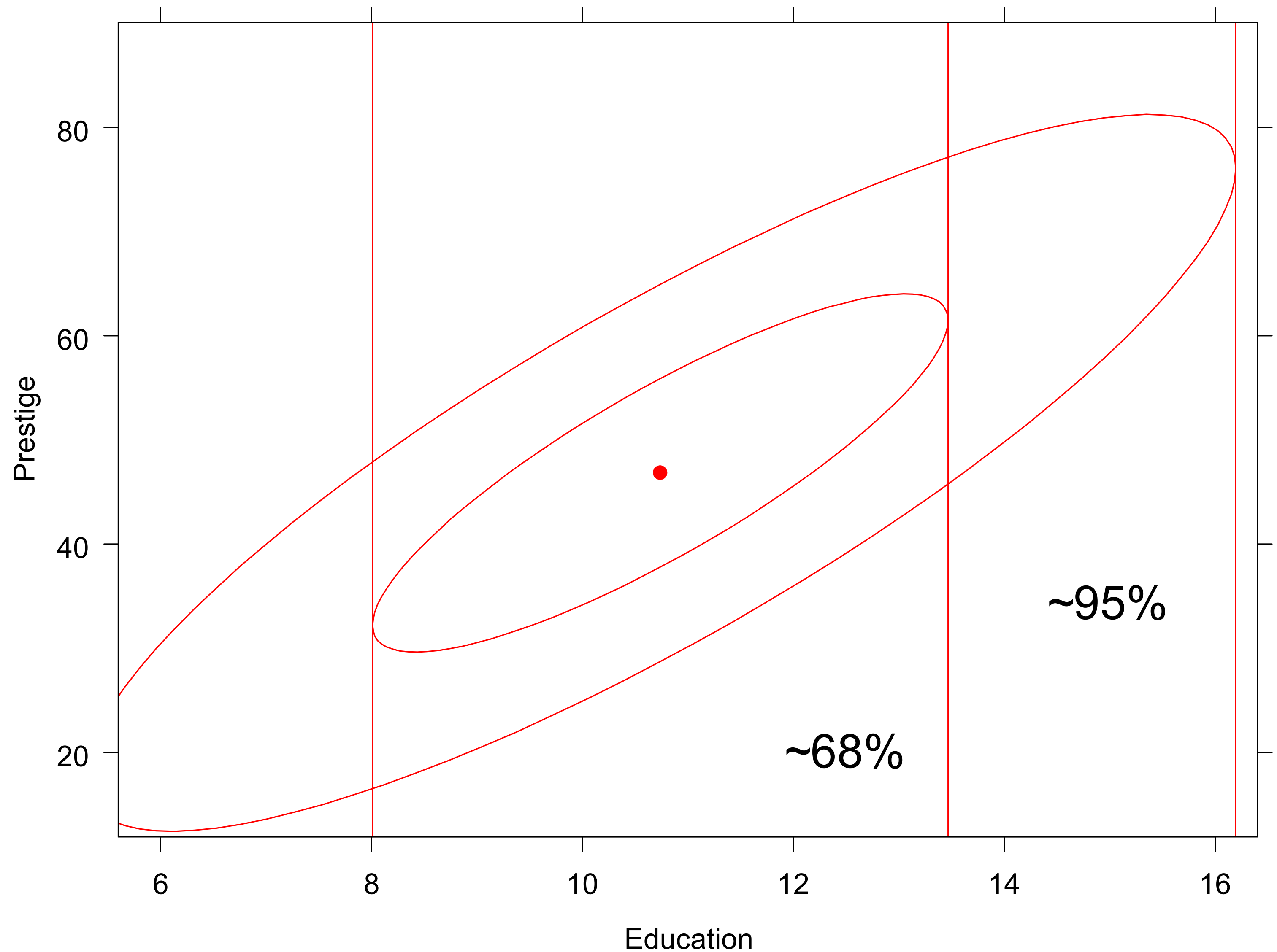


$s_{y \bullet x} = (1-r^2)^{1/2} s_y$
80
60
40
20
Prestige
$s_y$
$s_{y \bullet x}$
A
B
r = A/B
$s_x$
6
8
10
12
14
16
Education

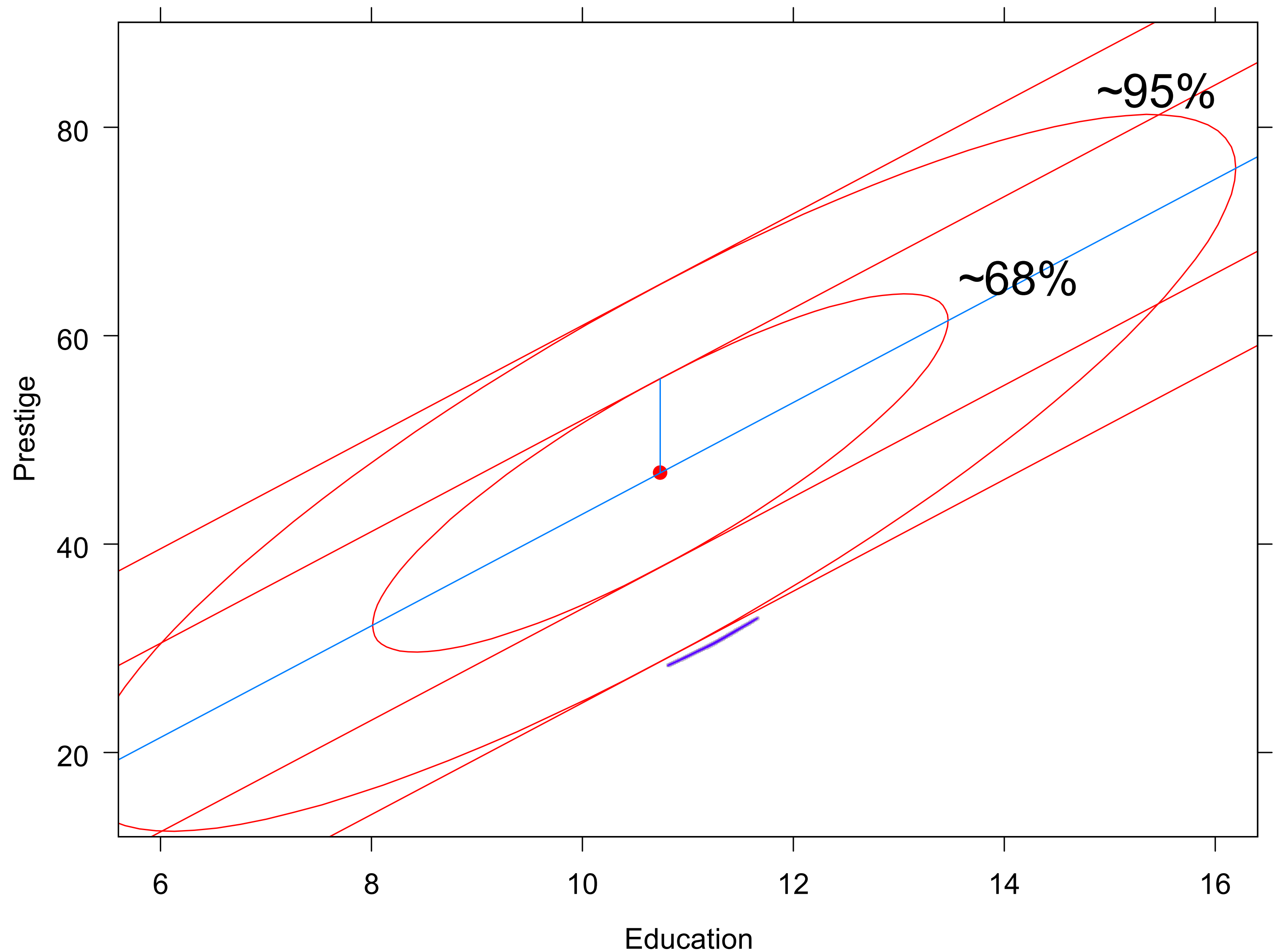


80
60
40
20
Prestige
6
8
10
12
14
16
Education


80
60
40
20
Prestige
6
8
10
12
14
16
Education
~68%



80
60
40
20
Prestige
6
8
10
12
14
16
Education
~95%
~68%


~95%
~68%
80
60
40
20
Prestige
6
8
10
12
14
16
Education

If the data cloud is approx. bivariate normal, then the proportion of points in a *band* containing $\mathcal{E}_r$ is approx:

$$\Pr(\chi_1^2 \leq r^2)$$

where $\chi_1^2$ has a $\chi^2$ distribution with $1$ degrees of freedom.

# 3 Visual 95% Confidence Interval

95% Confidence interval for slope:

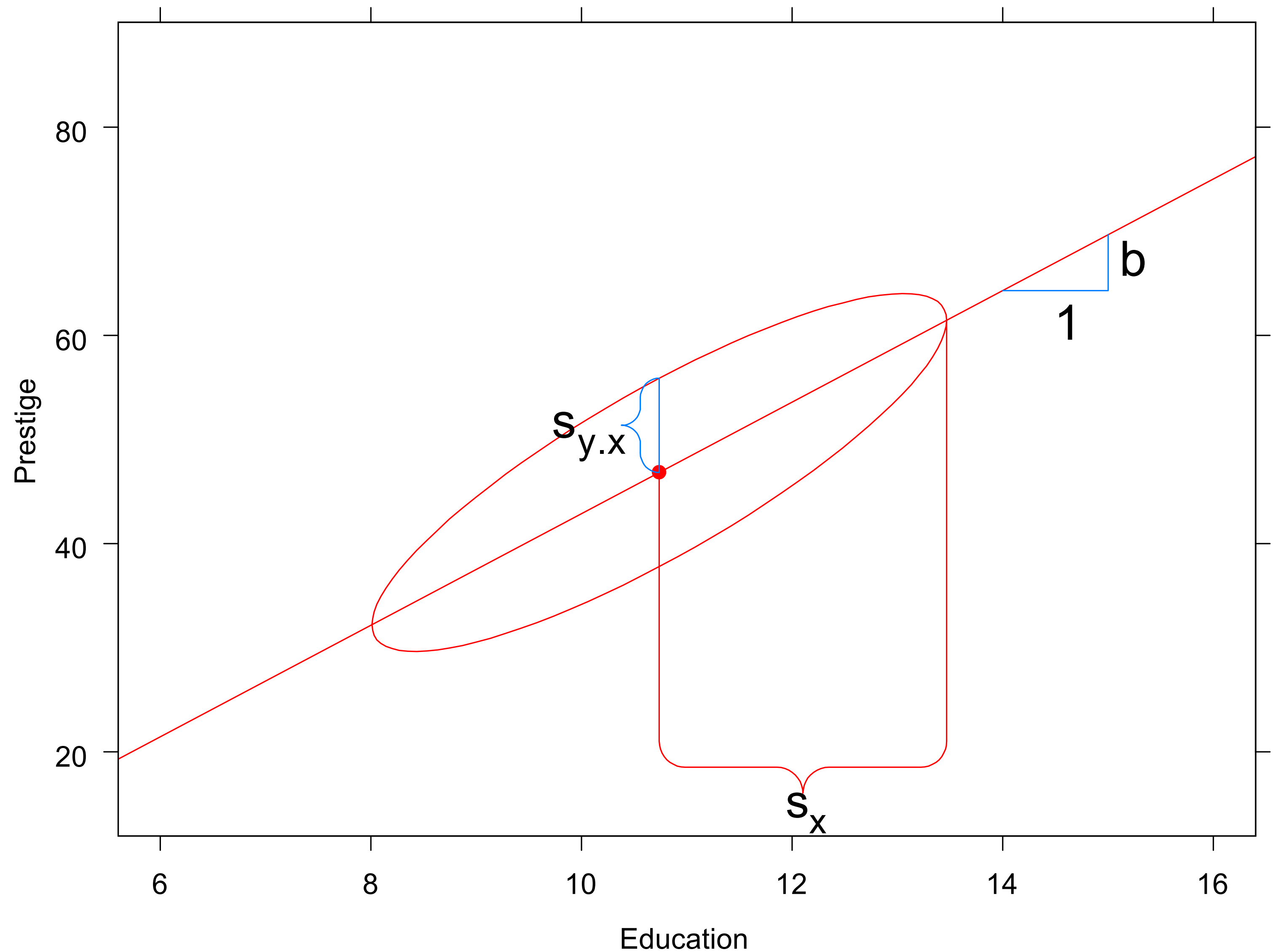
$$b \pm t_{0.975} \times SE(b)$$

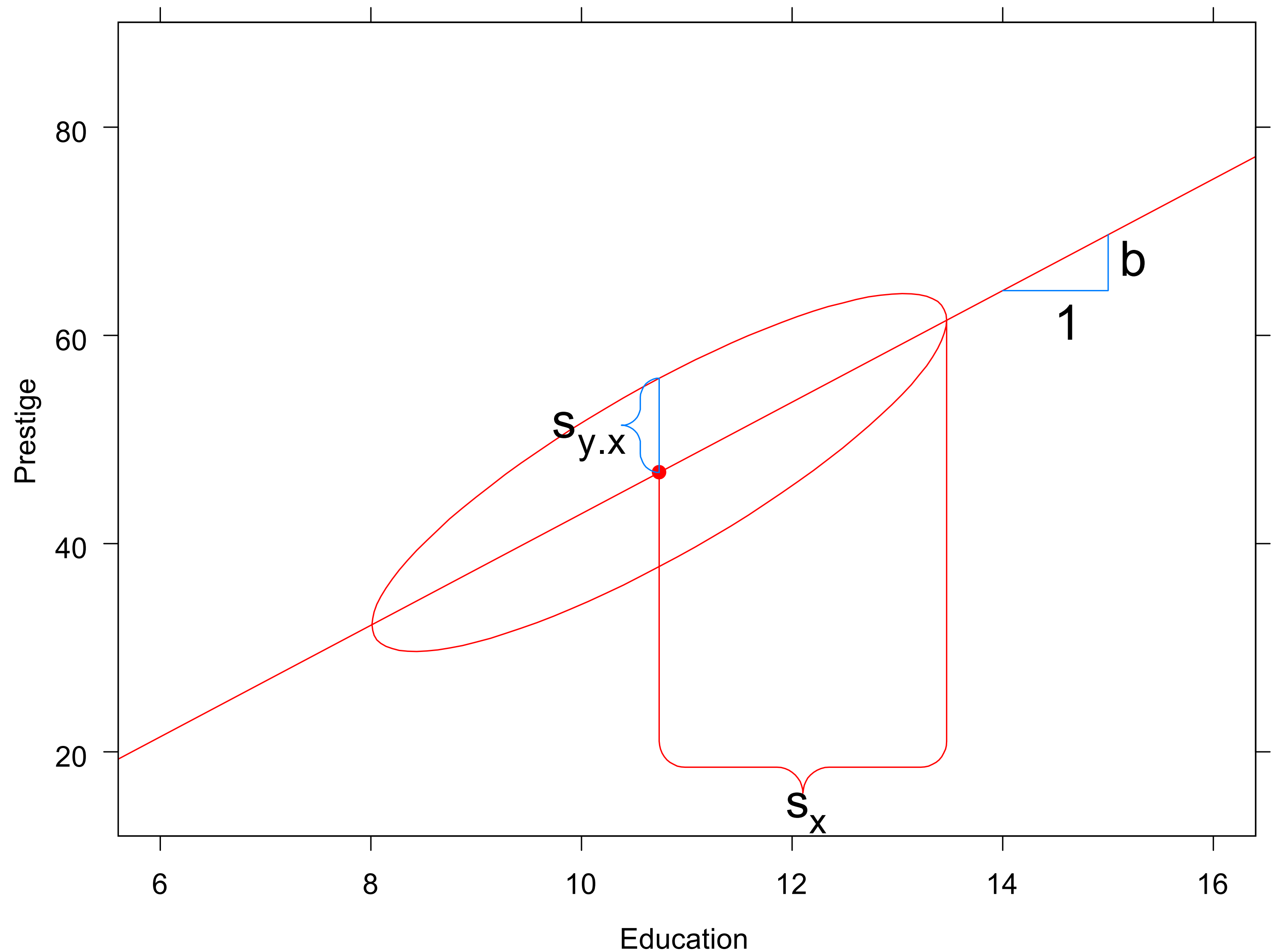
Now,

$$\begin{aligned} SE(b) &= \frac{1}{\sqrt{n}} \frac{s_{y.x}}{s_x} \times \frac{\sqrt{n}}{\sqrt{n-2}} \\ &\simeq \frac{1}{\sqrt{n}} \frac{s_{y.x}}{s_x} \end{aligned}$$

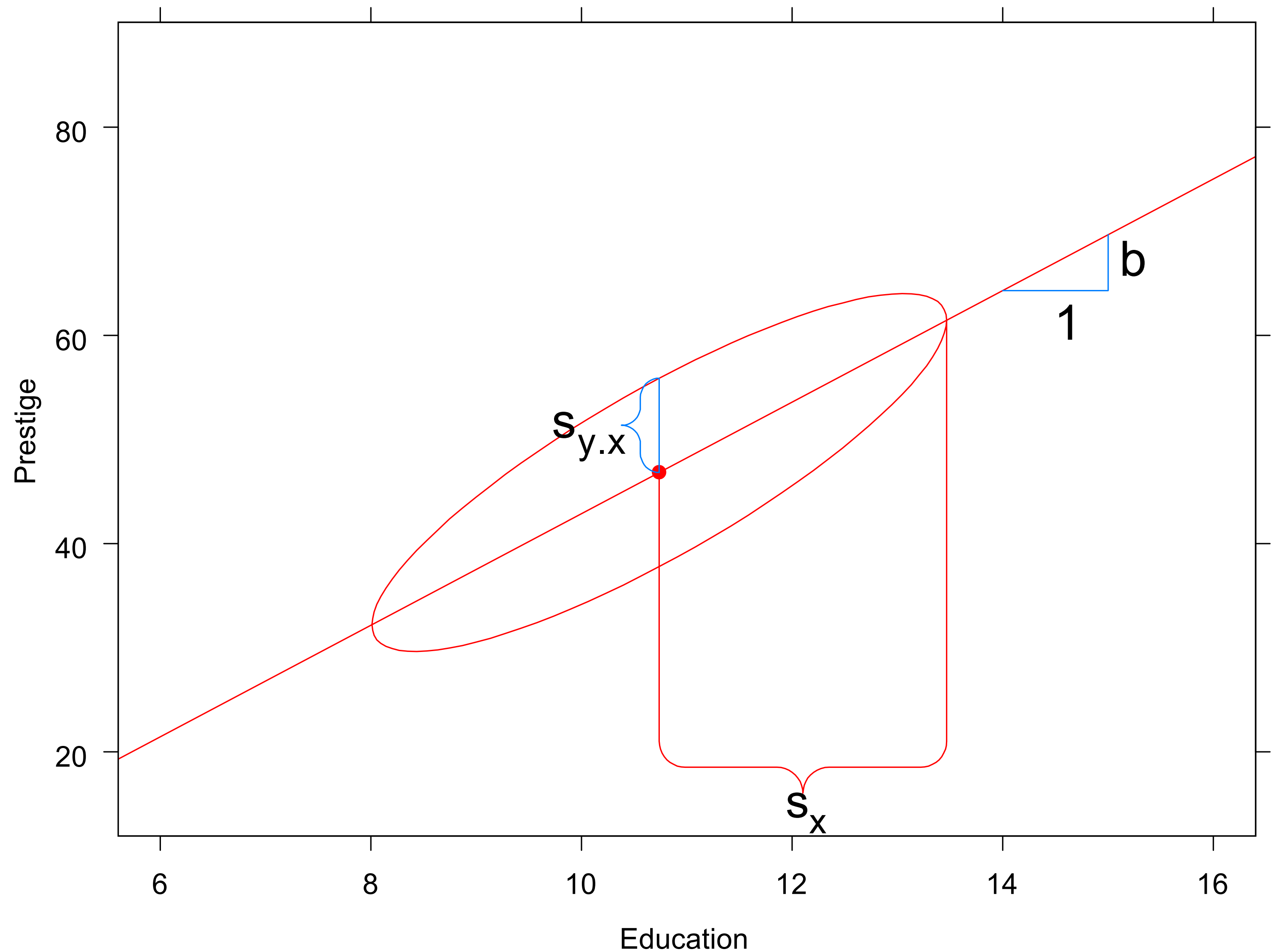
and taking $t_{0.975} \simeq 2$, we have an approximate 95% CI:

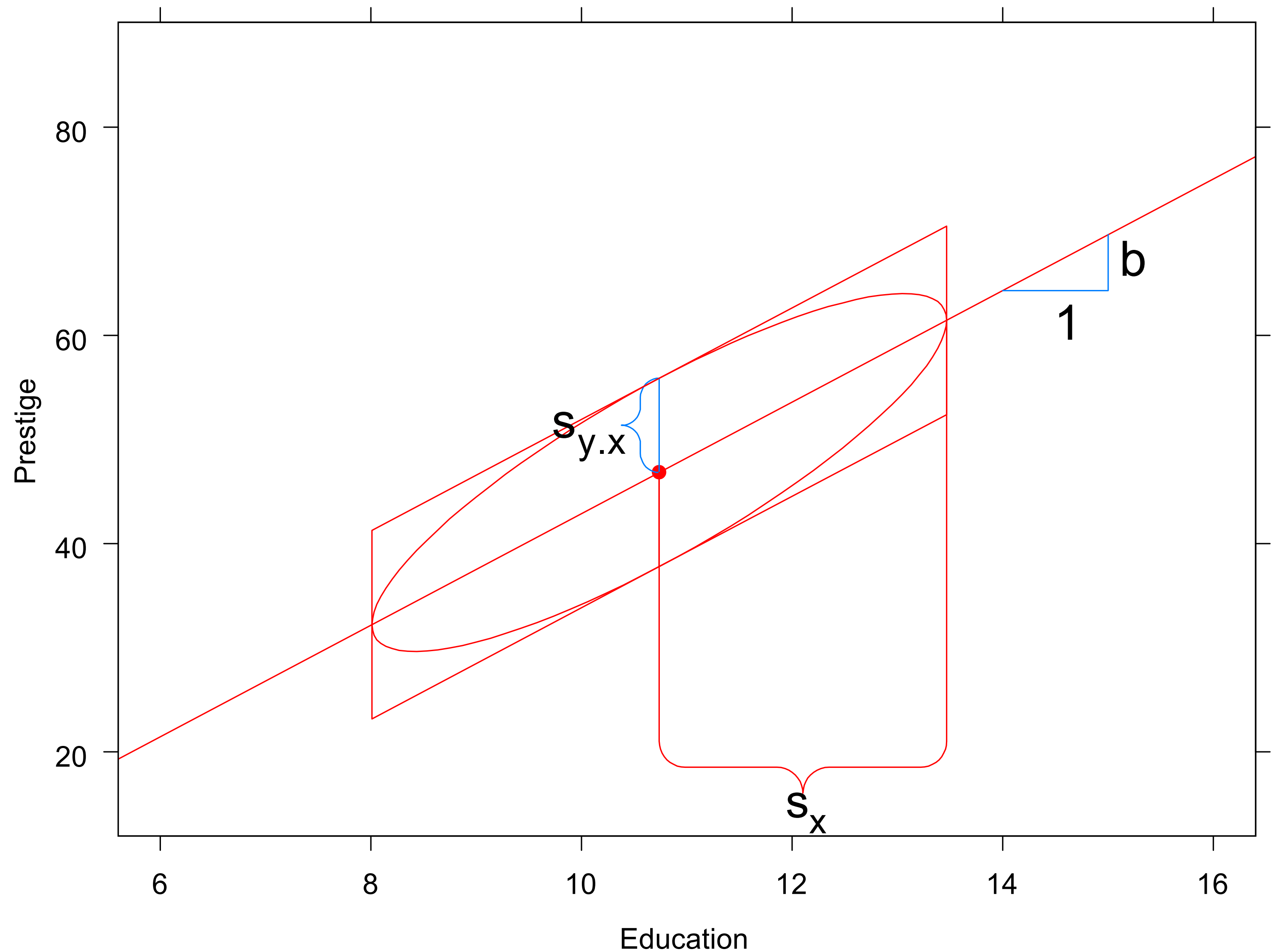
$$b \pm \frac{2}{\sqrt{n}} \times \frac{s_{y.x}}{s_x}$$

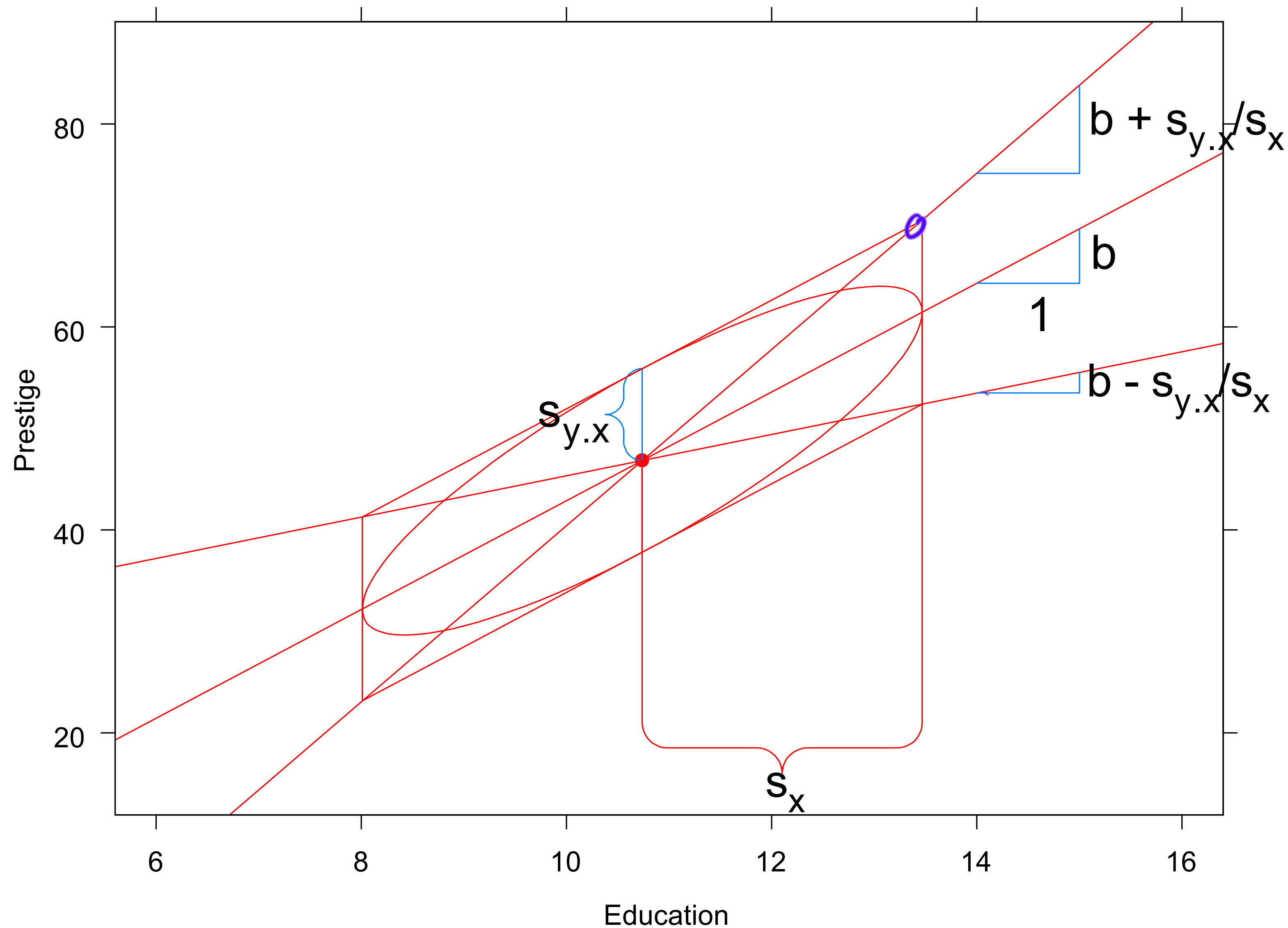

80
60
40
20
Prestige
6
8
10
12
14
16
Education
b
1
sy.x
sx



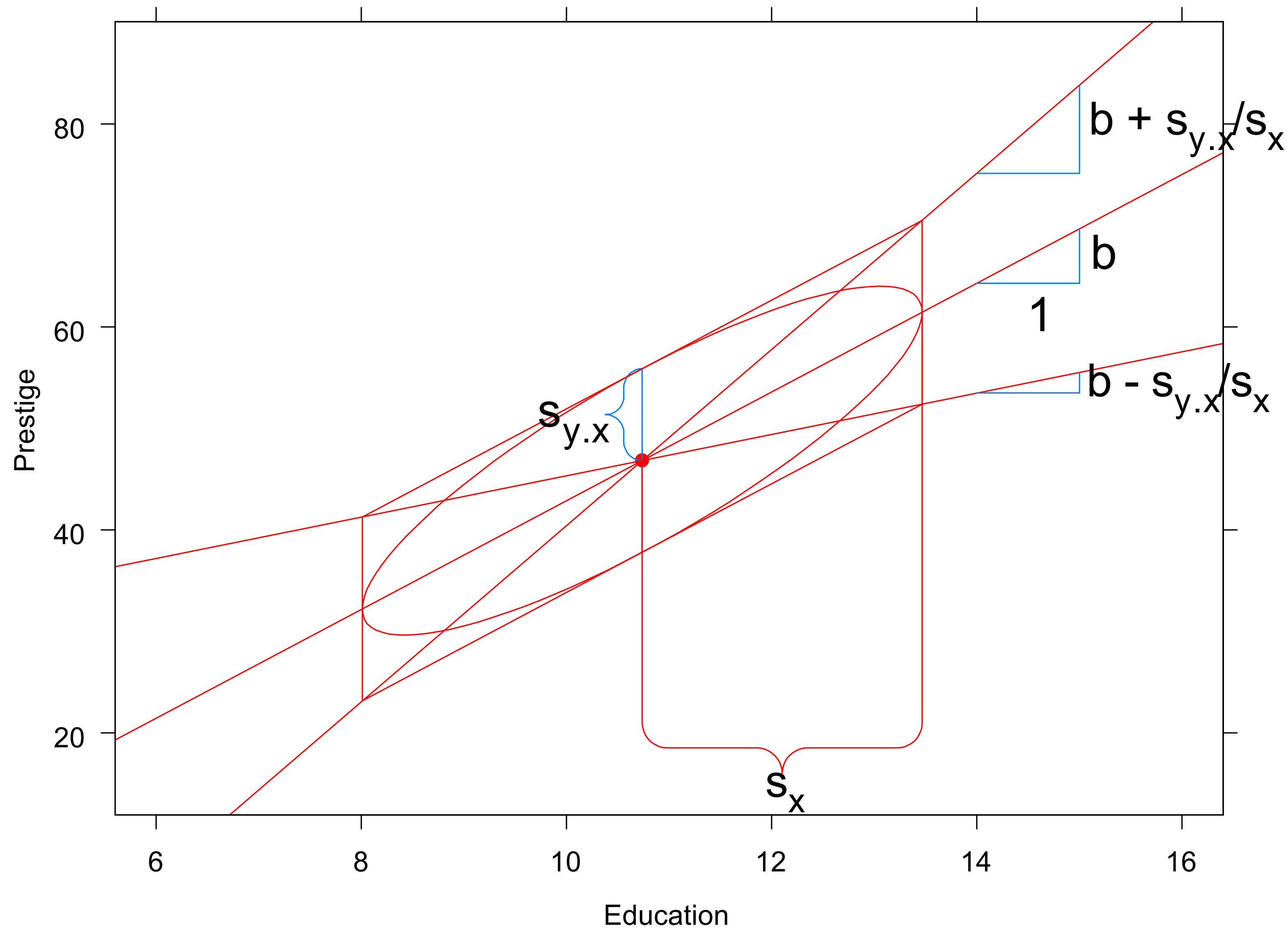
80
60
40
20
Prestige
6
8
10
12
14
16
Education
b
1
s y.x
s x


80
60
40
20
Prestige
6
8
10
12
14
16
Education
b
1
$s_{y.x}$
$s_x$

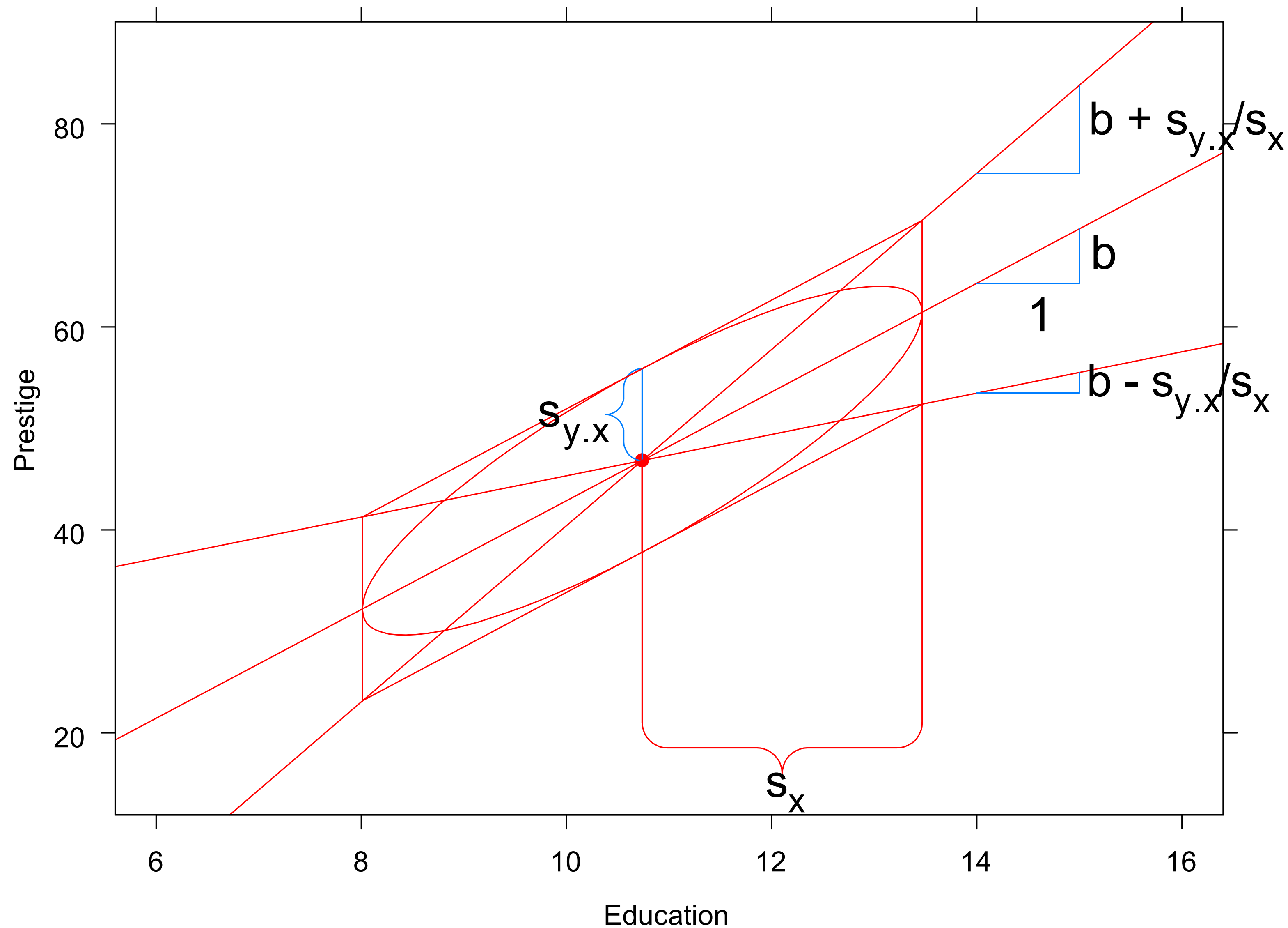

80
60
40
20
Prestige
6
8
10
12
14
16
Education
b
1
sy.x
sx

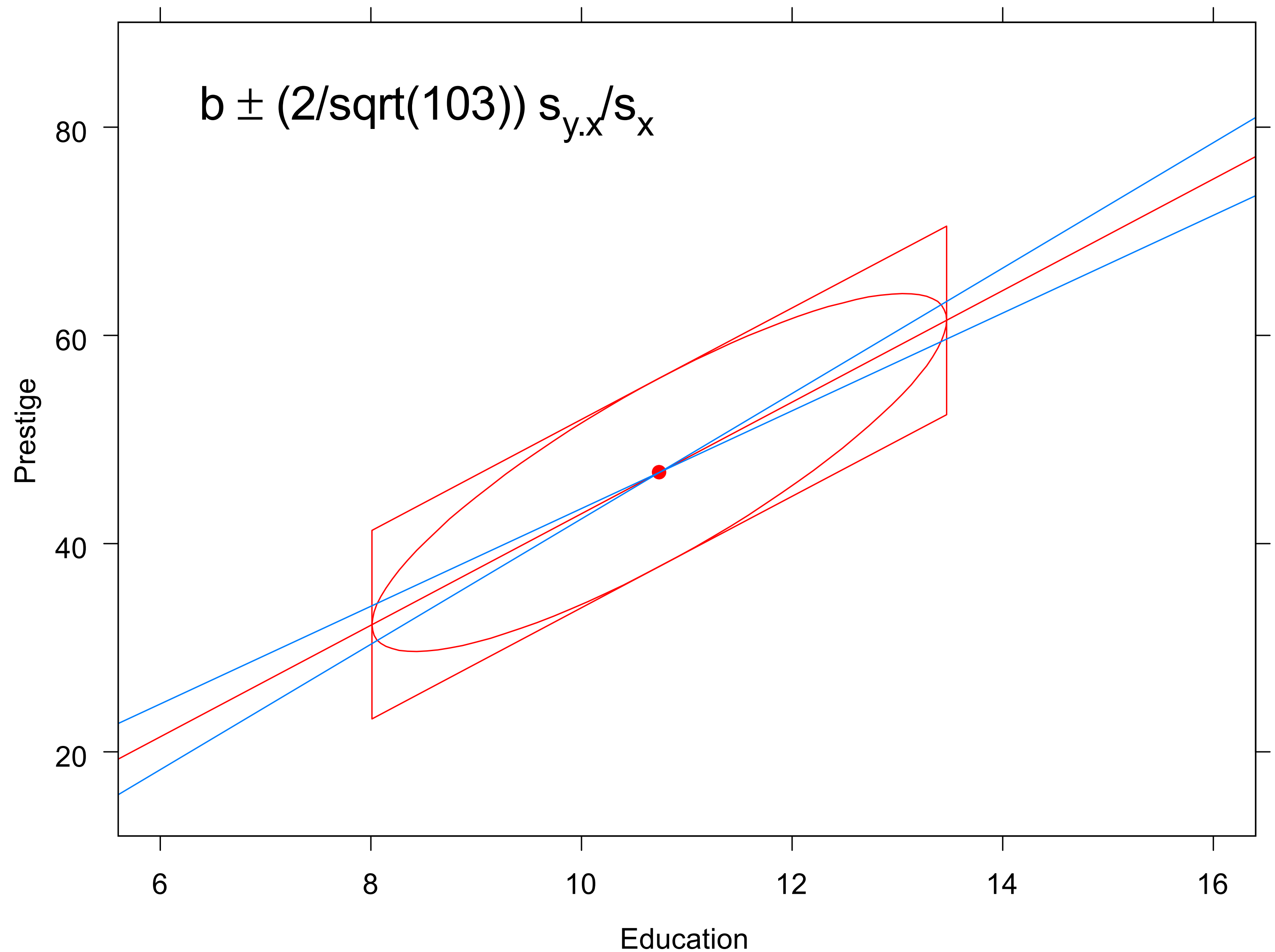


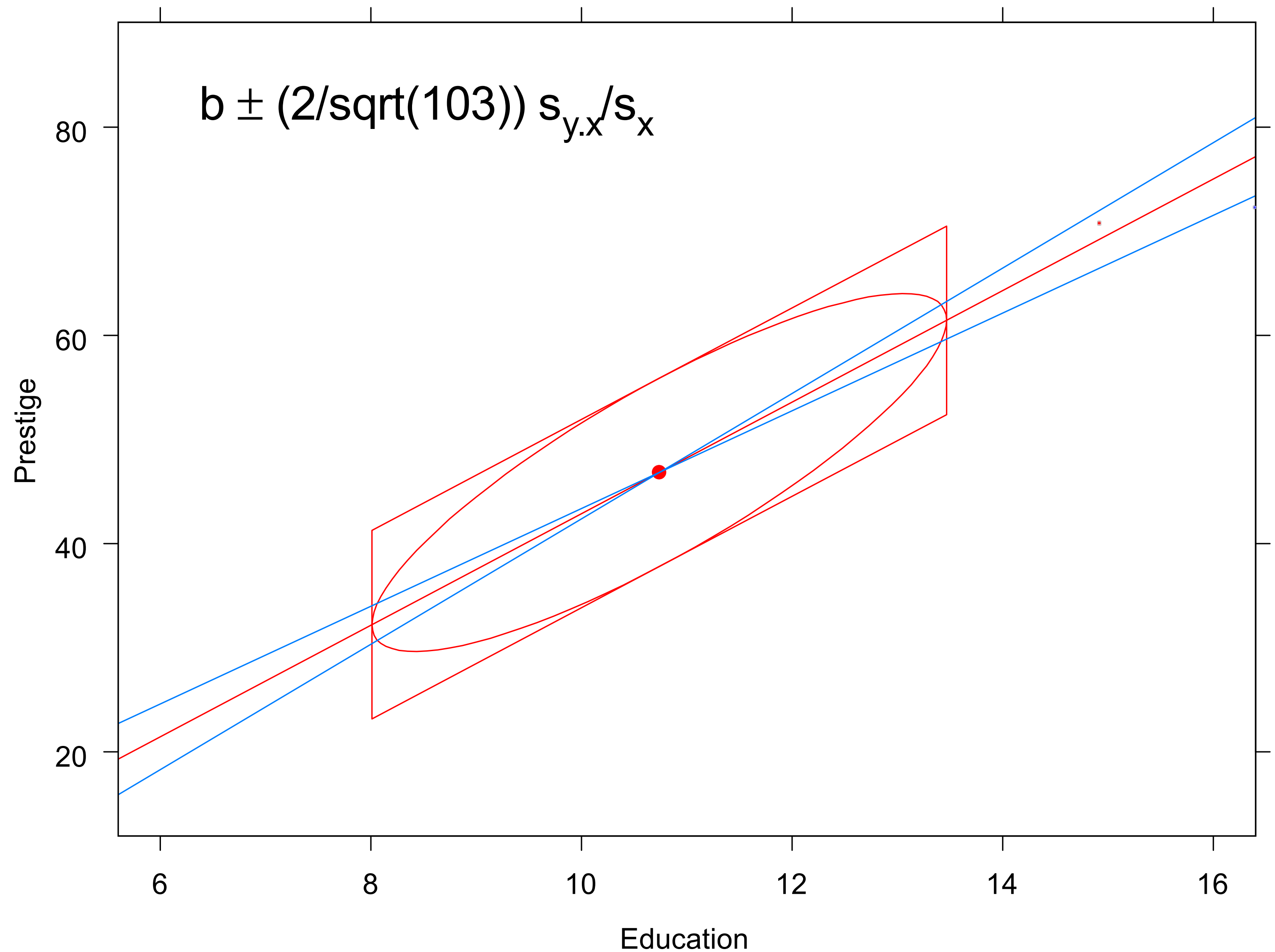
b + s
y.x
/s
x
b
1
b - s
y.x
/s
x
s
y.x
s
x
Prestige
Education
80
60
40
20
6
8
10
12
14
16



b + s_y.x/s_x
b
1
b - s_y.x/s_x
s_y.x
s_x
Prestige
Education
20
40
60
80
6
8
10
12
14
16

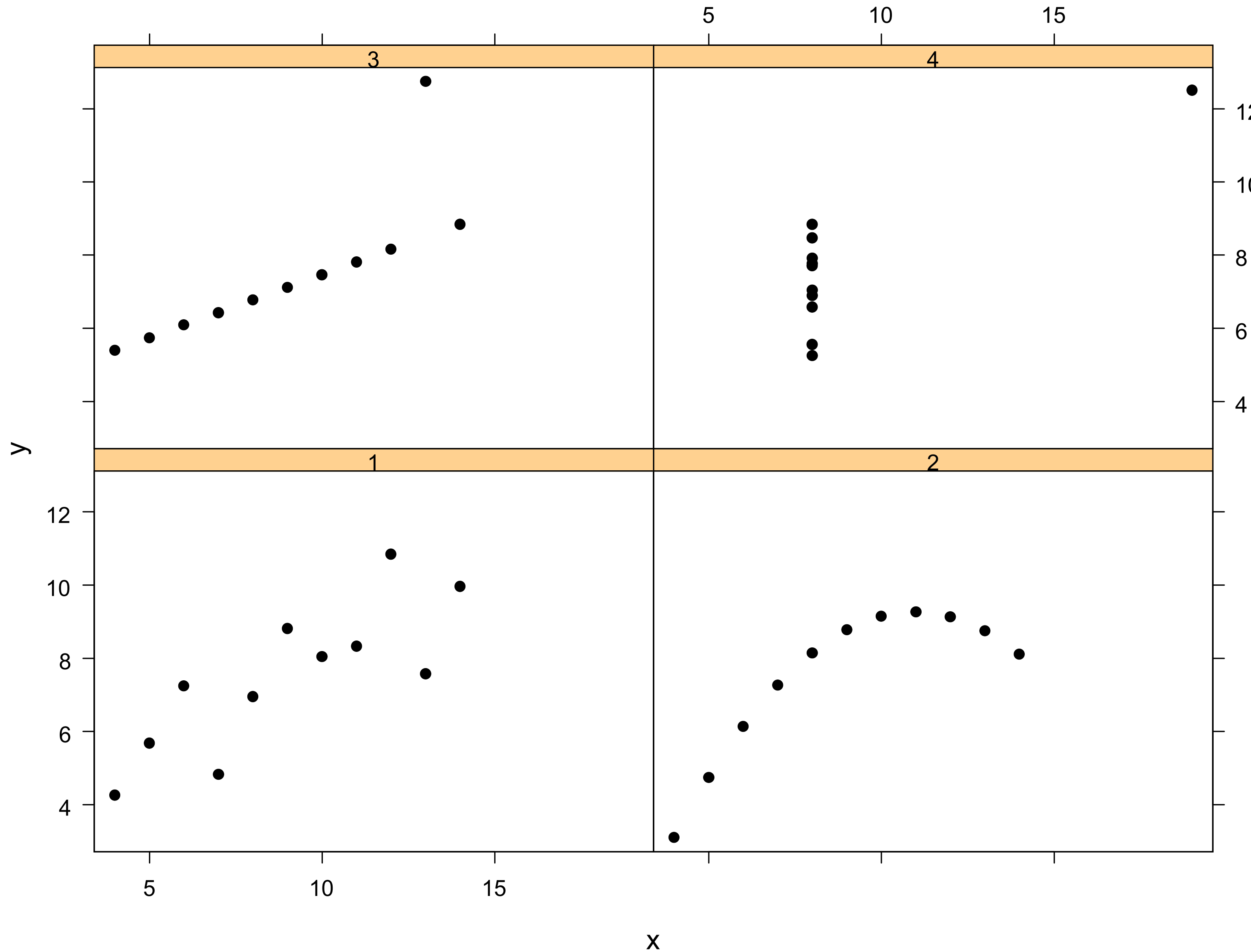

b + $s_{y.x}/s_x$
b
1
b - $s_{y.x}/s_x$
$s_{y.x}$
$s_x$
Prestige
Education
80
60
40
20
6
8
10
12
14
16


$b \pm (2/\text{sqrt}(103))\ s_{y.x}/s_x$
Prestige
80
60
40
20
6
8
10
12
14
16
Education


b ± (2/sqrt(103)) $s_{y.x}/s_x$
Prestige
Education
80
60
40
20
6
8
10
12
14
16

# 4 Anscombe Examples

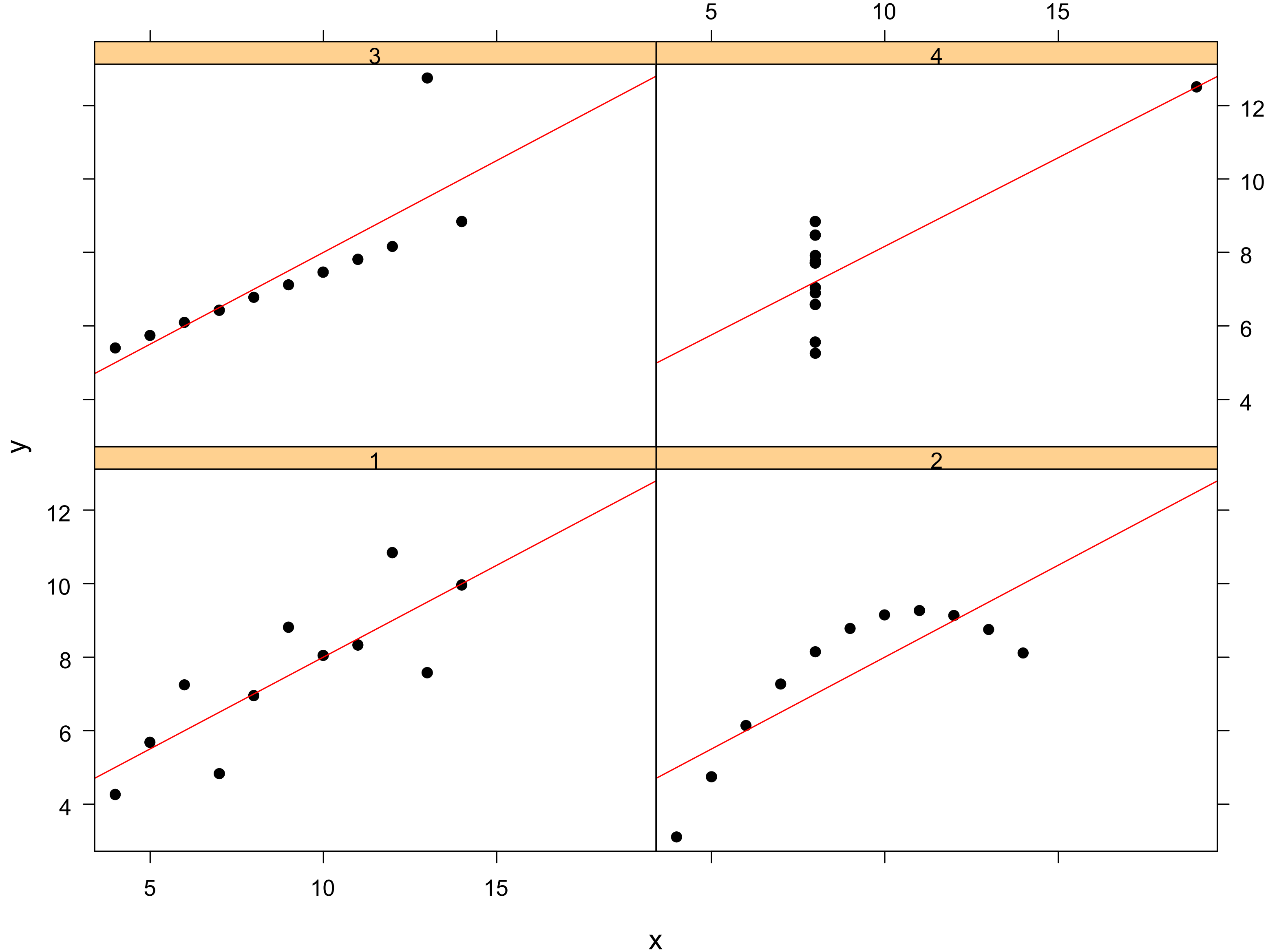
Four datasets:
Same least-squares regression but very different stories



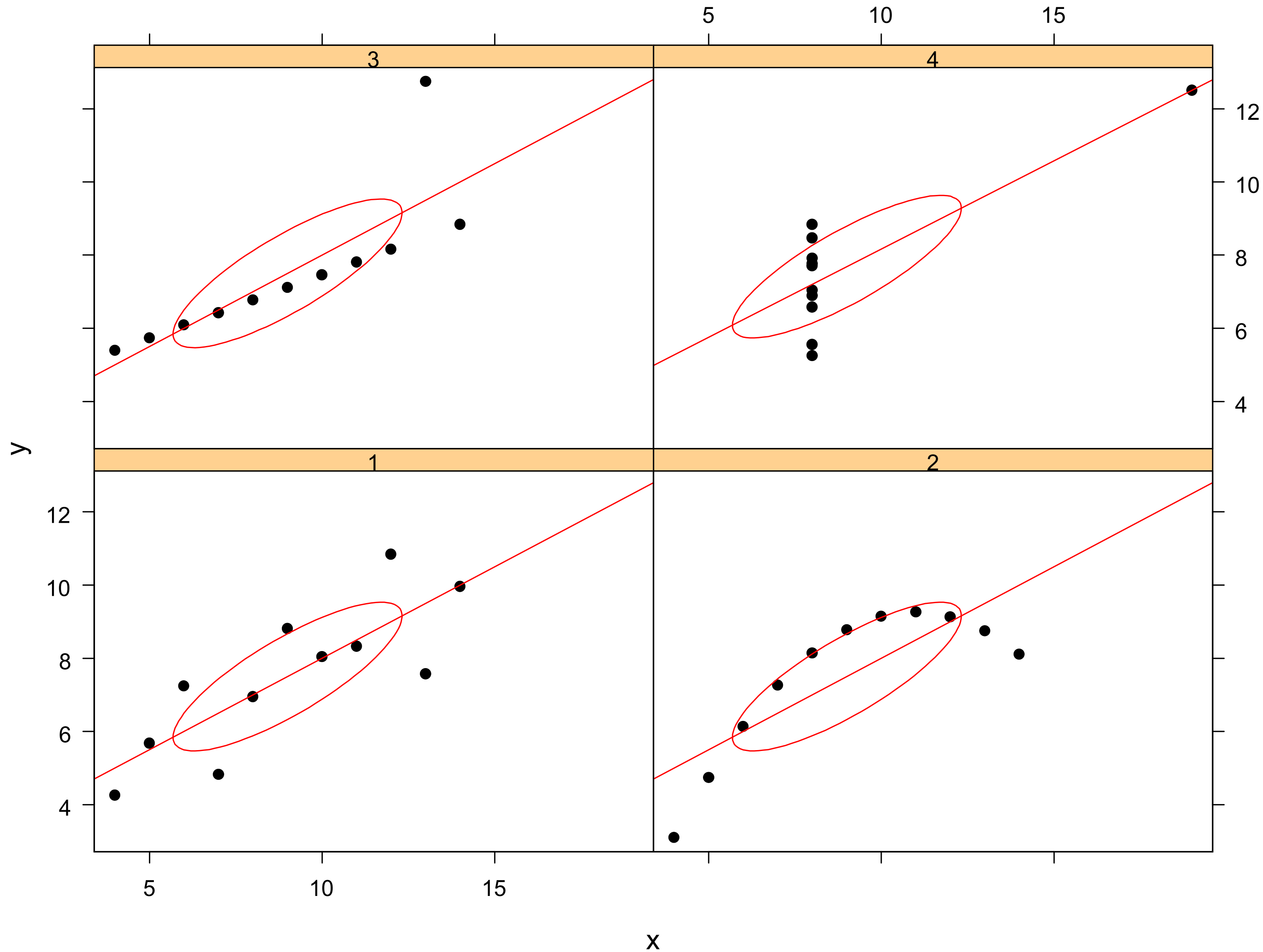
3
4
1
2
5
10
15
4
6
8
10
12
x
y

# Same means, variances and covariances... so same least-squares regression results:

```
Call: lm(formula = y ~ x, data = Anscombe, subset = type == 1)
Residuals:
    Min      1Q   Median     3Q   Max
 -1.921 -0.4558 -0.04136 0.7094 1.839

Coefficients:
             Value Std. Error t value Pr(>|t|)
(Intercept) 3.0001 1.1247     2.6673  0.0257
          x 0.5001 0.1179     4.2415  0.0022

Residual standard error: 1.237 on 9 degrees of freedom
Multiple R-Squared: 0.6665
F-statistic: 17.99 on 1 and 9 degrees of freedom,
       the p-value is 0.00217
```



3
4
1
2
5
10
15
12
10
8
6
4
y
x

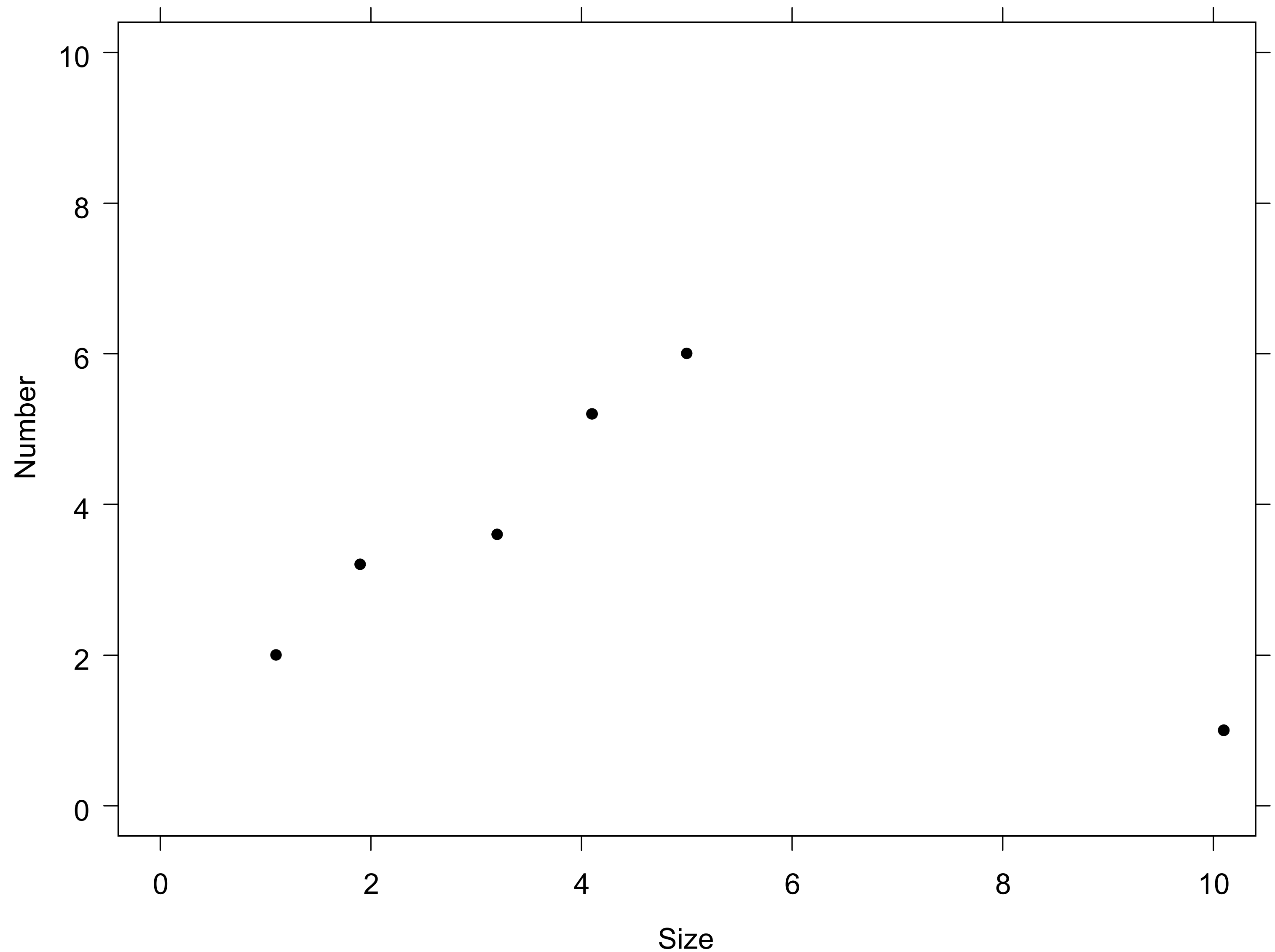

3
4
1
2
5
10
15
12
10
8
6
4
y
x

# 5 Influence and Andrews' Configuration

Darwin's data on Galapagos Islands:
X: size of island
Y: number of finch species

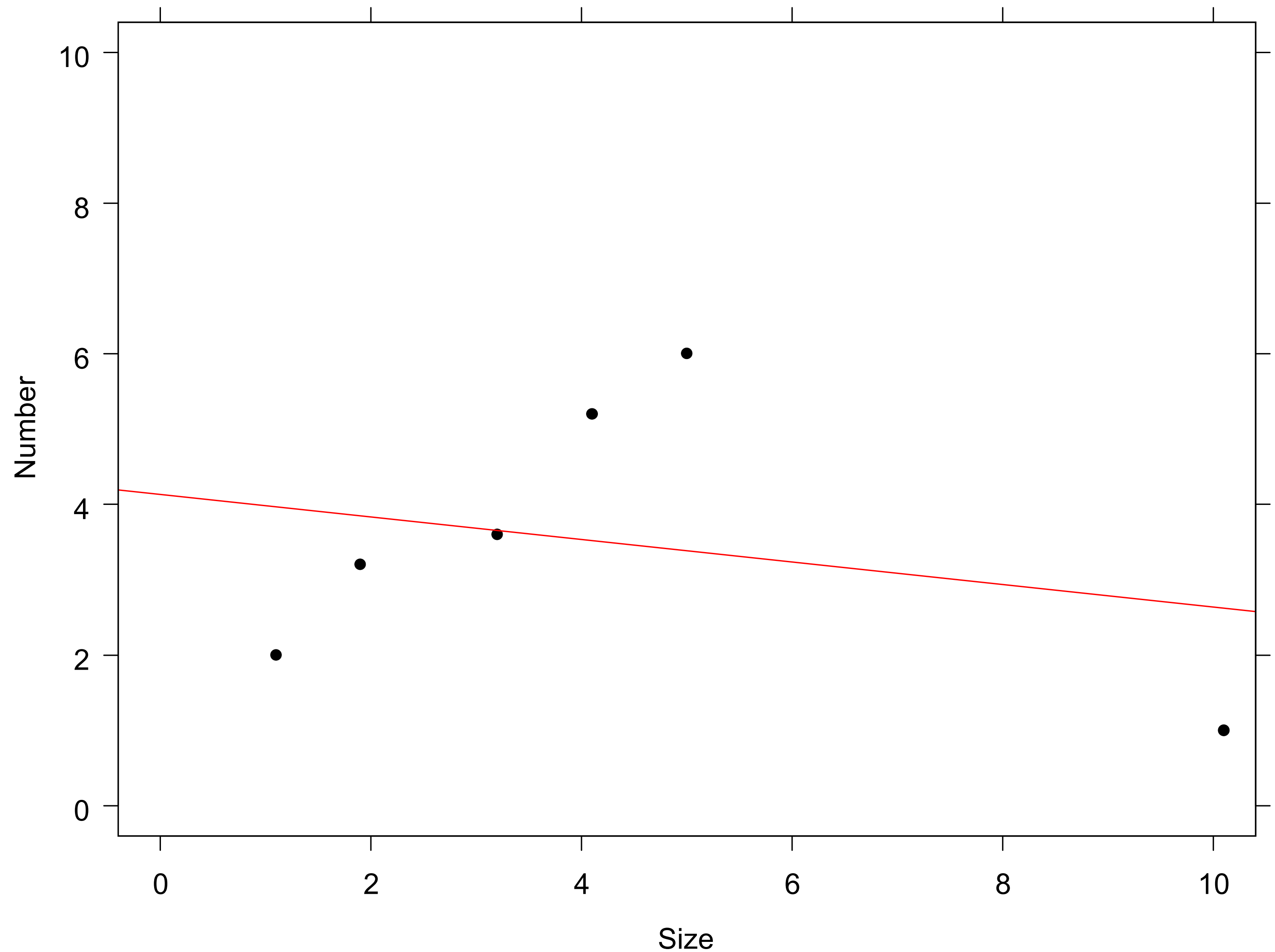
Hypothesis: X and Y should have a positive relationship if a larger area provides more evolutionary niches

Idealized version of Darwin's data = Andrews Configuration

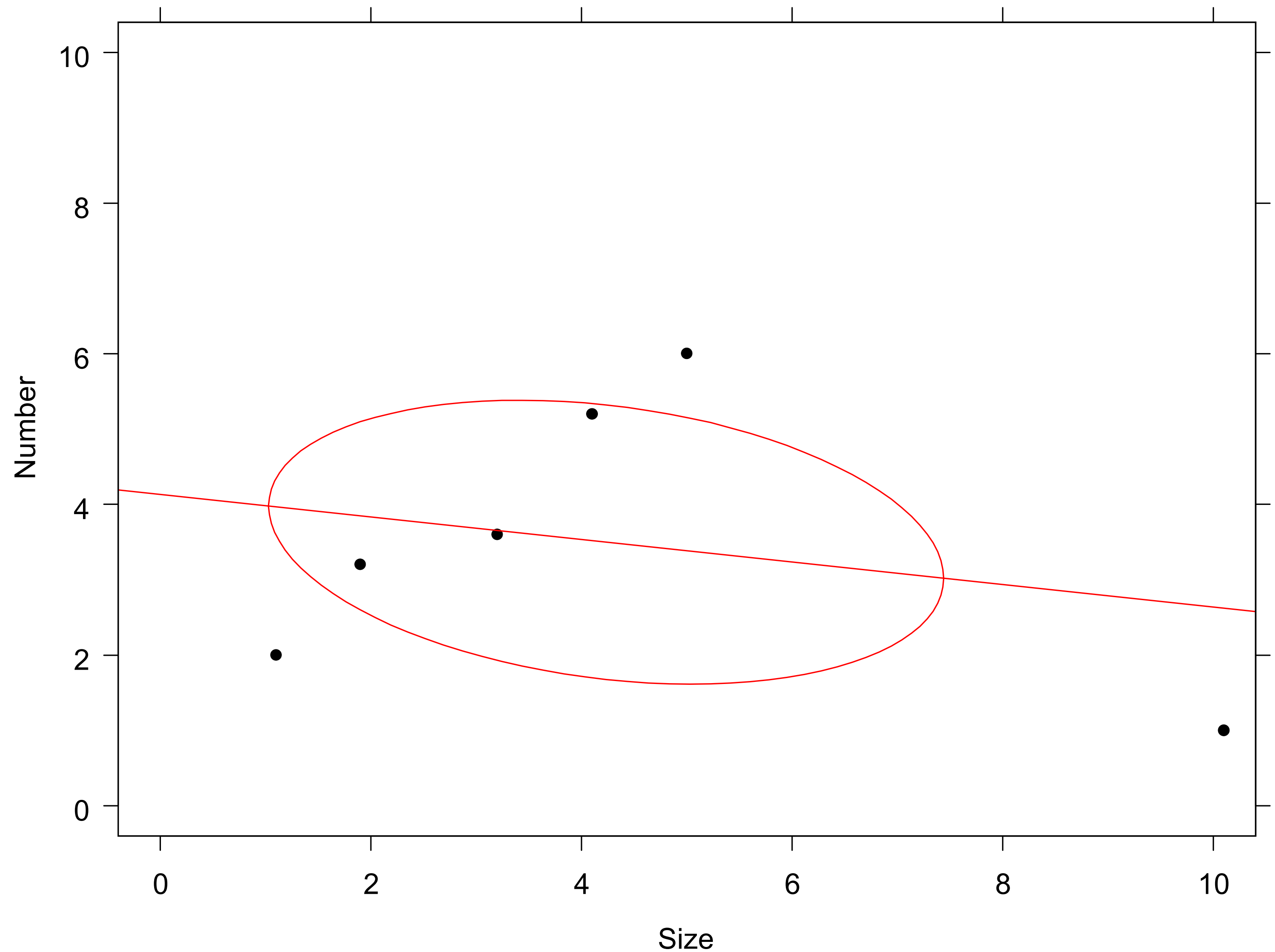

Number
Size
0
2
4
6
8
10

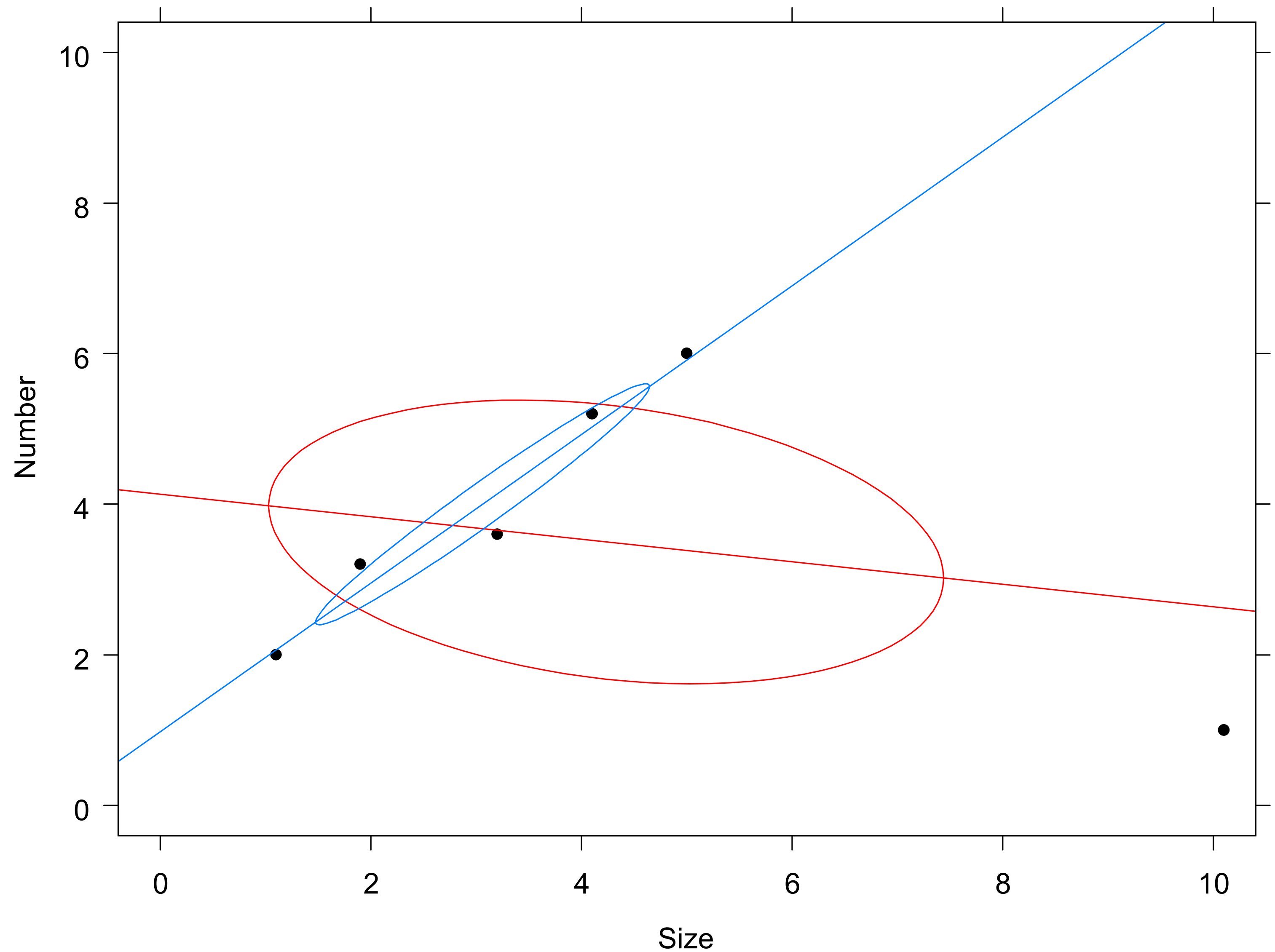
# Least-squares regression:

```
> fit <- lm(Number ~Size, Andrews)
> summary(fit)

Call: lm(formula = Number ~Size, data = Andrews)
Residuals:
      1      2        3    4     5      6
 -1.969 -0.649 -0.05454 1.68 2.615 -1.623

Coefficients:
              Value Std. Error t value Pr(>|t|)
(Intercept)  4.1331  1.4625     2.8261  0.0475
       Size -0.1496  0.2842    -0.5262  0.6266

Residual standard error: 2.037 on 4 degrees of freedom
Multiple R-Squared: 0.06474
F-statistic: 0.2769 on 1 and 4 degrees of freedom,
   the p-value is 0.6266
```



Number
Size
0
2
4
6
8
10


10
8
6
4
2
0
Number
0
2
4
6
8
10
Size

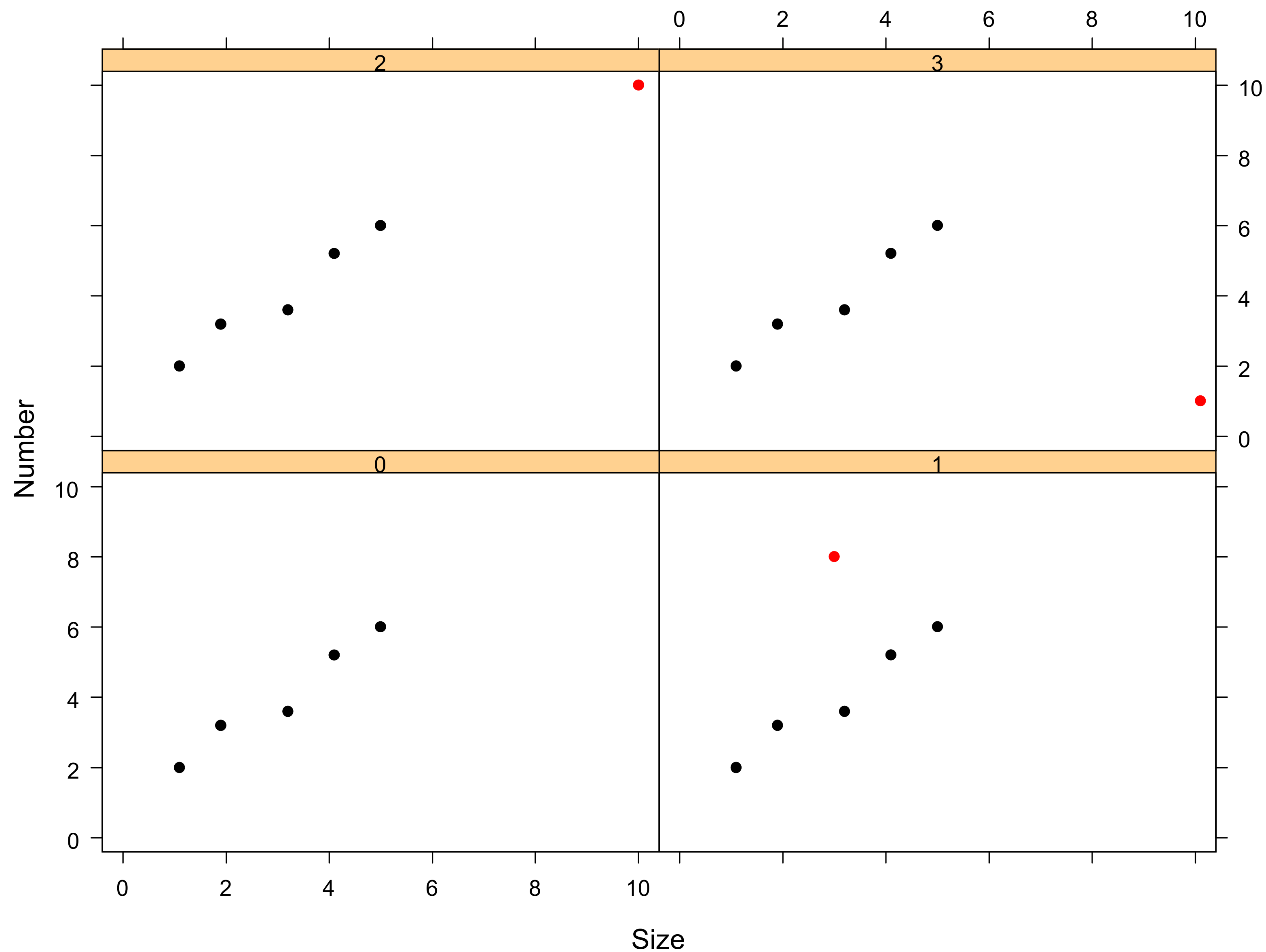

Number
Size
0
2
4
6
8
10
0
2
4
6
8
10

# 6 Influence and Leverage

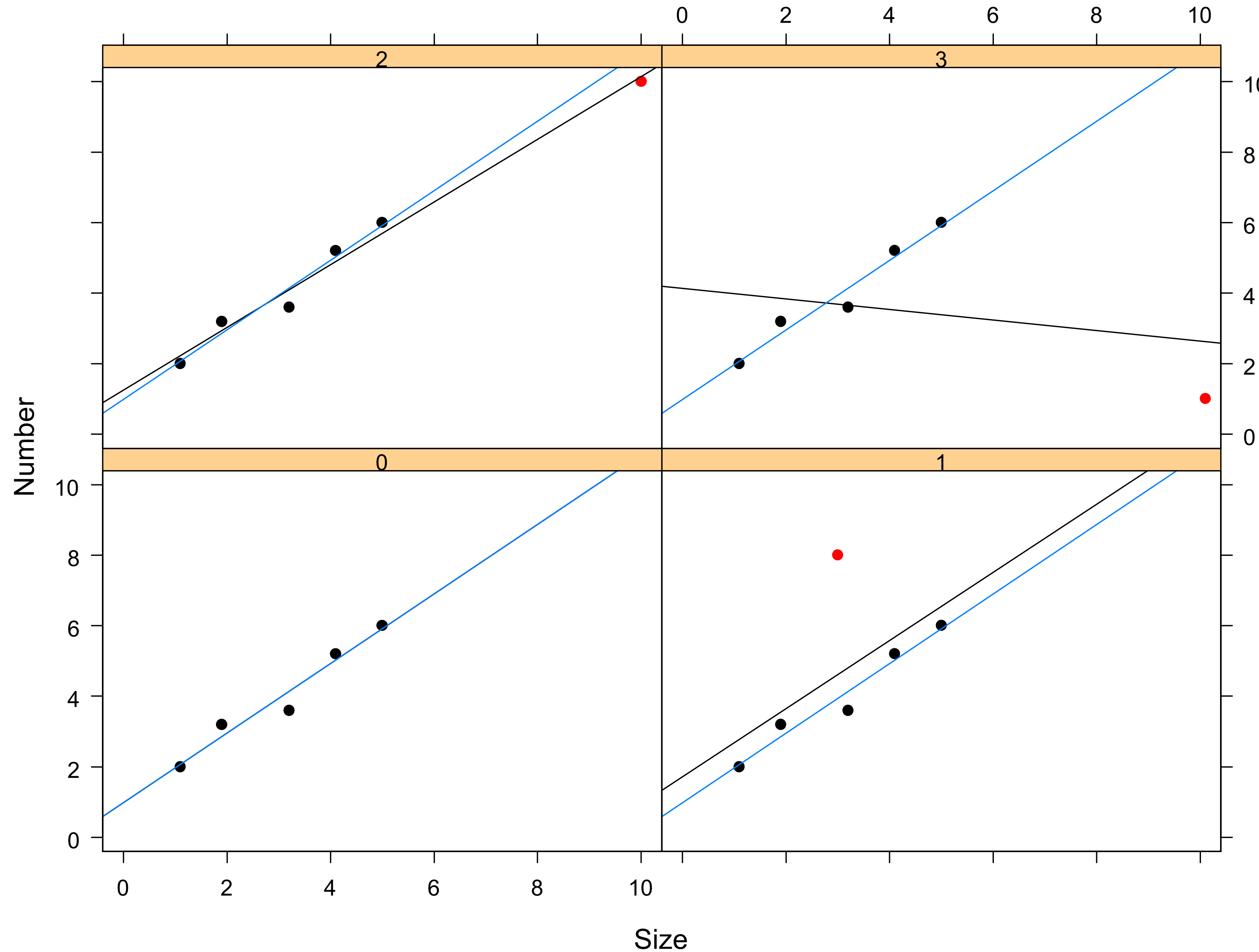
What happens when you take good data and add a wild point?

Three archetypal wild points:

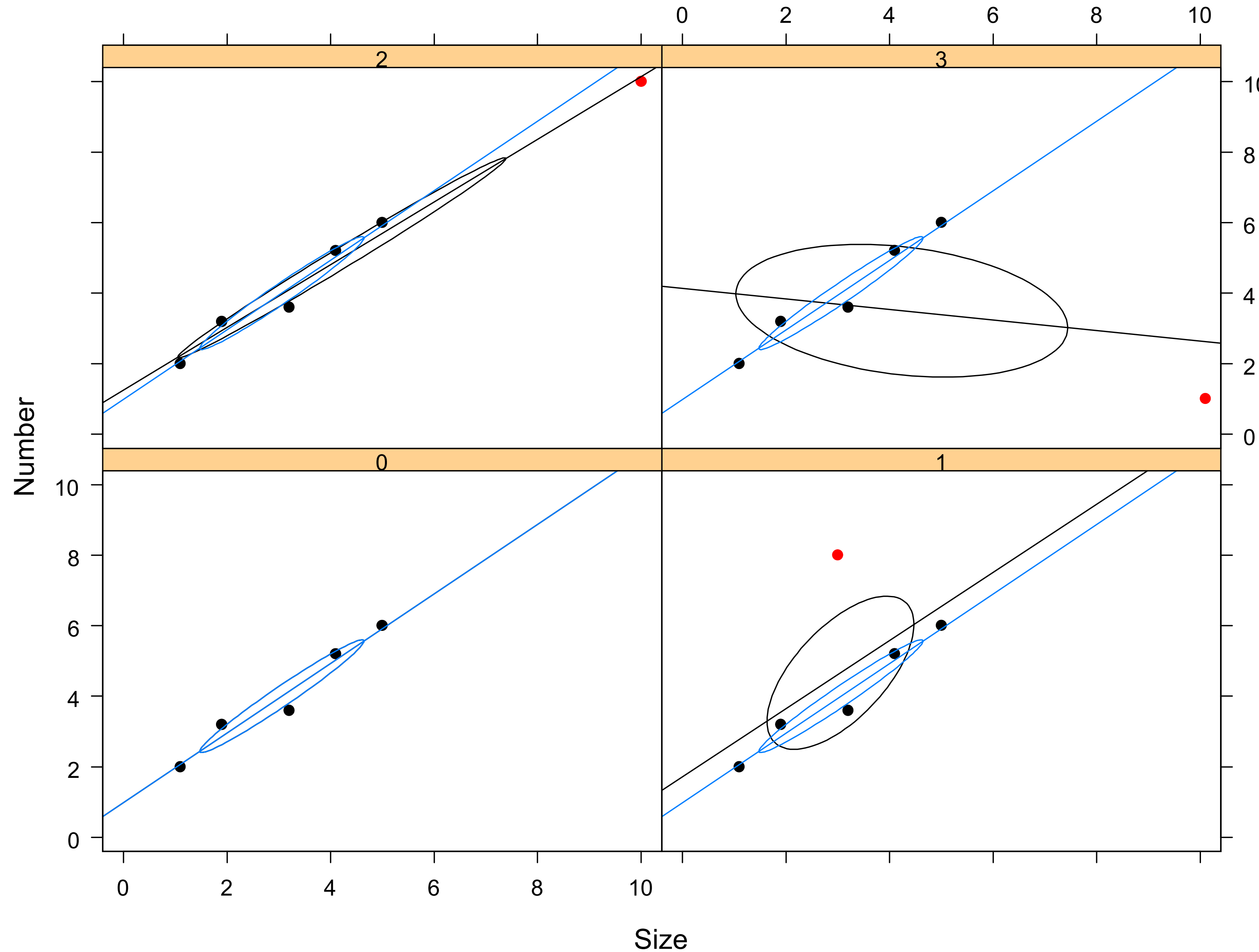
- Typical X, bad fit
- Unusual X, good fit
- Unusual X, bad fit

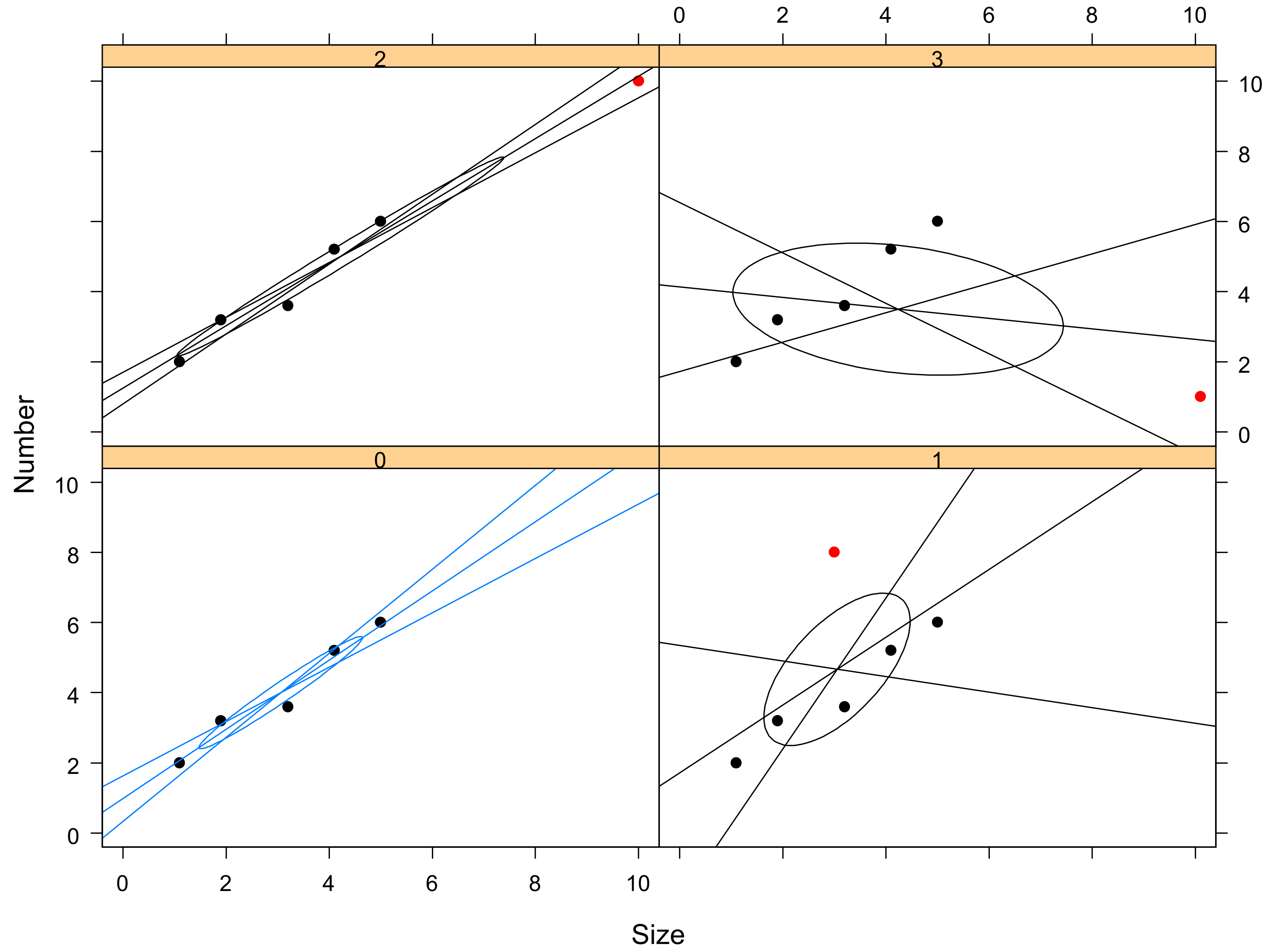


2
3
0
1
Number
Size
0
2
4
6
8
10



2
3
0
1
Number
Size
0
2
4
6
8
10


2
3
0
1
Number
Size
0
2
4
6
8
10

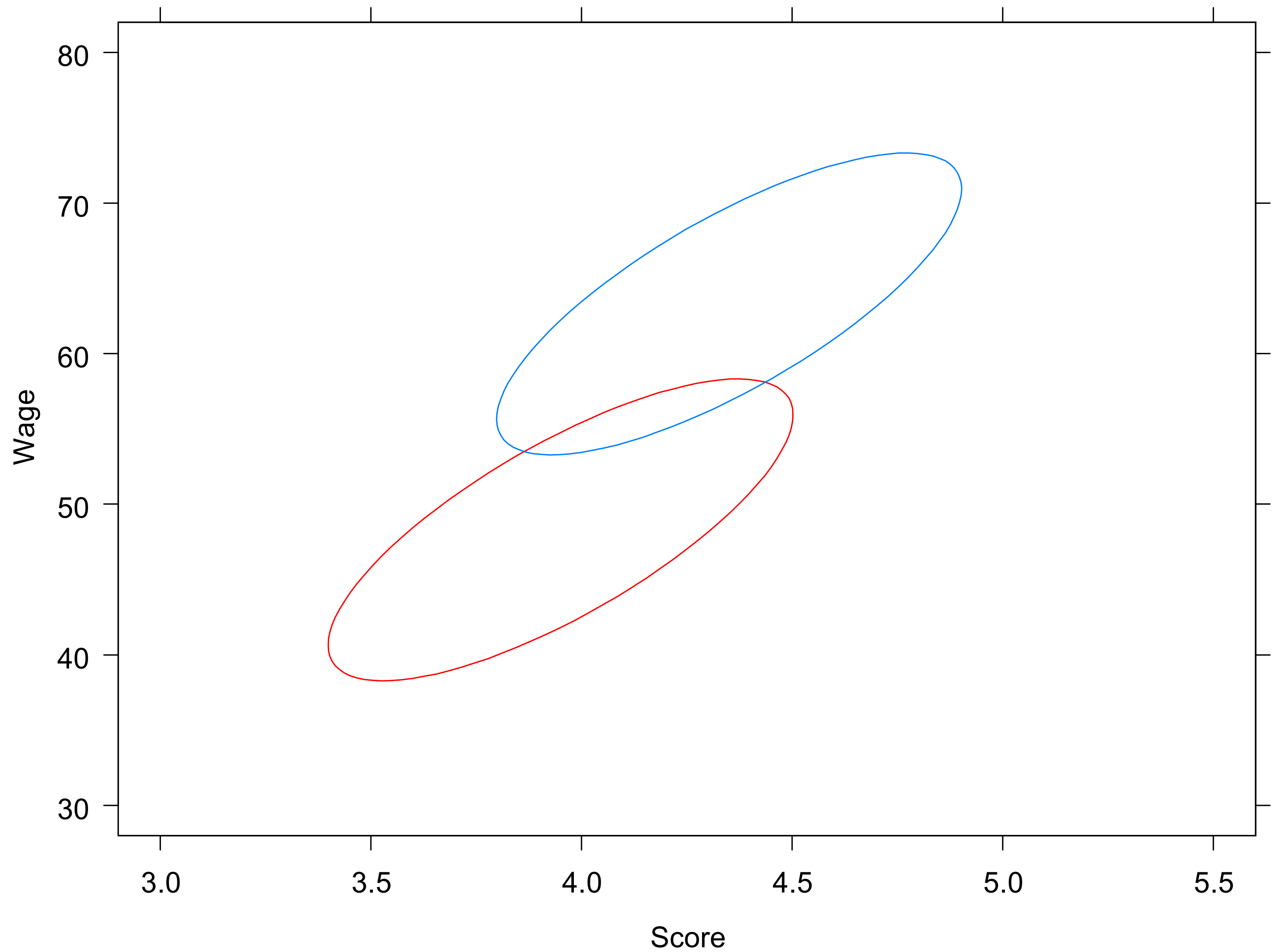

2
3
0
1
Number
Size
0
2
4
6
8
10

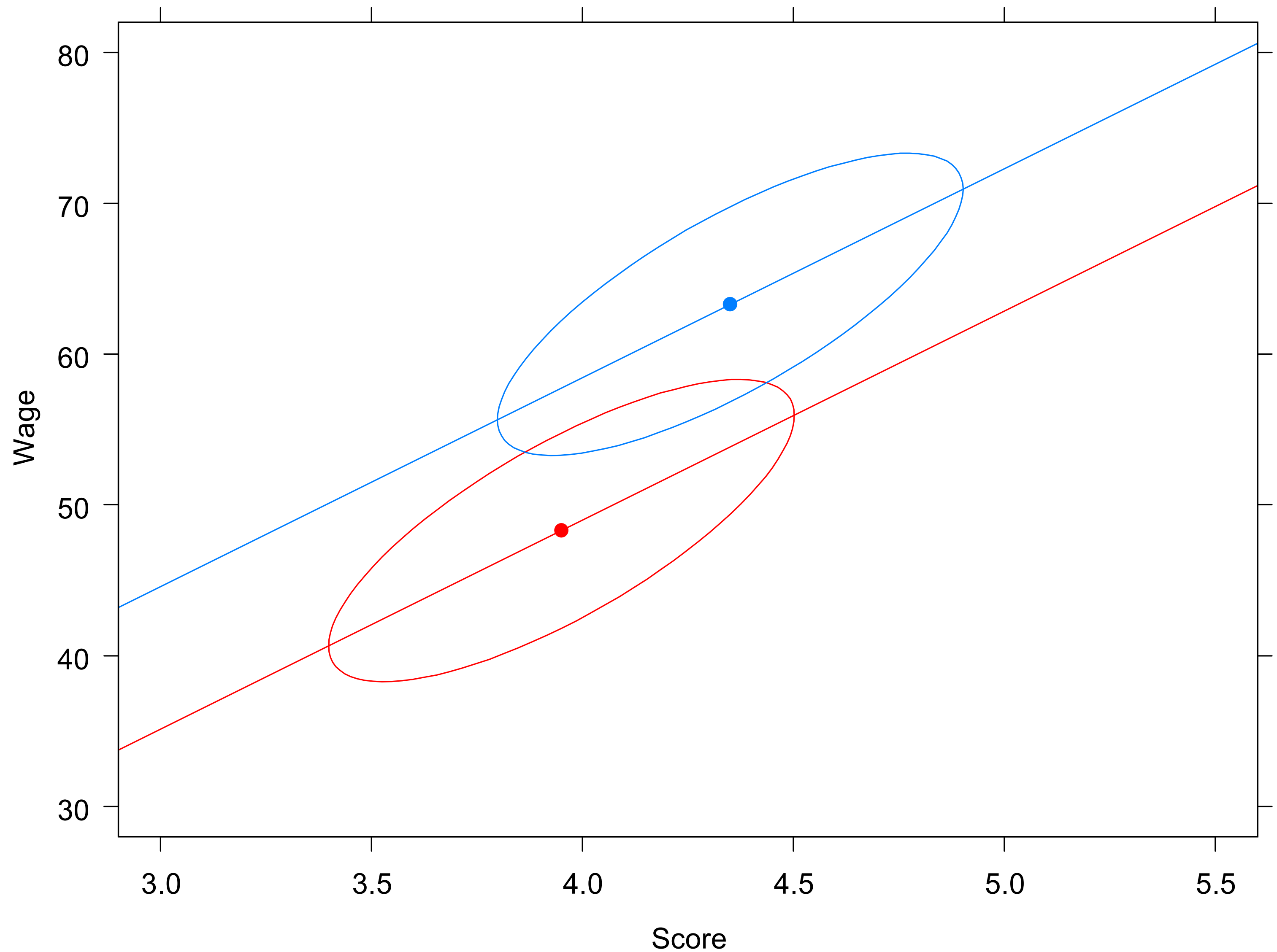
# 7 Measurement error

What happens when a covariate is measured with error?

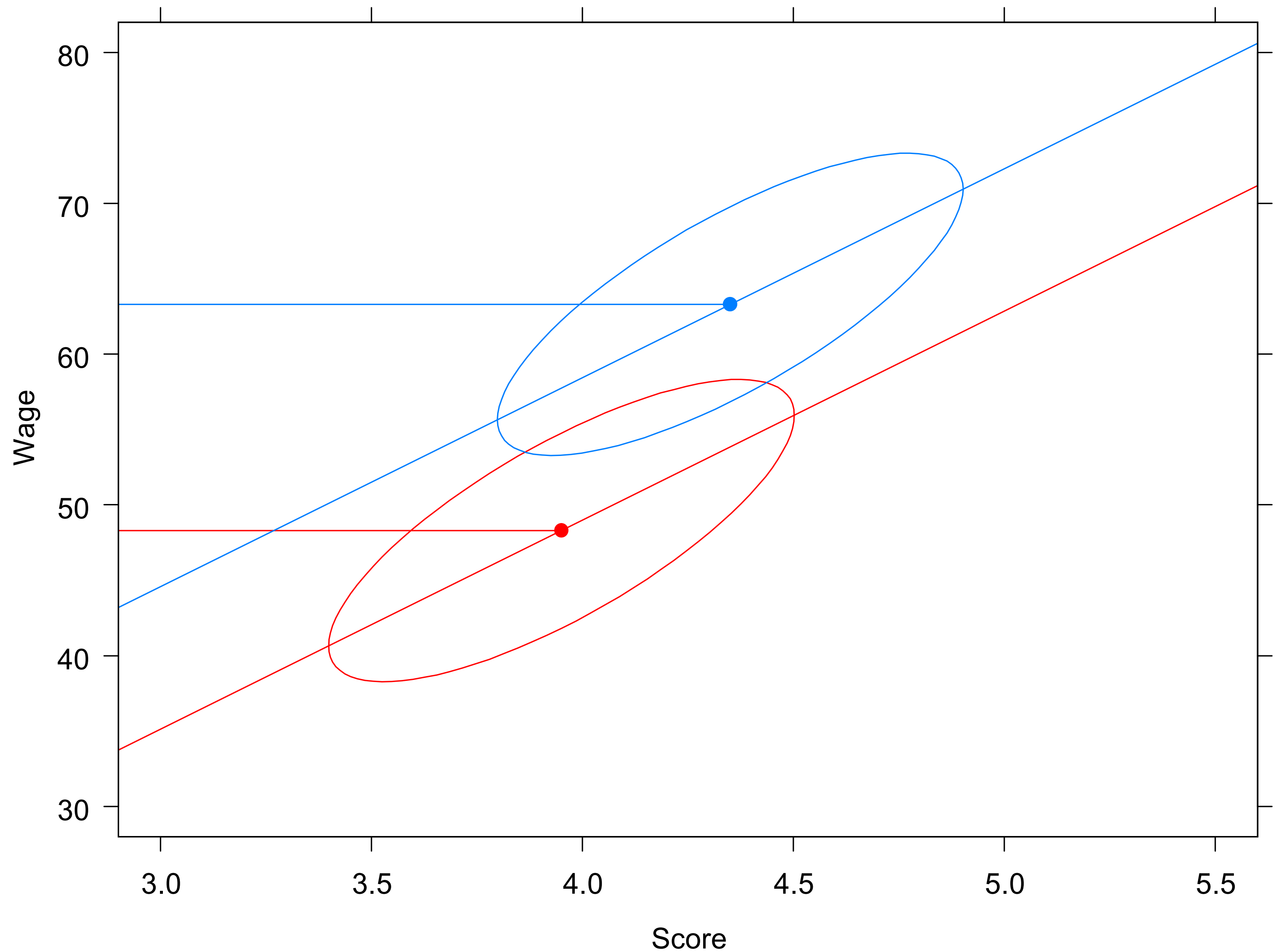
Example: Pay Equity – Estimating the Gender Gap

Wages vs Job Value by Gender

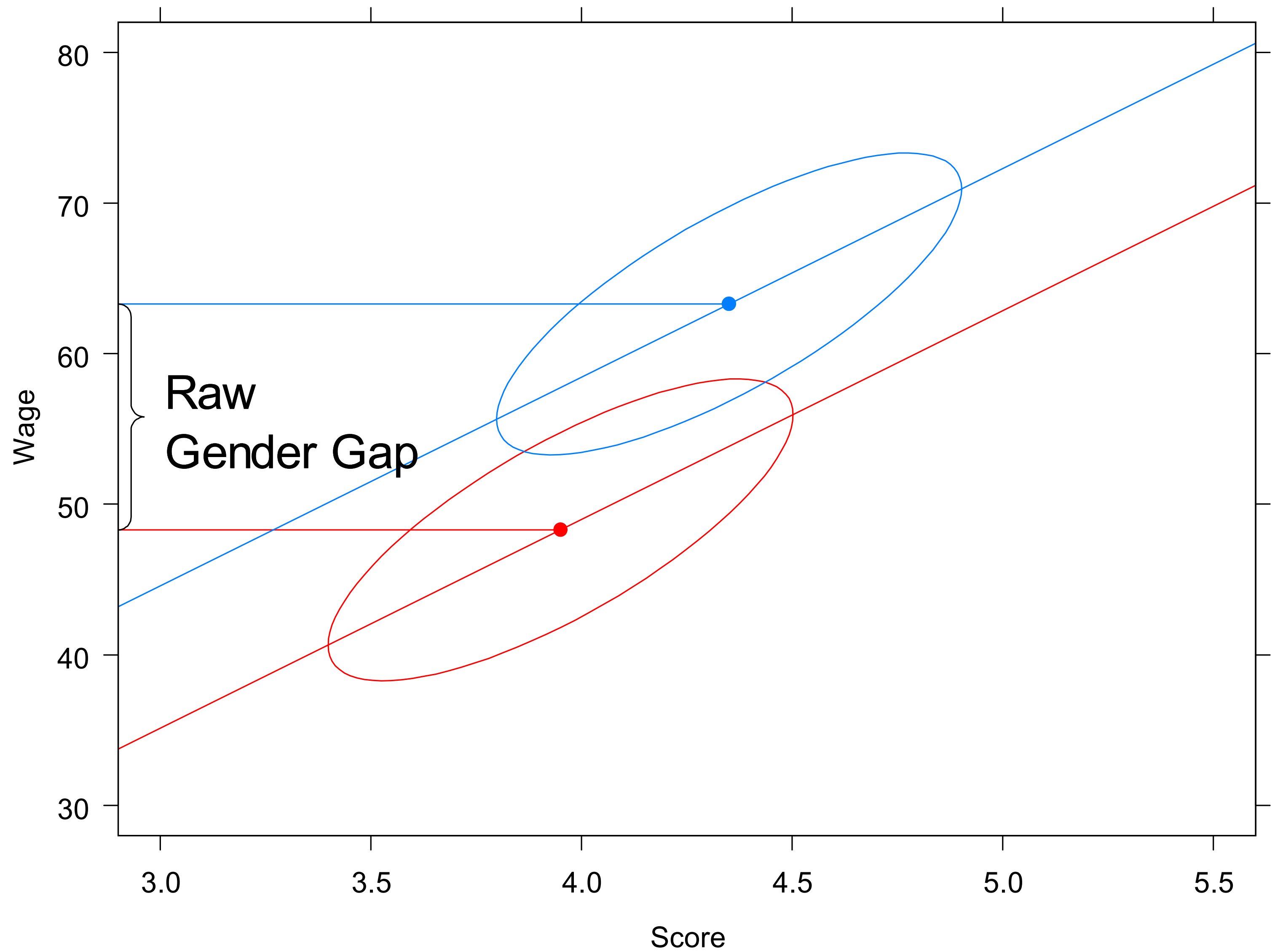

Wage
80
70
60
50
40
30
3.0
3.5
4.0
4.5
5.0
5.5
Score

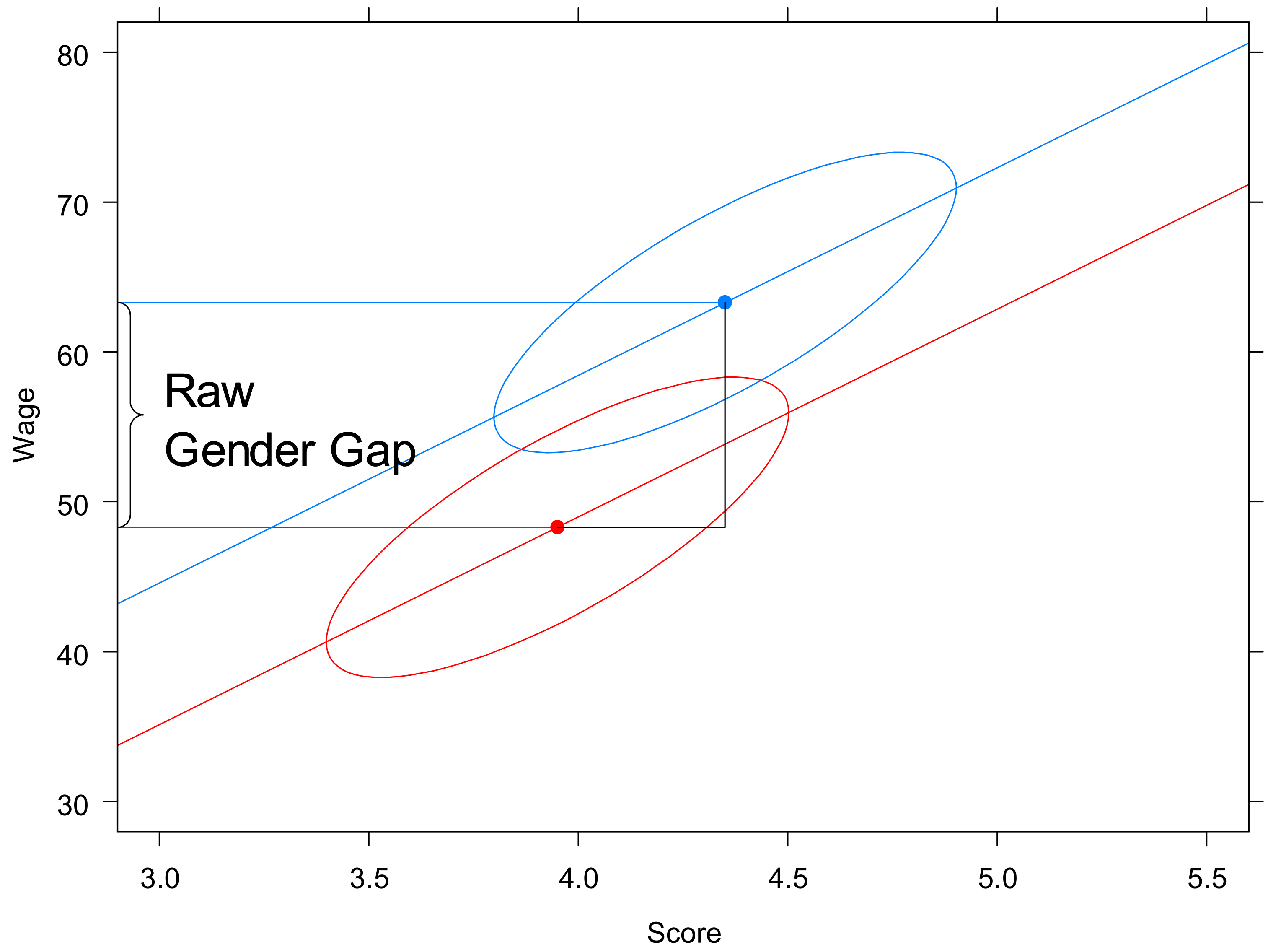


80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score

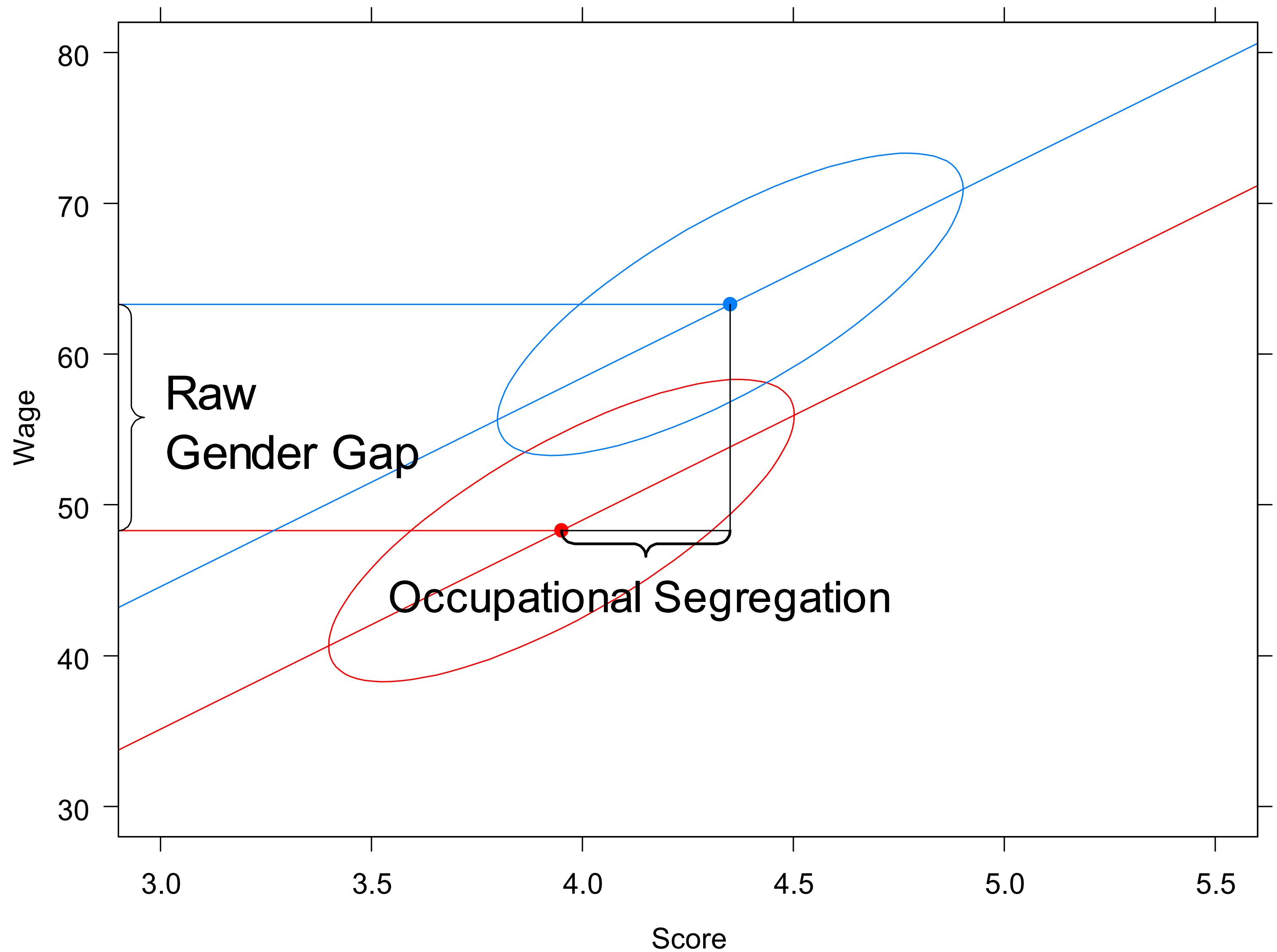


Wage
80
70
60
50
40
30
3.0
3.5
4.0
4.5
5.0
5.5
Score

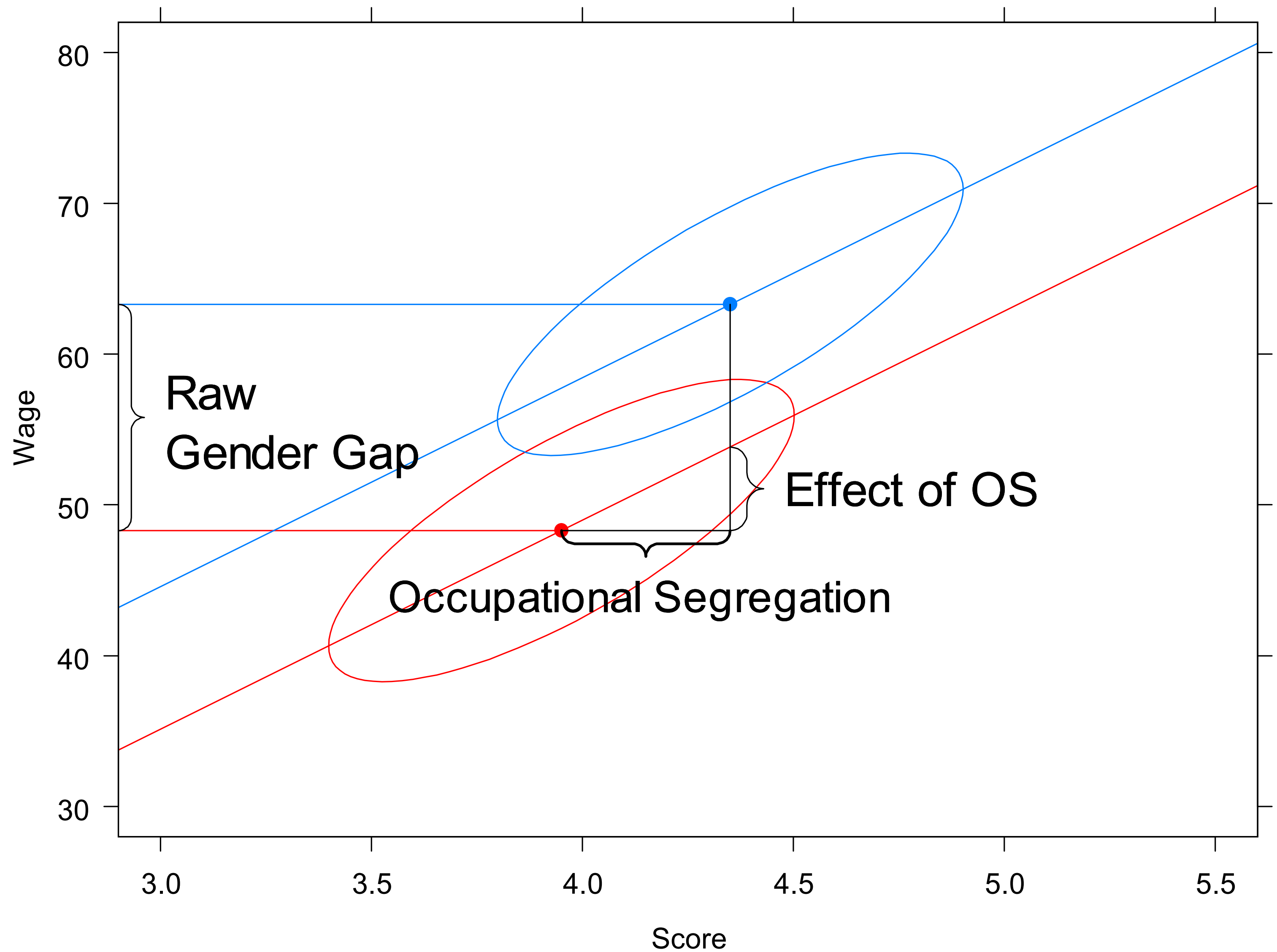

Raw
Gender Gap
Wage
Score
80
70
60
50
40
30
3.0
3.5
4.0
4.5
5.0
5.5



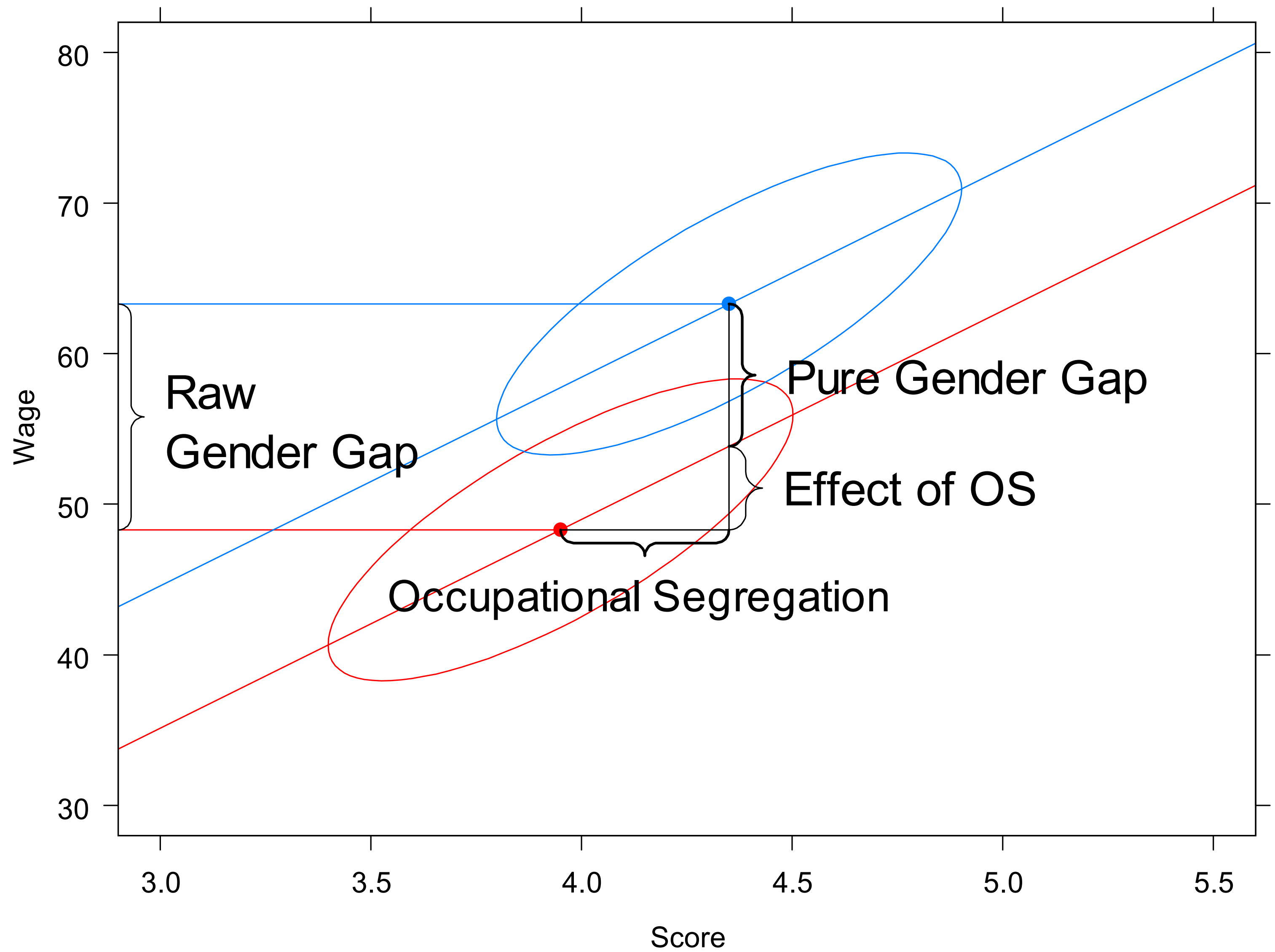
Raw
Gender Gap
Wage
80
70
60
50
40
30
3.0
3.5
4.0
4.5
5.0
5.5
Score

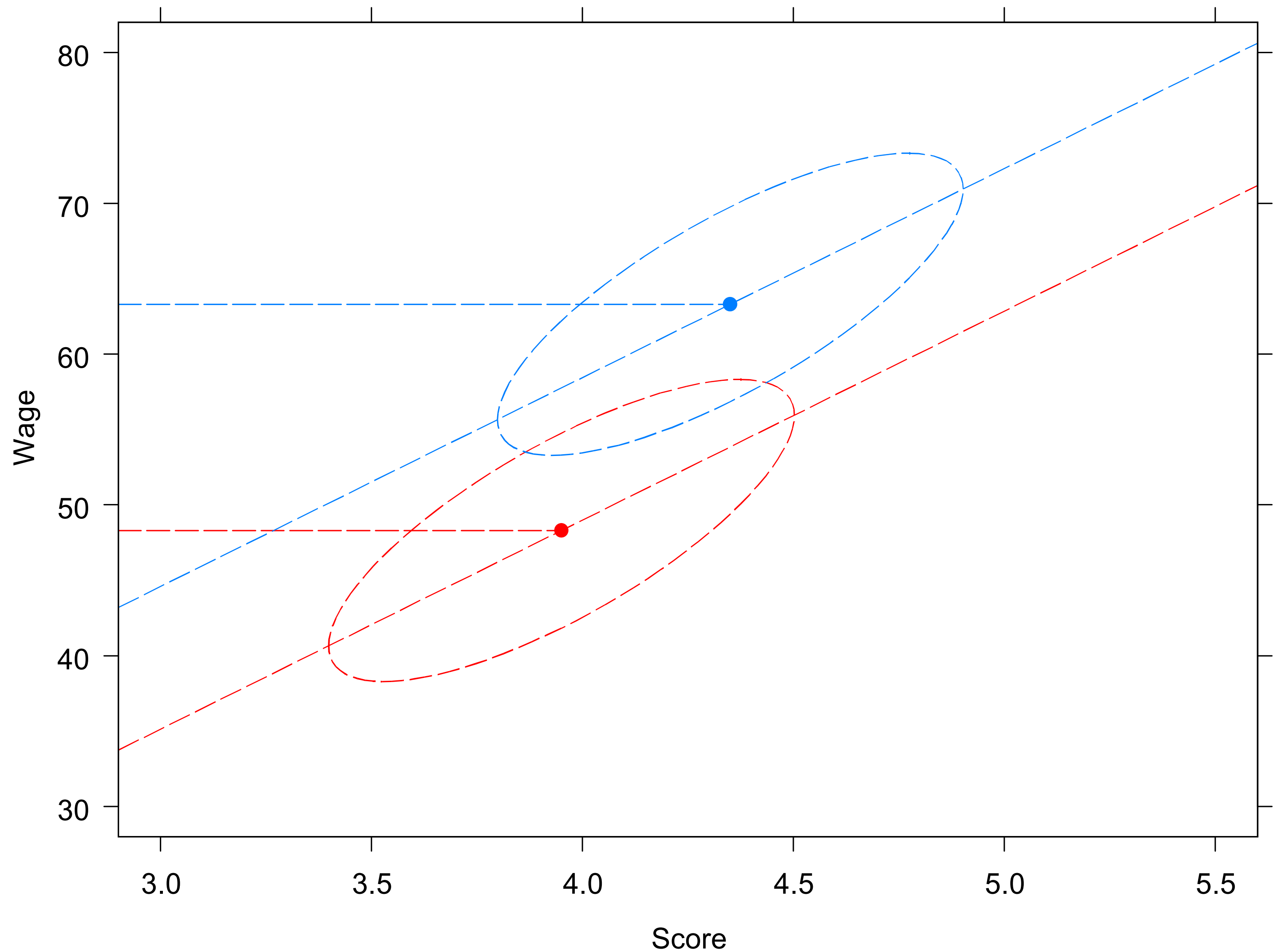


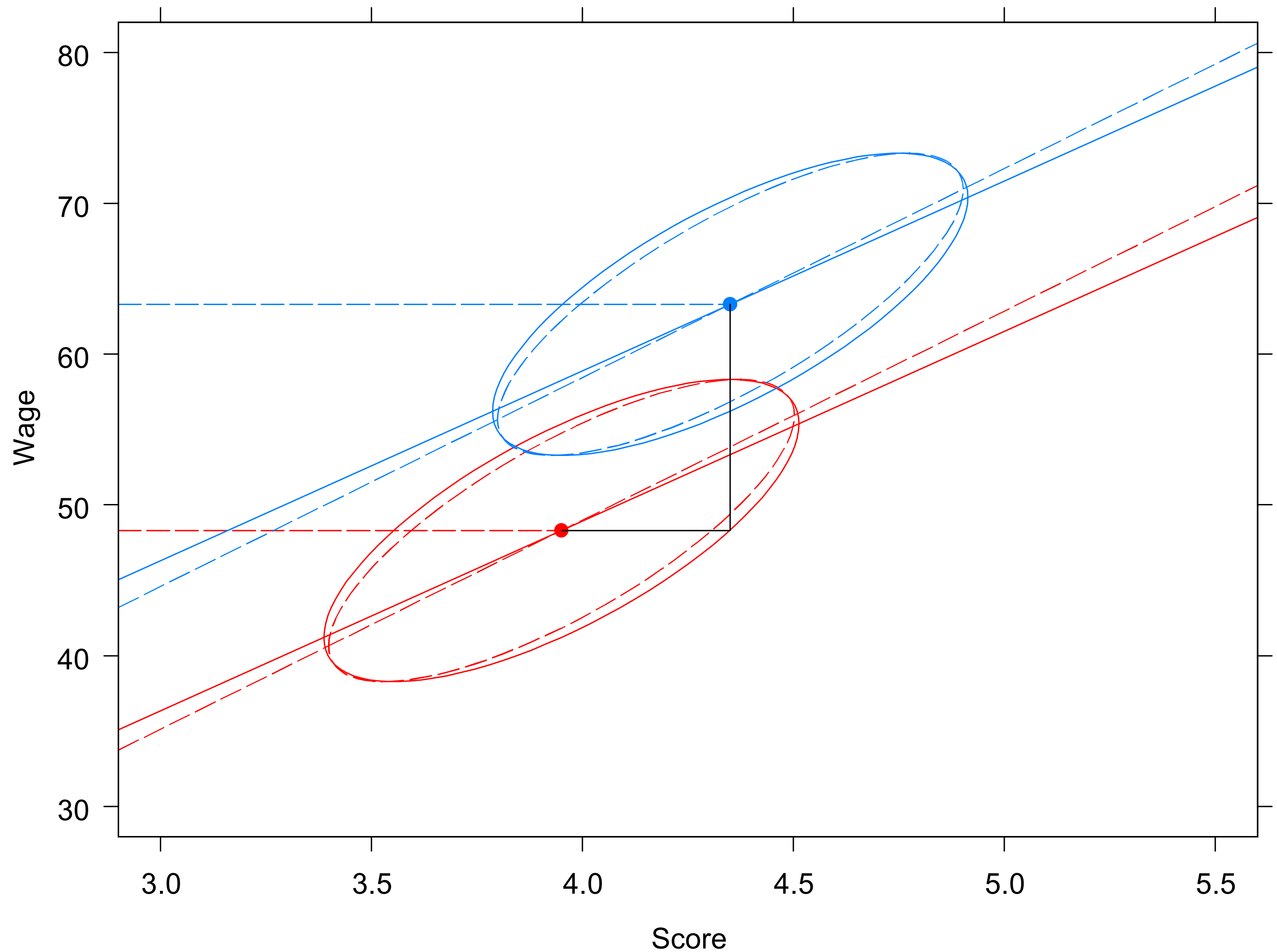
80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score
Raw
Gender Gap
Occupational Segregation

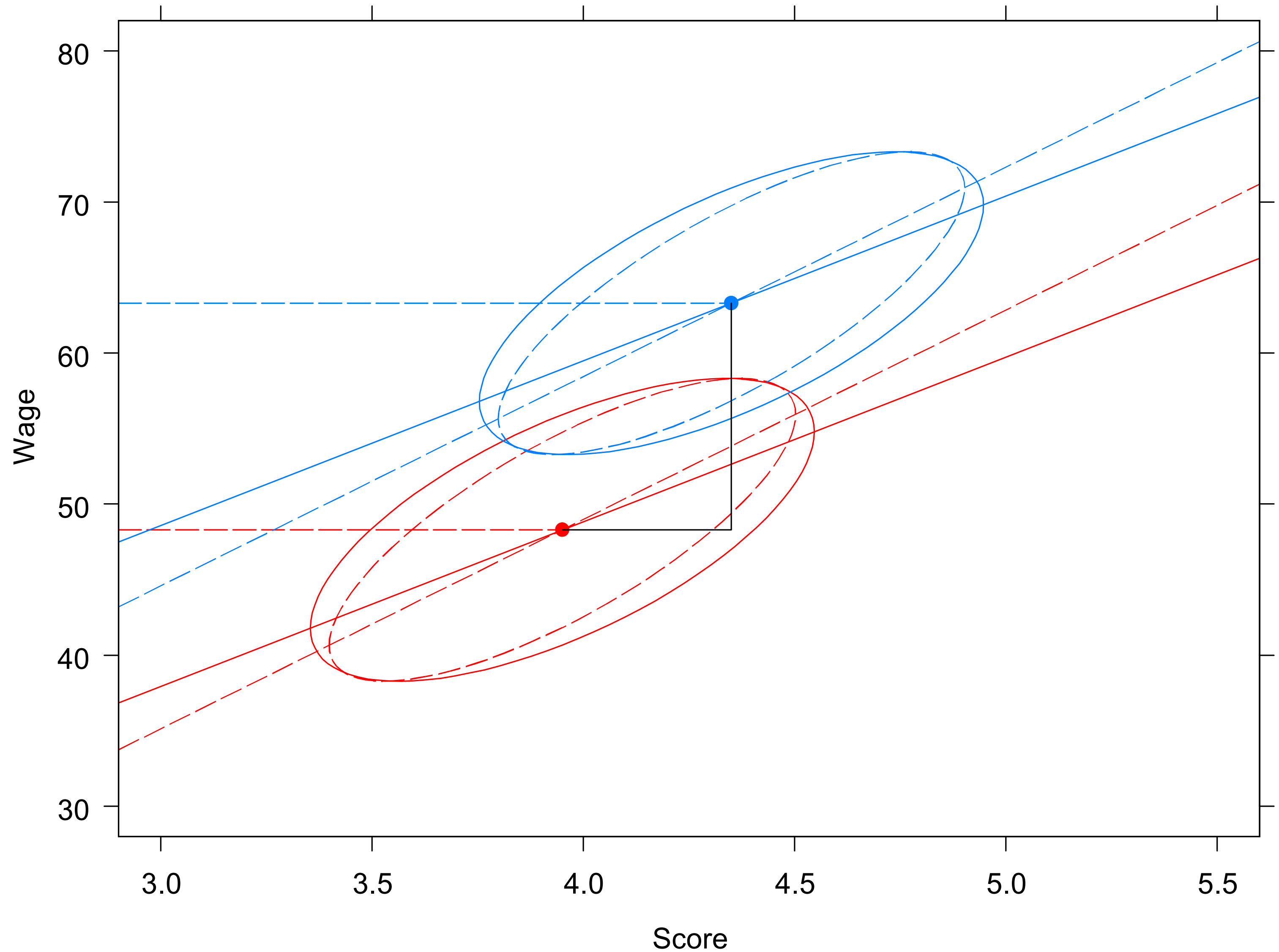


80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score
Raw
Gender Gap
Effect of OS
Occupational Segregation

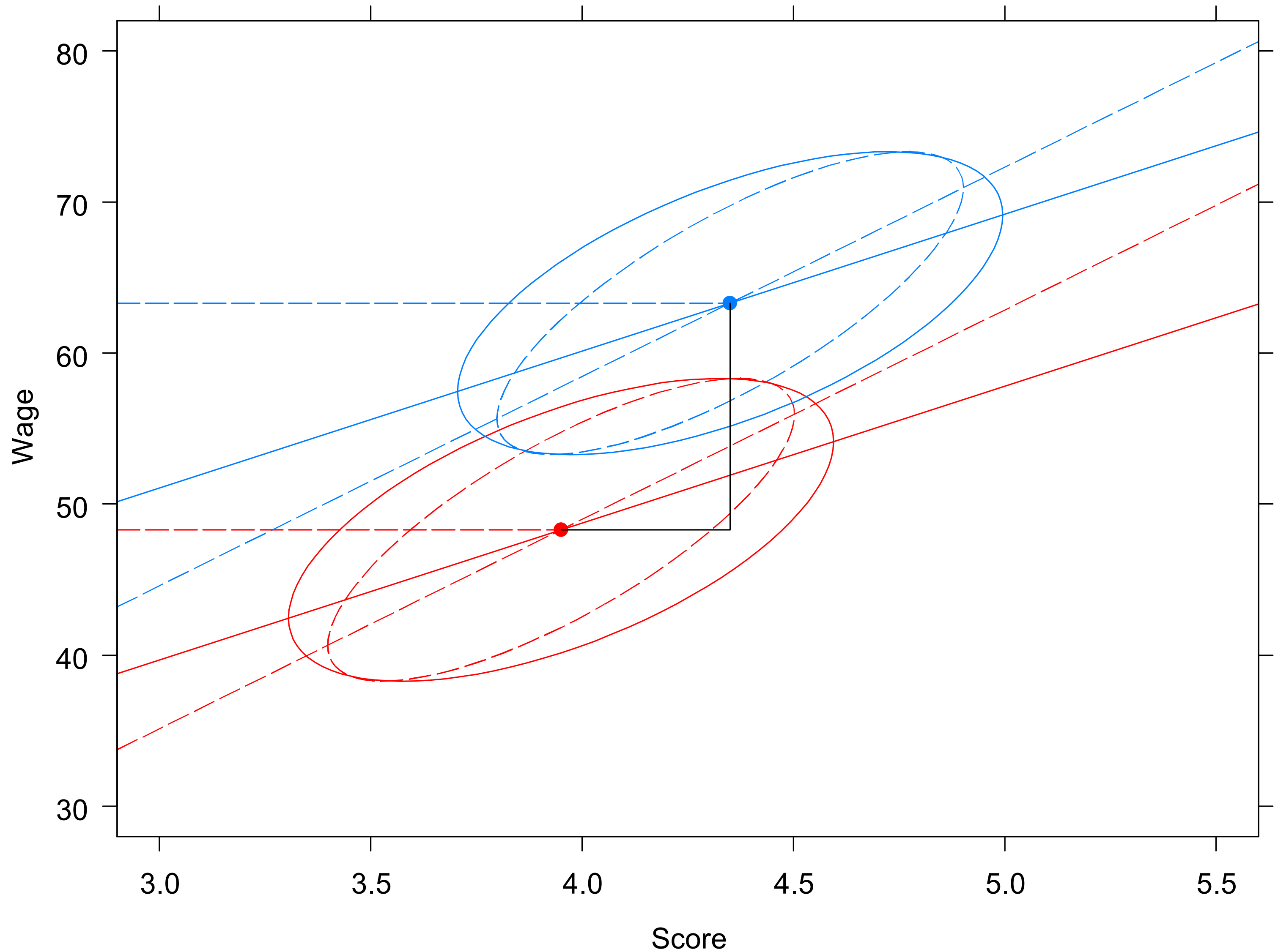

Raw
Gender Gap
Pure Gender Gap
Effect of OS
Occupational Segregation
Wage
Score
80
70
60
50
40
30
3.0
3.5
4.0
4.5
5.0
5.5

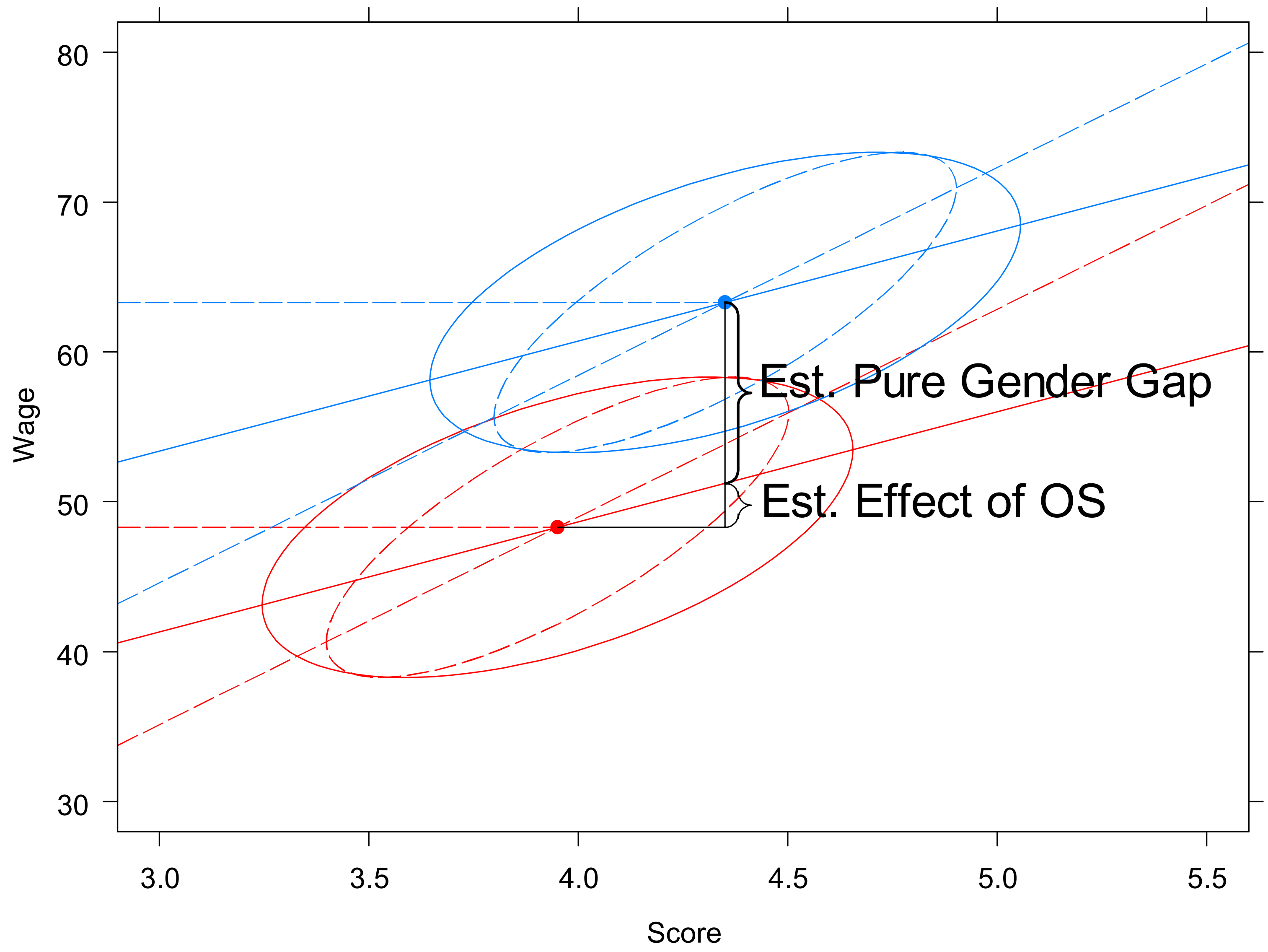

Wage
80
70
60
50
40
30
3.0
3.5
4.0
4.5
5.0
5.5
Score


80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score



80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score

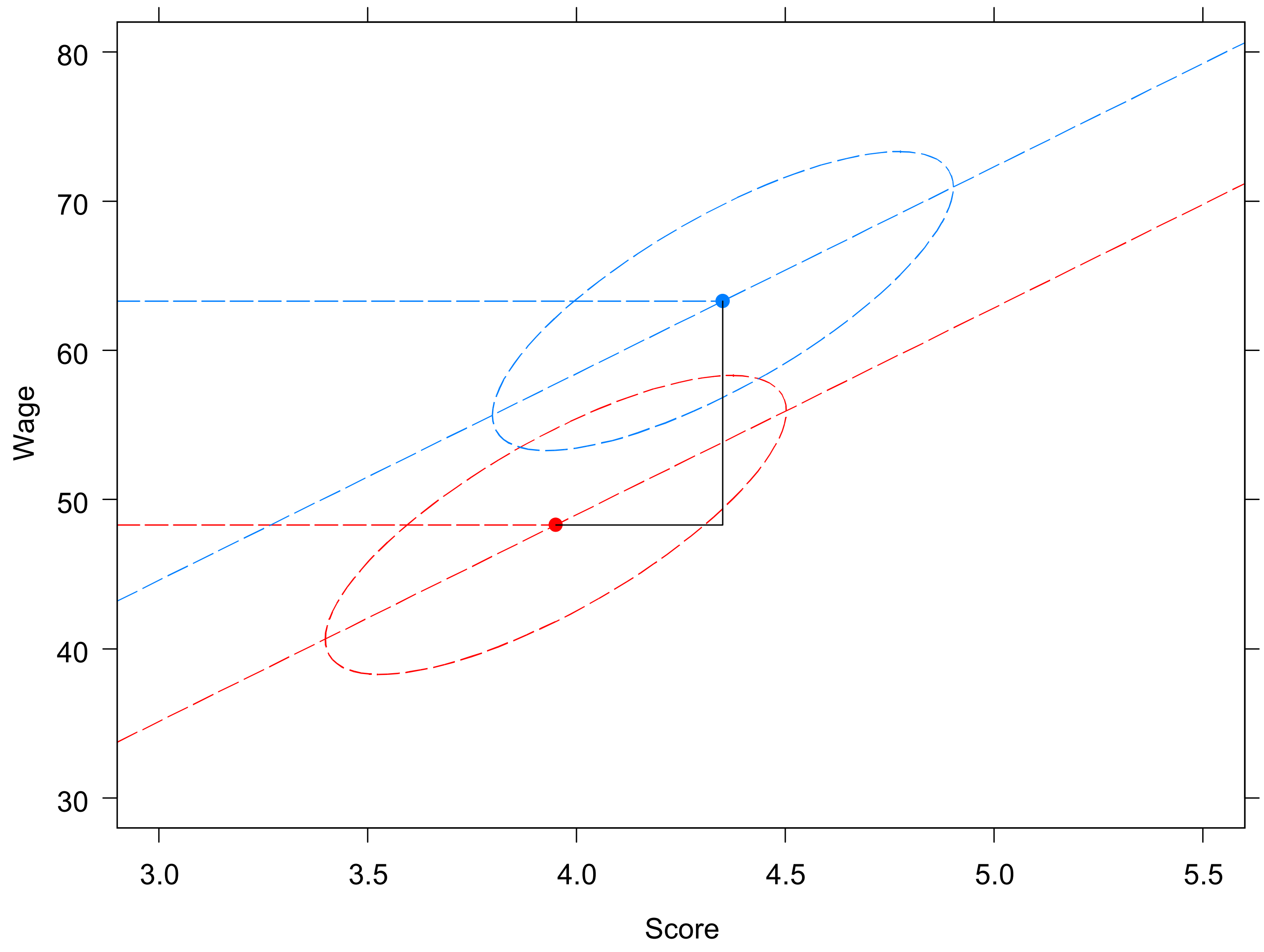

Wage
Score
80
70
60
50
40
30
3.0
3.5
4.0
4.5
5.0
5.5

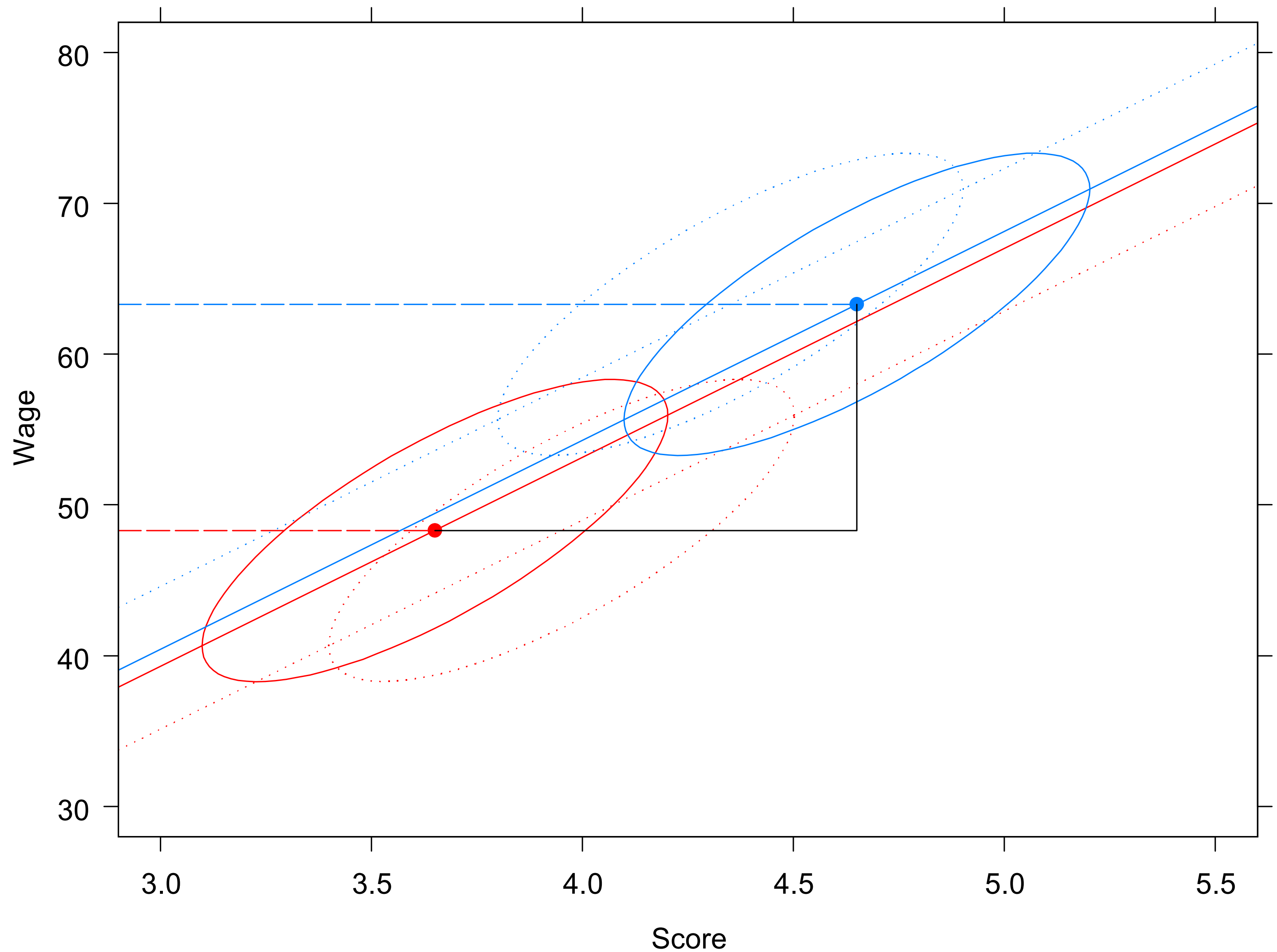

Est. Pure Gender Gap
Est. Effect of OS
Wage
Score
80
70
60
50
40
30
3.0
3.5
4.0
4.5
5.0
5.5

# 8 Bias in X: Job Evaluation Score

Measurement error leads to overestimation of gender gap

But bias might work in the opposite direction


Wage
80
70
60
50
40
30
3.0
3.5
4.0
4.5
5.0
5.5
Score


80
70
60
50
40
30
Wage
3.0
3.5
4.0
4.5
5.0
5.5
Score

# 9 Simpson's (Yule's) Paradox

Some types of association between variables:

- Marginal
- Conditional
- Partial
- Ecological

Paradoxes:

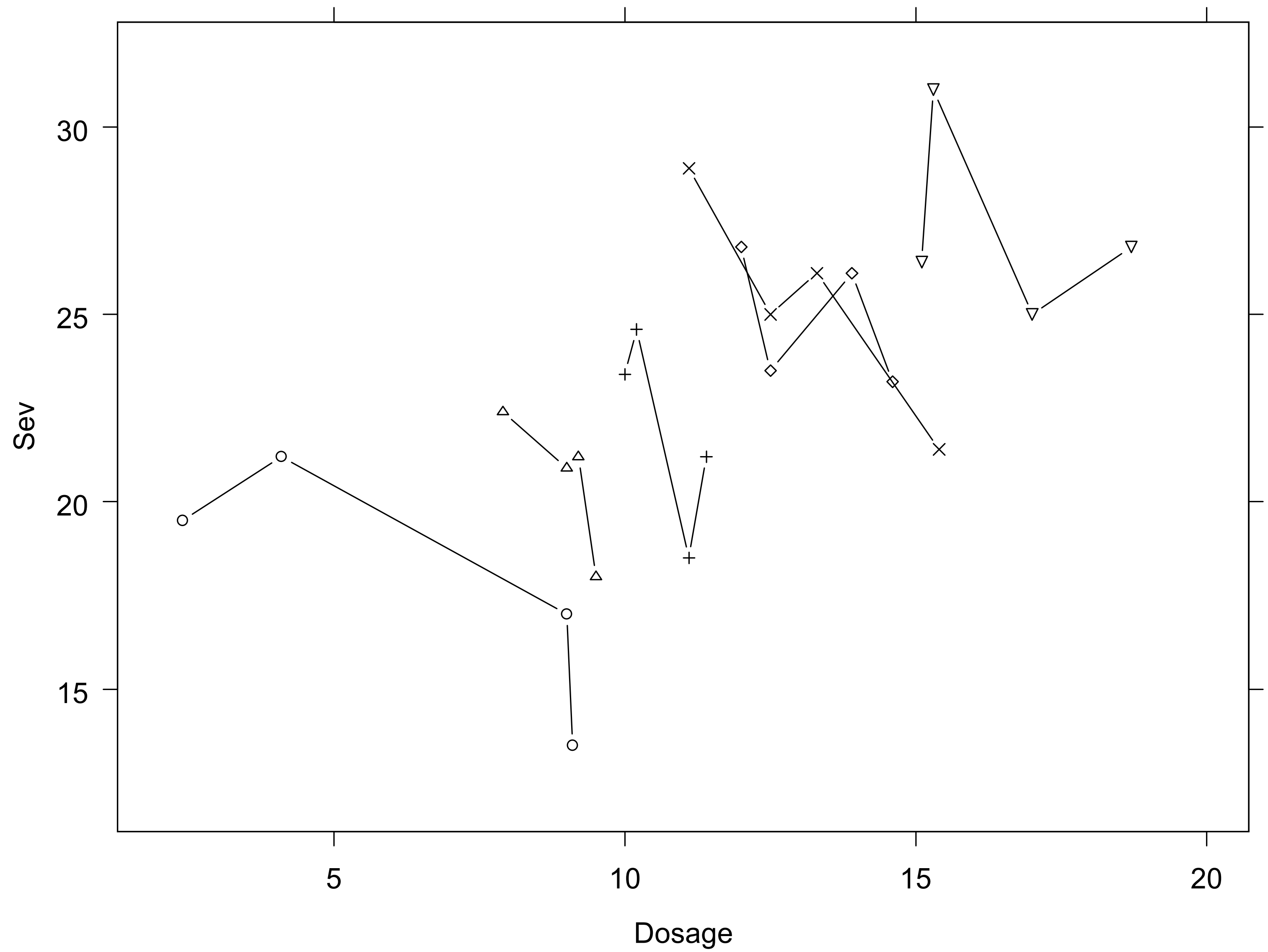
- Simpson's: Marginal and Conditional can have opposite signs
- Robinson's: Ecological and Conditional can have opposite signs

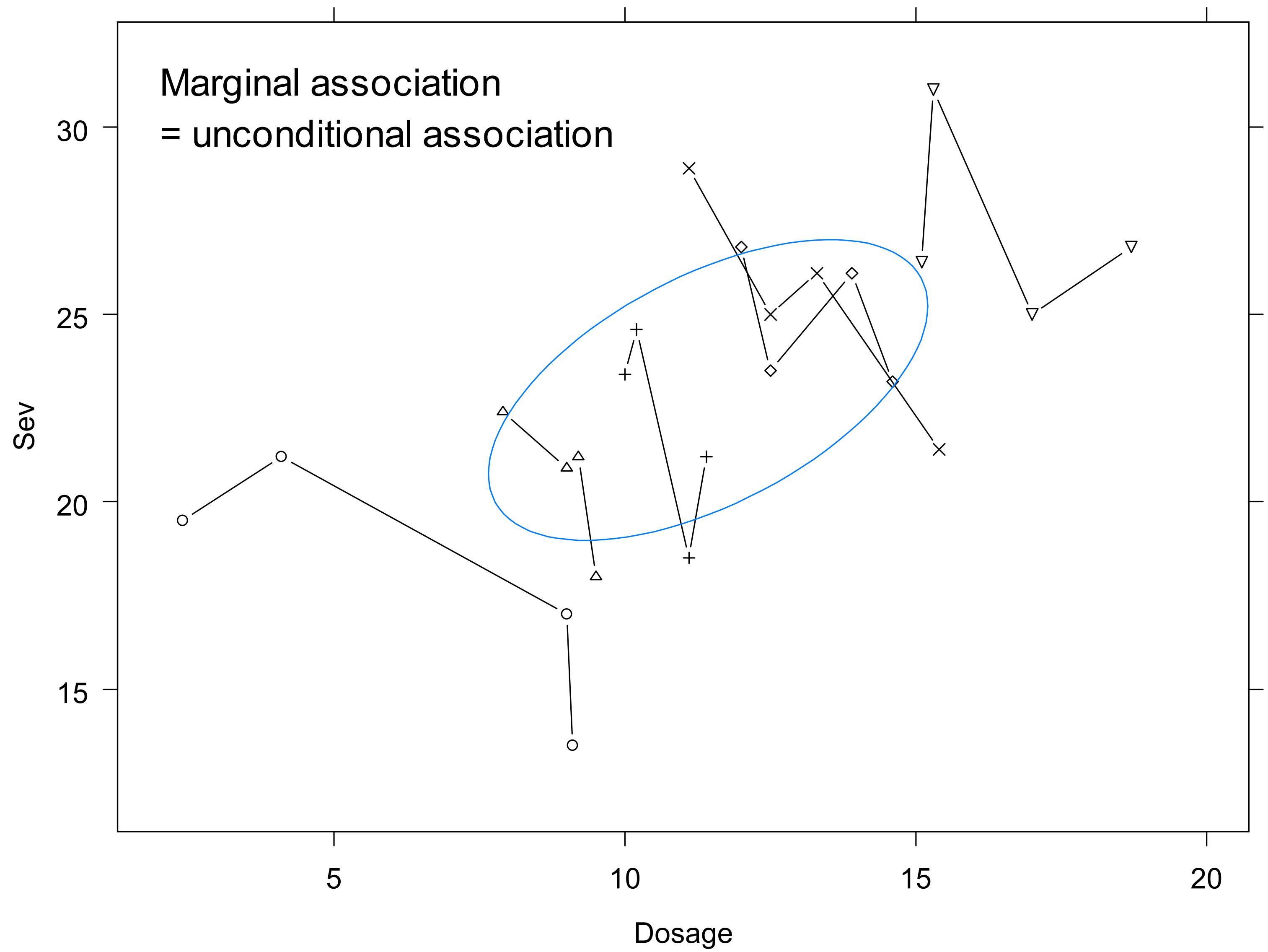
Example:

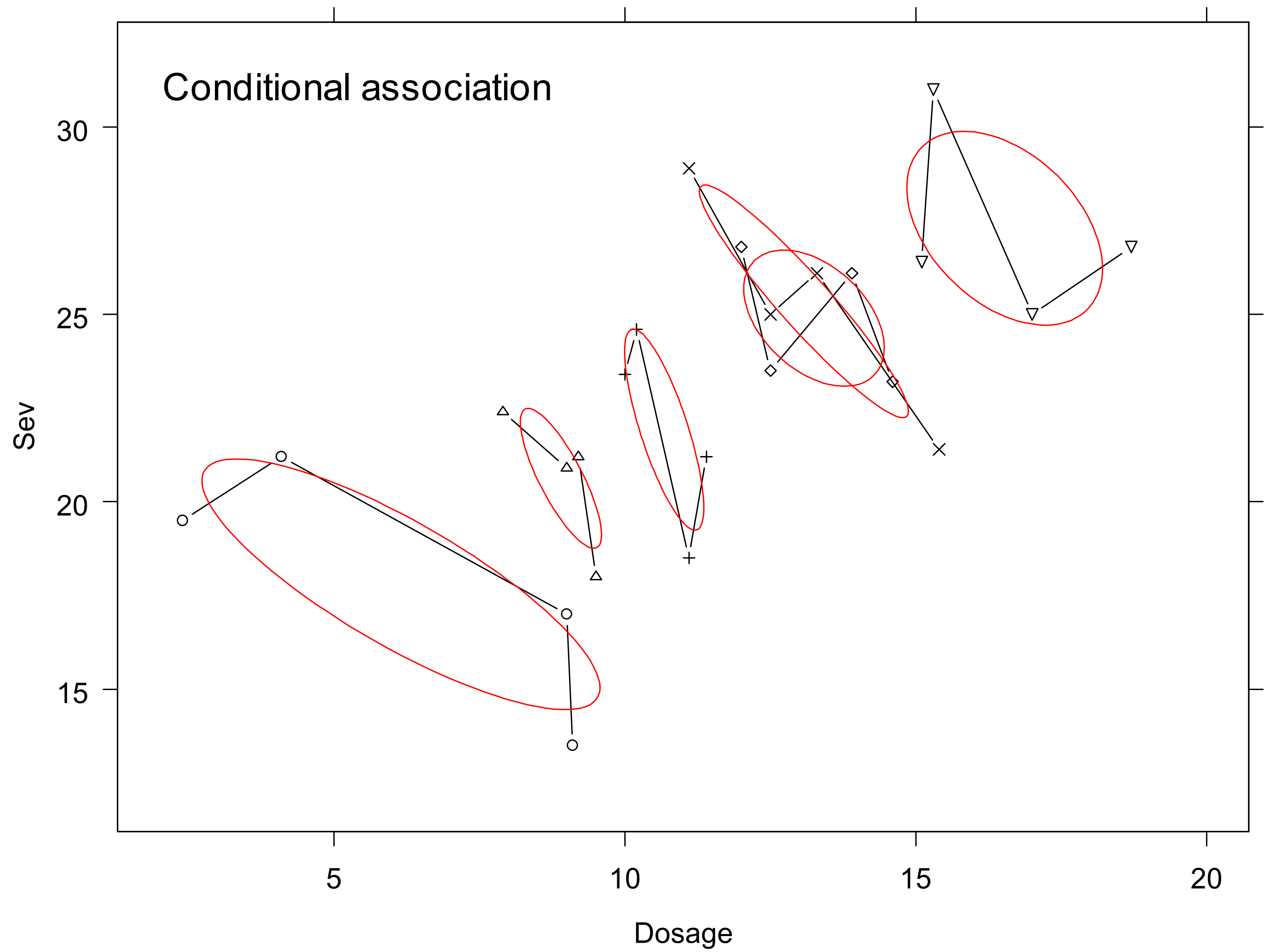
- Symptoms vs. Dosage of Drug: 6 patients at 4 dosages

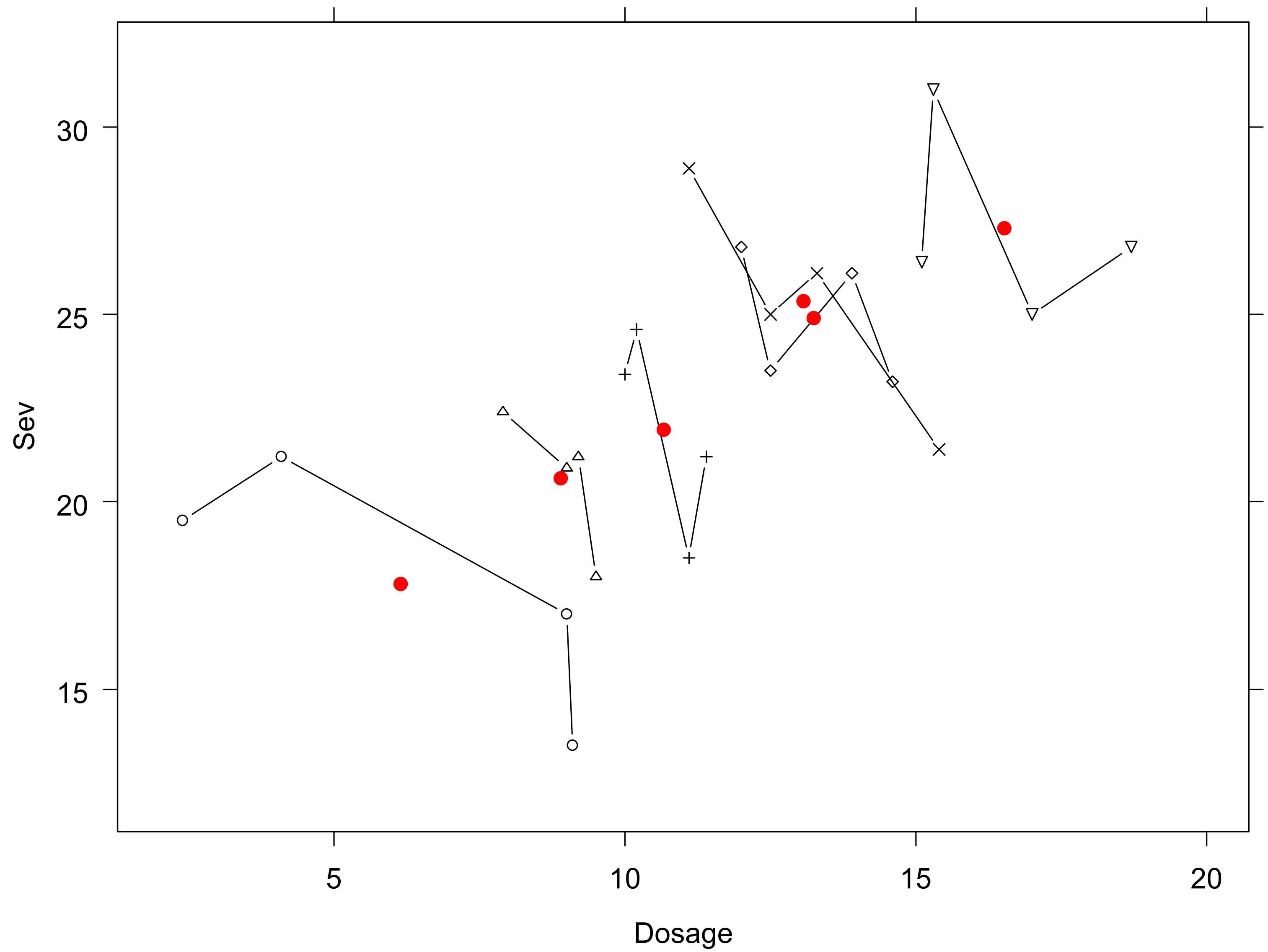
```
> Drug
   ID Dosage  Sev
 1  A   10.3 22.4
 2  A   10.6 22.8
 3  A   11.3 21.7
 4  A   11.3 20.9
 5  B   11.1 23.1
 6  B   11.3 22.7
 7  B   11.3 22.8
 8  B   11.4 22.0
 9  C   11.4 23.3
10  C   11.5 23.6
11  C   11.6 22.1
12  C   11.6 22.8
13  D   11.6 24.7
14  D   11.8 23.8
15  D   11.9 24.0
16  D   12.2 22.8
17  E   11.7 24.2
18  E   11.8 23.4
19  E   12.0 24.0
```

| | | | |
|---|---|---|---|
| 20 | E | 12.1 | 23.3 |
| 21 | F | 12.2 | 24.1 |
| 22 | F | 12.2 | 25.3 |
| 23 | F | 12.4 | 23.7 |
| 24 | F | 12.7 | 24.2 |

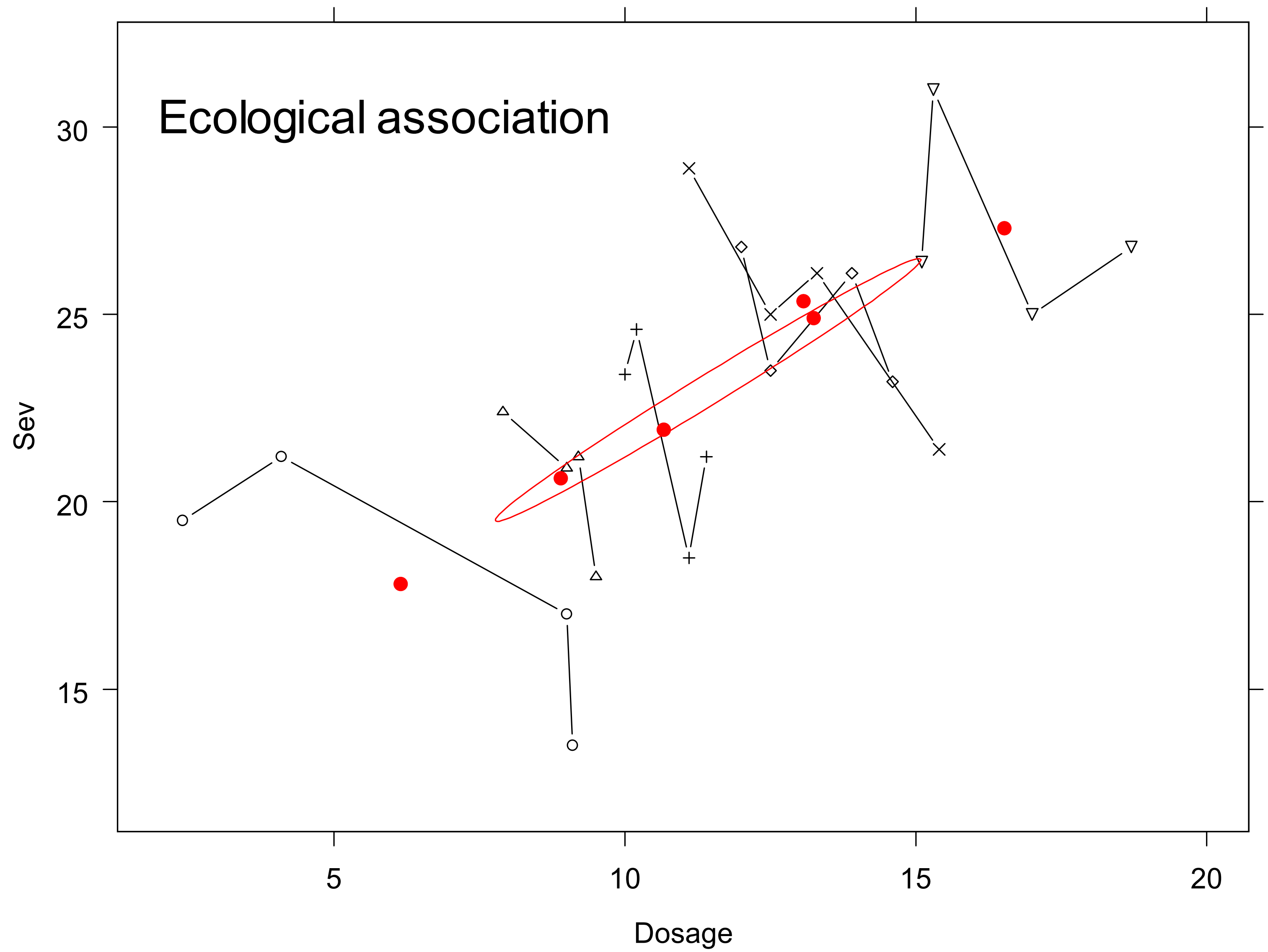

Sev
30
25
20
15
5
10
15
20
Dosage


Marginal association
= unconditional association
Sev
Dosage
30
25
20
15
5
10
15
20


Conditional association
Sev
Dosage
30
25
20
15
5
10
15
20



Sev
Dosage
30
25
20
15
5
10
15
20


Ecological association
Sev
Dosage
30
25
20
15
5
10
15
20

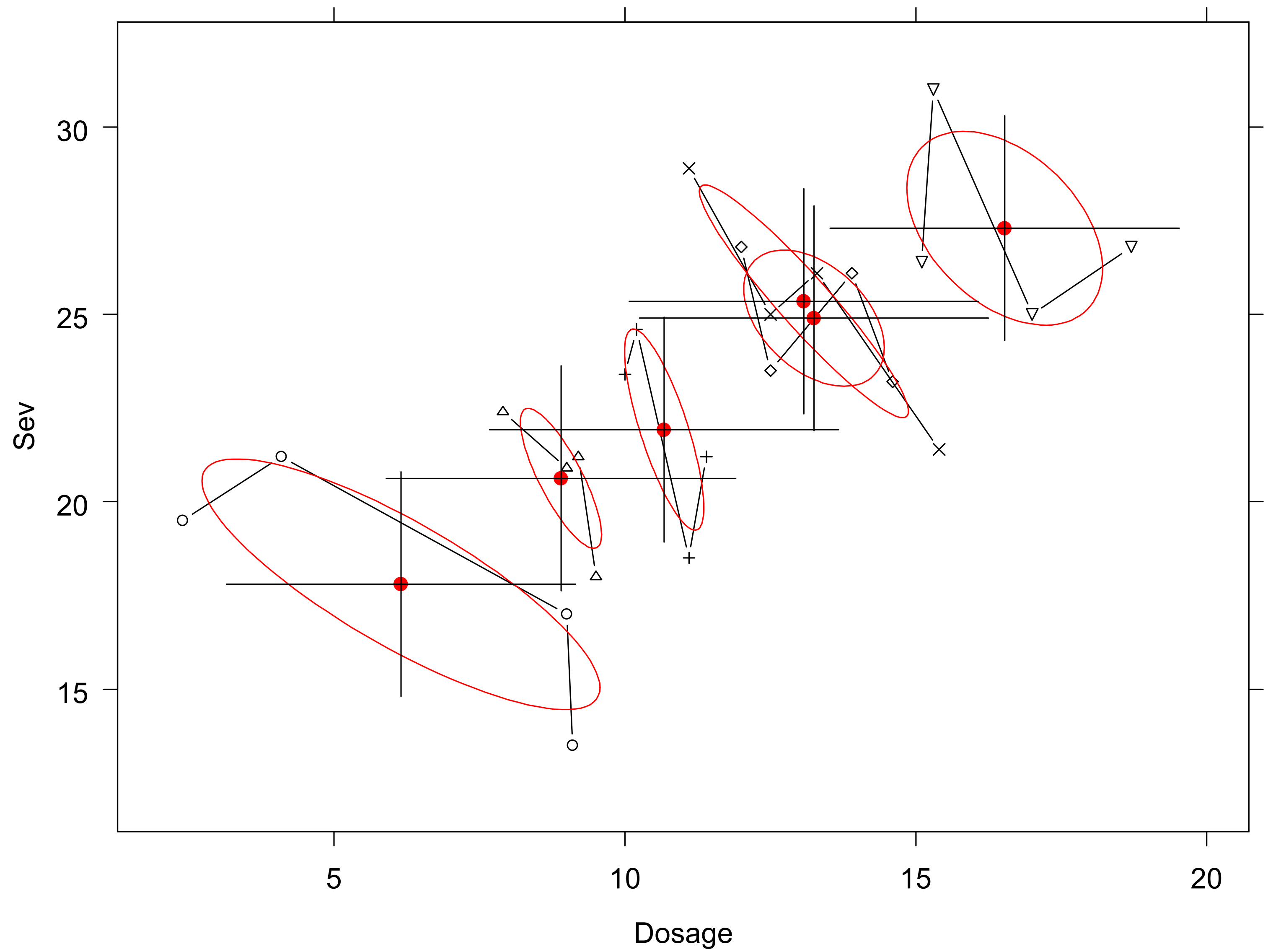
# Simple regression = marginal association

```
> fit <- lm(Sev ~Dosage, Drug)
> summary(fit)

Call: lm(formula = Sev ~Dosage, data = Drug)
Residuals:
    Min      1Q Median     3Q    Max
 -8.103 -1.688 0.5618 1.902 6.112

Coefficients:
              Value Std. Error t value Pr(>|t|)
(Intercept) 16.2110  2.2596     7.1744  0.0000
     Dosage  0.5925  0.1881     3.1500  0.0046

Residual standard error: 3.406 on 22 degrees of freedom
Multiple R-Squared: 0.3108
F-statistic: 9.922 on 1 and 22 degrees of freedom,
         the p-value is 0.004648
```

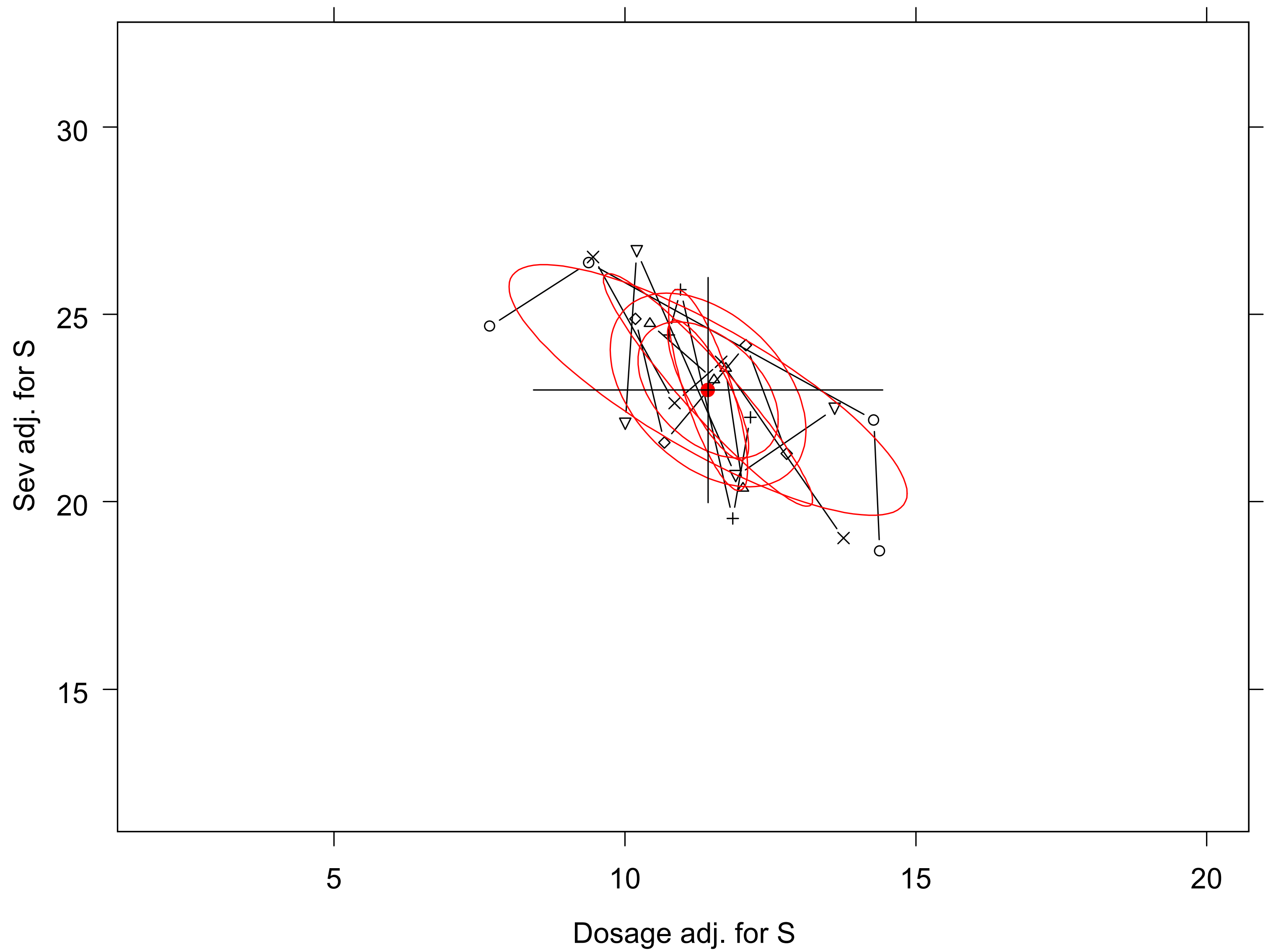
# Multiple regression = Analysis of covariance (with these data)

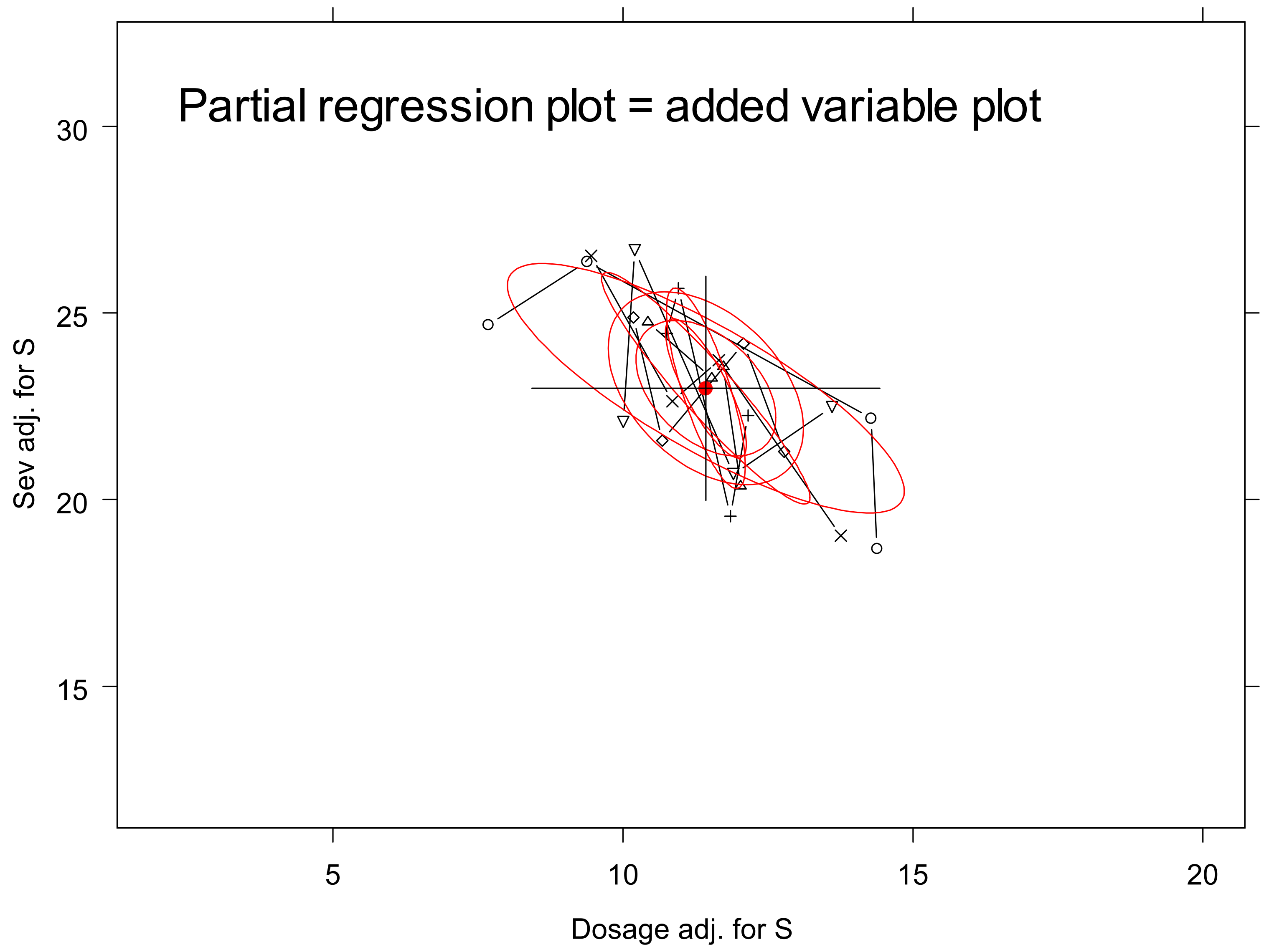
```
> fit <- lm(Sev ~Dosage + ID, Drug)
> summary(fit)

Call: lm(formula = Sev ~Dosage + ID, data = Drug)
Residuals:
    Min      1Q Median     3Q    Max
 -3.005 -1.696  0.519 1.429 2.489

Coefficients:
               Value Std. Error  t value Pr(>|t|)
(Intercept)  34.2840   2.9082    11.7886   0.0000
     Dosage  -0.9888   0.2520    -3.9230   0.0011
        ID1   2.7720   0.7748     3.5776   0.0023
        ID2   1.9424   0.4797     4.0490   0.0008
        ID3   2.4207   0.4005     6.0434   0.0000
```

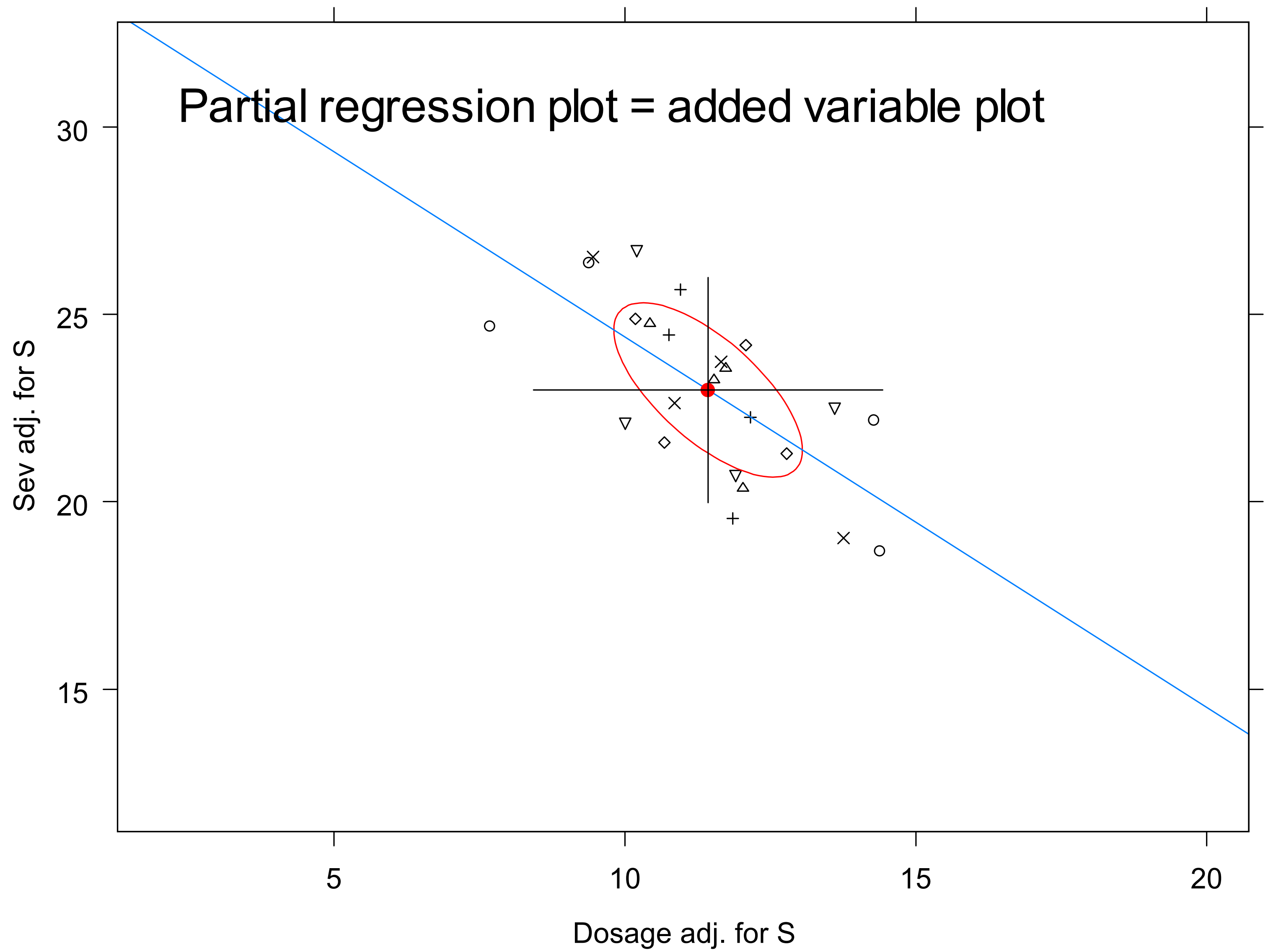
```
      ID4   1.3970   0.2829     4.9377   0.0001
      ID5   1.8710   0.3130     5.9769   0.0000

Residual standard error: 1.96 on 17 degrees of freedom
Multiple R-Squared: 0.8236
F-statistic: 13.23 on 6 and 17 degrees of freedom,
             the p-value is 0.00001393
```



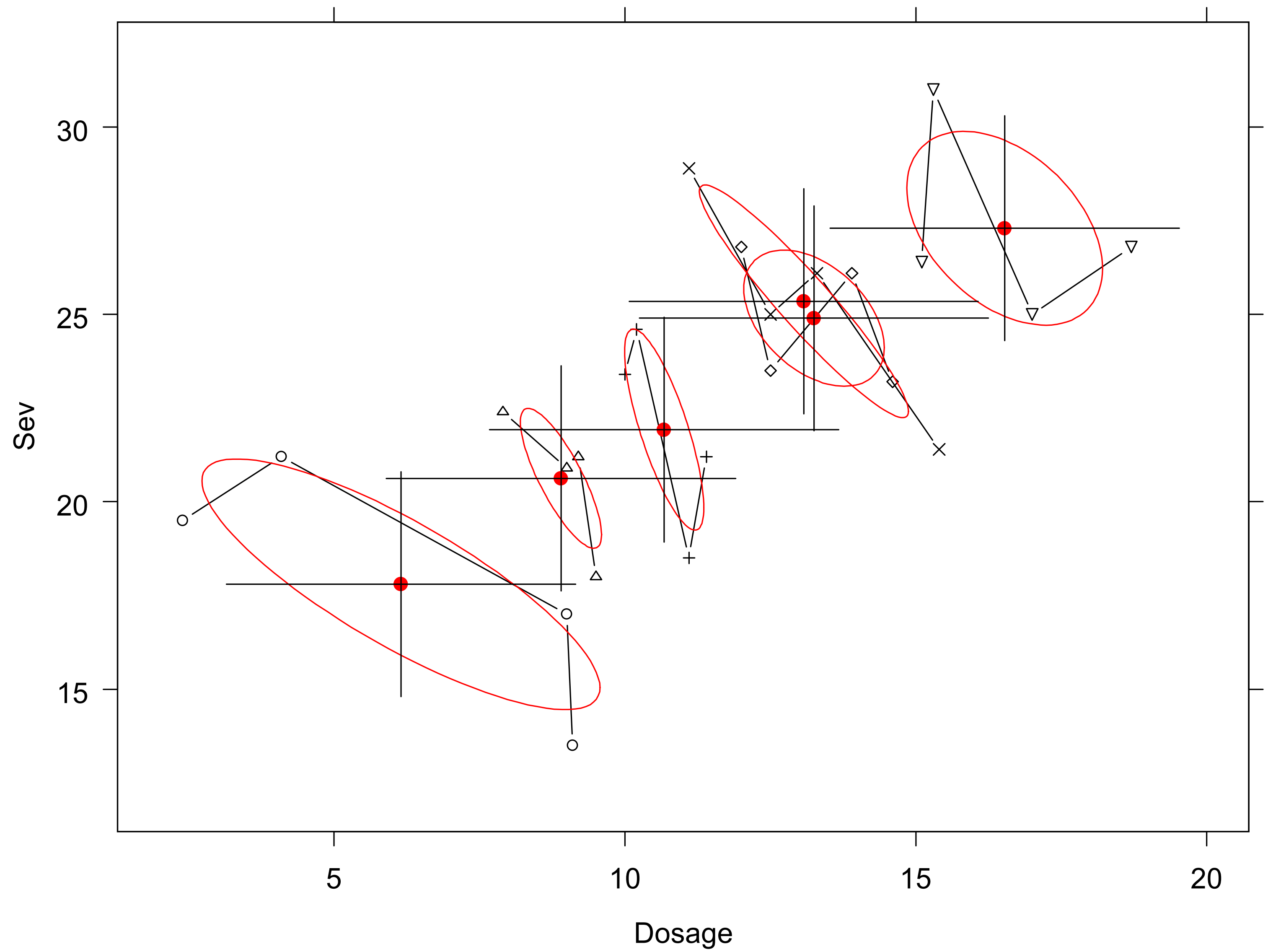
30
25
20
15
Sev
5
10
15
20
Dosage

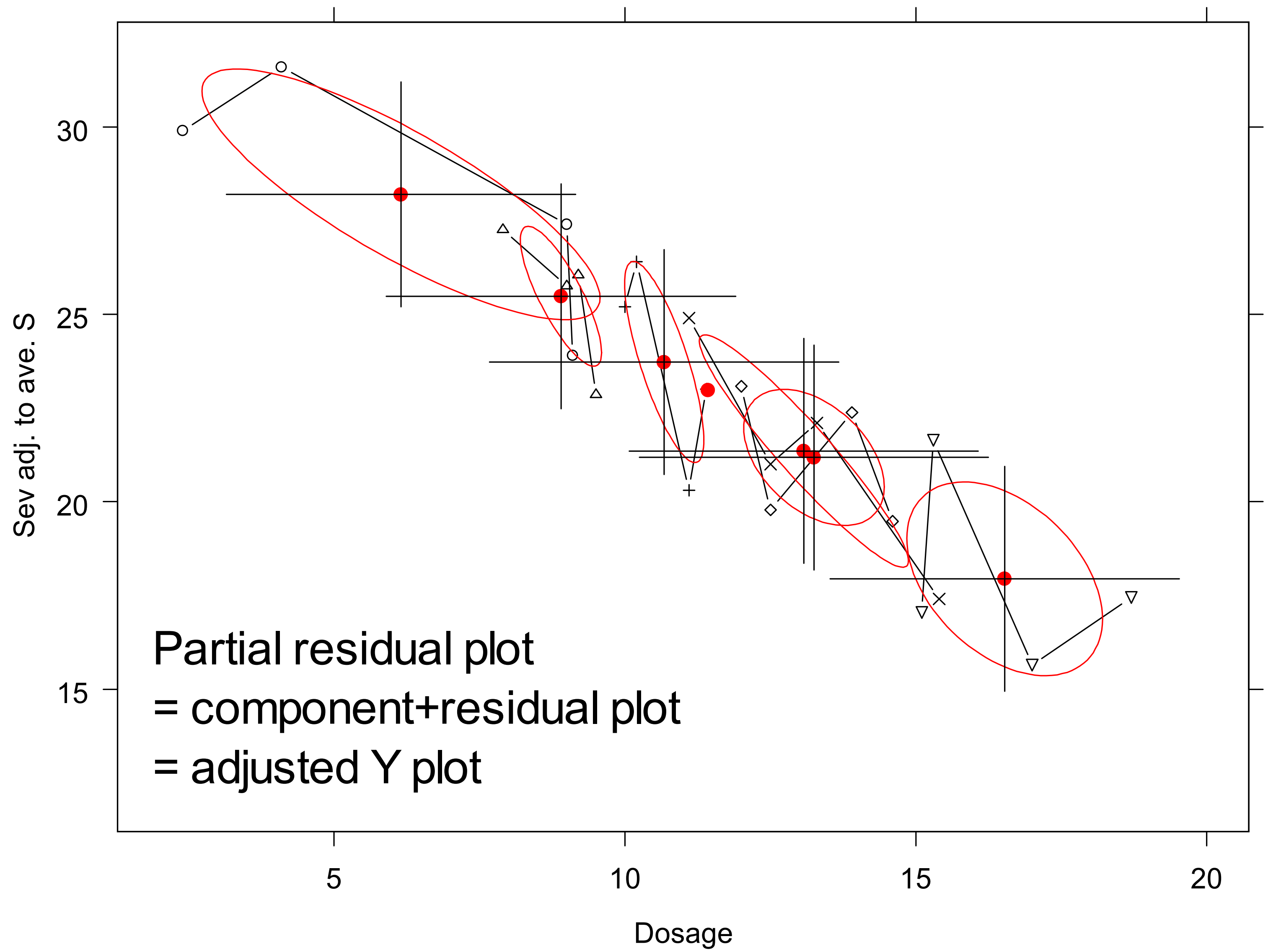

30
25
20
15
5
10
15
20
Sev adj. for S
Dosage adj. for S

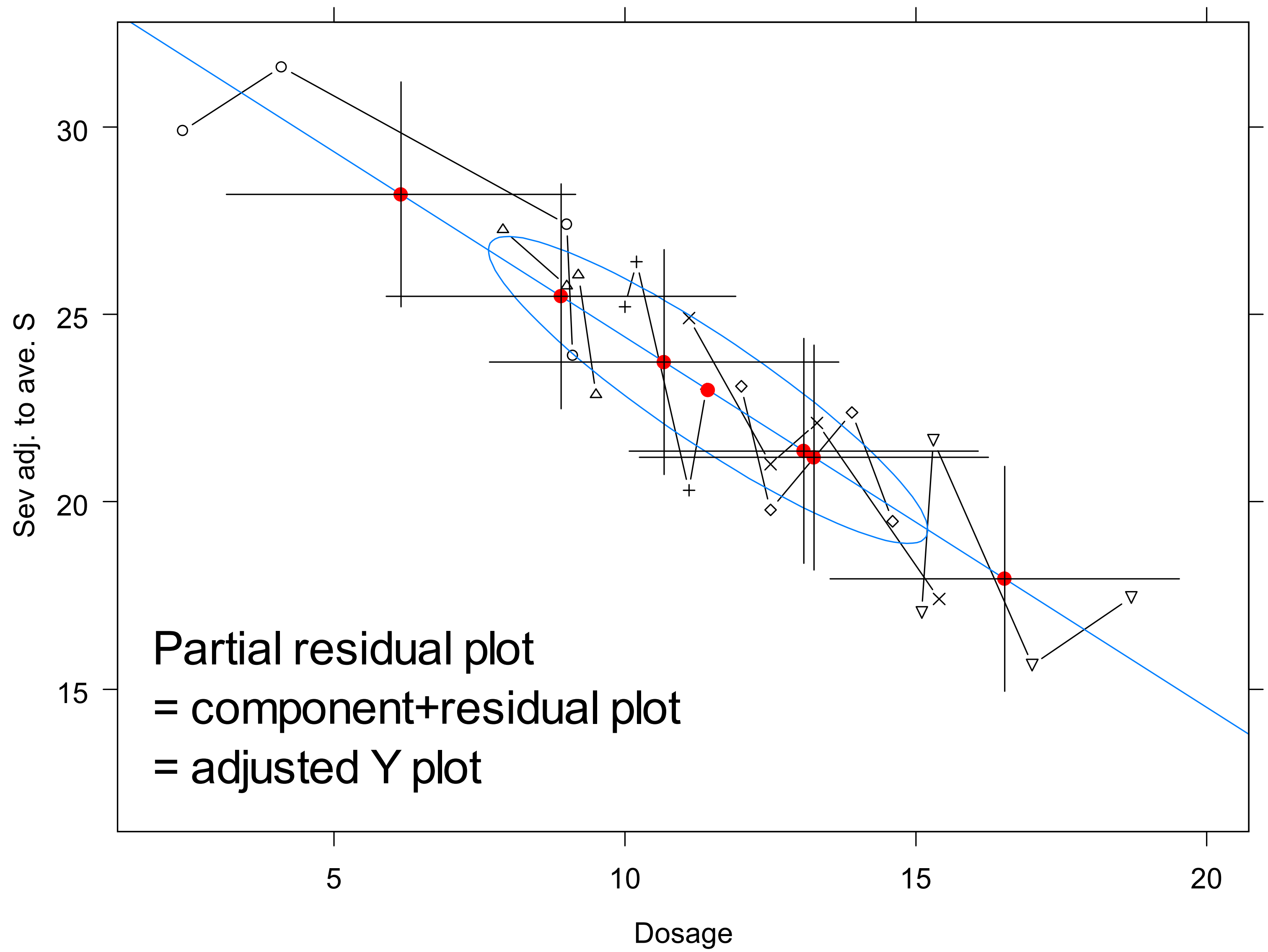

Partial regression plot = added variable plot
Sev adj. for S
Dosage adj. for S
30
25
20
15
5
10
15
20



Partial regression plot = added variable plot
30
25
20
15
5
10
15
20
Sev adj. for S
Dosage adj. for S


30
25
20
15
Sev
5
10
15
20
Dosage


Partial residual plot
= component+residual plot
= adjusted Y plot
Sev adj. to ave. S
Dosage
30
25
20
15
5
10
15
20


Partial residual plot
= component+residual plot
= adjusted Y plot
Sev adj. to ave. S
Dosage
30
25
20
15
5
10
15
20

# 10 Added variable plot

Ideal tool for statistical sleuthing

## 10.1

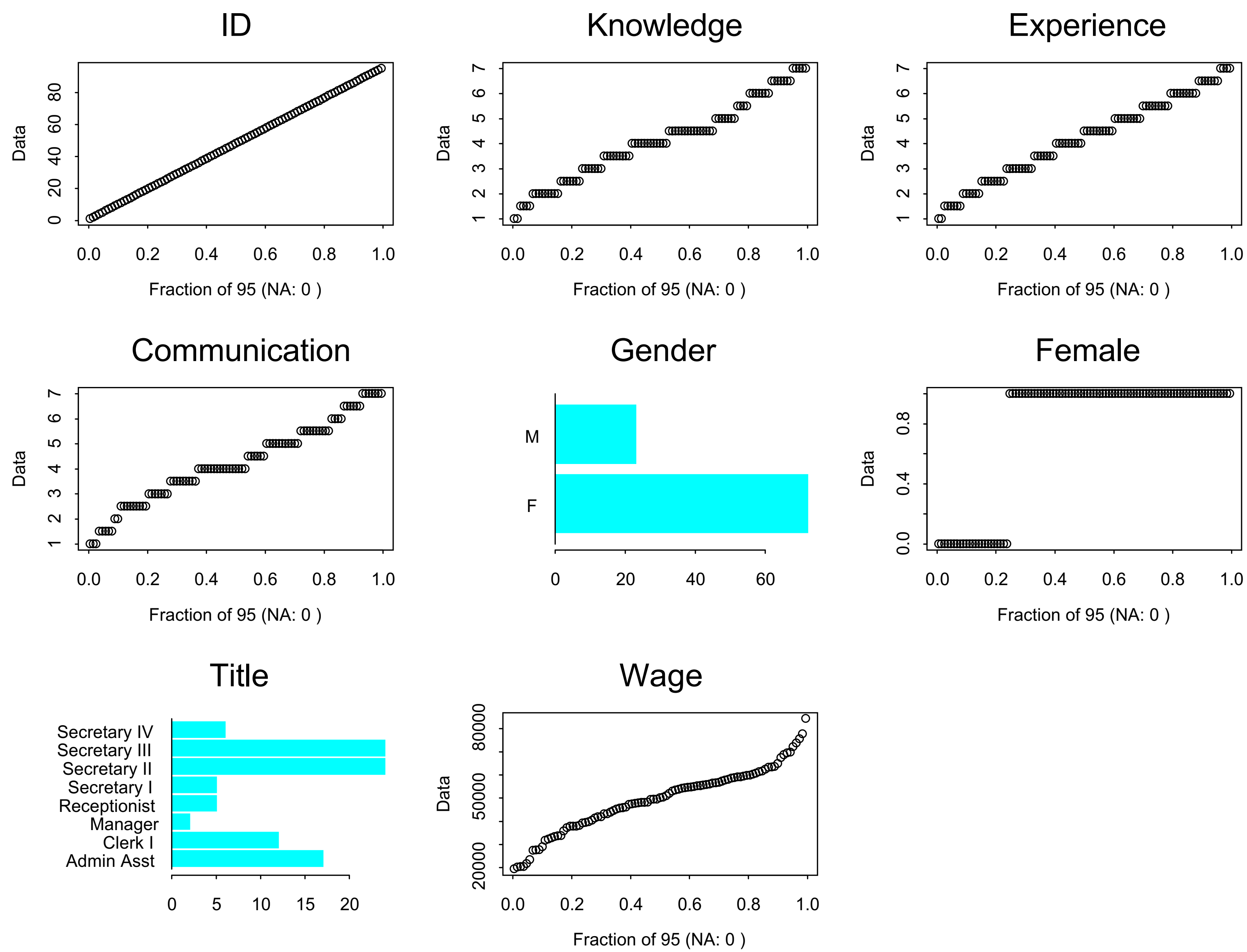
# Pay equity in a large law firm

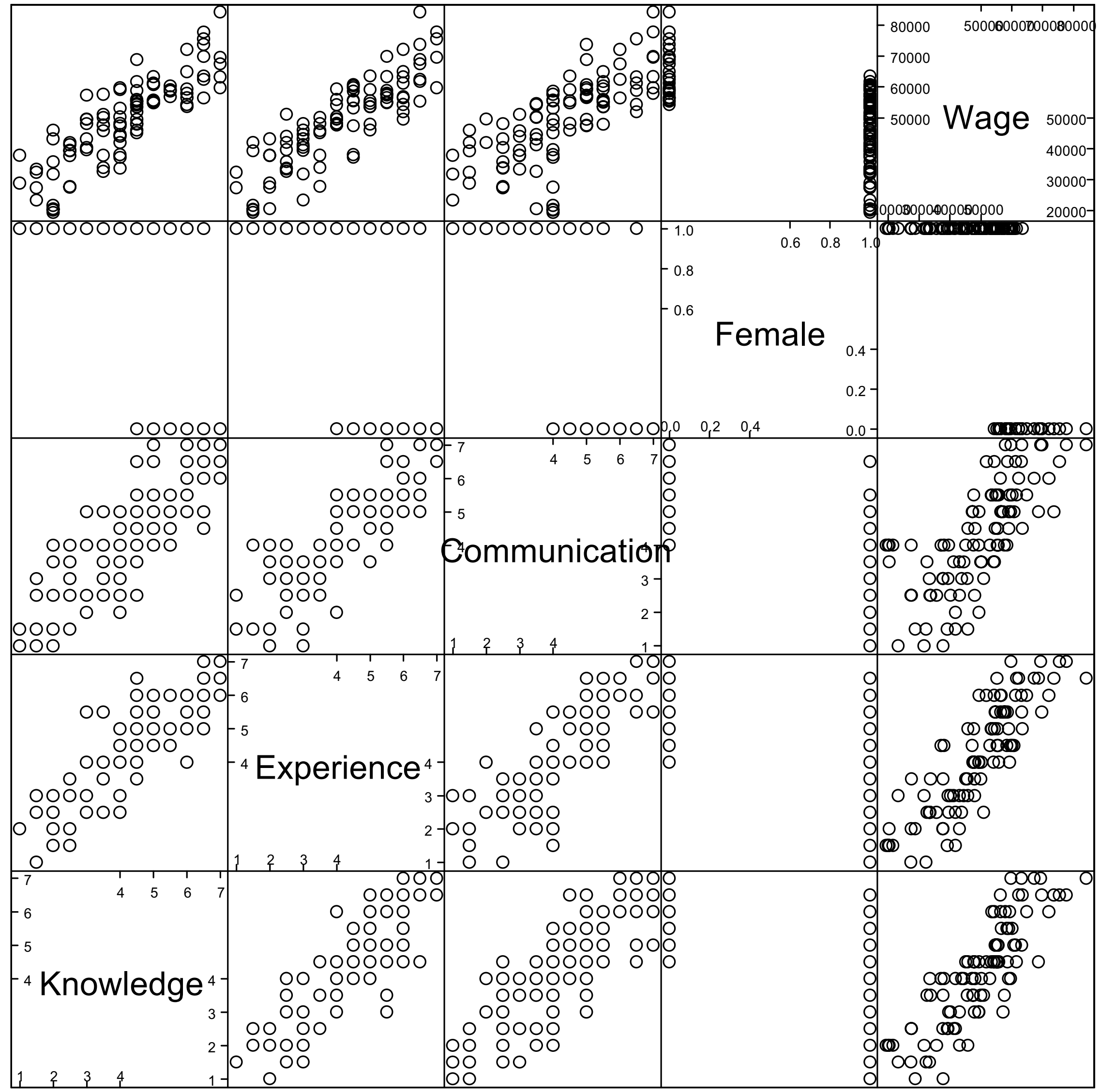
| | Title | Knowledge | Experience | Communication | Gender |
|---|---|---|---|---|---|
| 58 | Secretary III | 5.0 | 4.5 | 4.0 | F |
| 70 | Admin Asst | 6.5 | 5.5 | 7.0 | M |
| 36 | Secretary II | 3.5 | 3.5 | 2.5 | F |
| 74 | Secretary III | 4.5 | 5.0 | 3.5 | F |
| 78 | Secretary III | 3.0 | 5.5 | 5.0 | F |
| 25 | Secretary II | 3.0 | 3.0 | 2.5 | F |
| 19 | Admin Asst | 7.0 | 7.0 | 7.0 | M |
| 42 | Admin Asst | 5.0 | 6.0 | 6.5 | M |
| 87 | Clerk I | 1.5 | 1.0 | 2.5 | F |
| 32 | Secretary I | 2.5 | 3.5 | 2.5 | F |
| 66 | Admin Asst | 4.5 | 6.0 | 6.5 | F |
| 47 | Admin Asst | 7.0 | 7.0 | 7.0 | M |
| 34 | Secretary III | 4.5 | 6.5 | 5.5 | F |
| 69 | Secretary II | 3.0 | 4.0 | 4.0 | F |
| 21 | Secretary III | 5.0 | 5.0 | 4.5 | F |
| 52 | Secretary II | 3.5 | 3.5 | 3.0 | F |
| 95 | Receptionist | 2.5 | 2.0 | 4.0 | F |

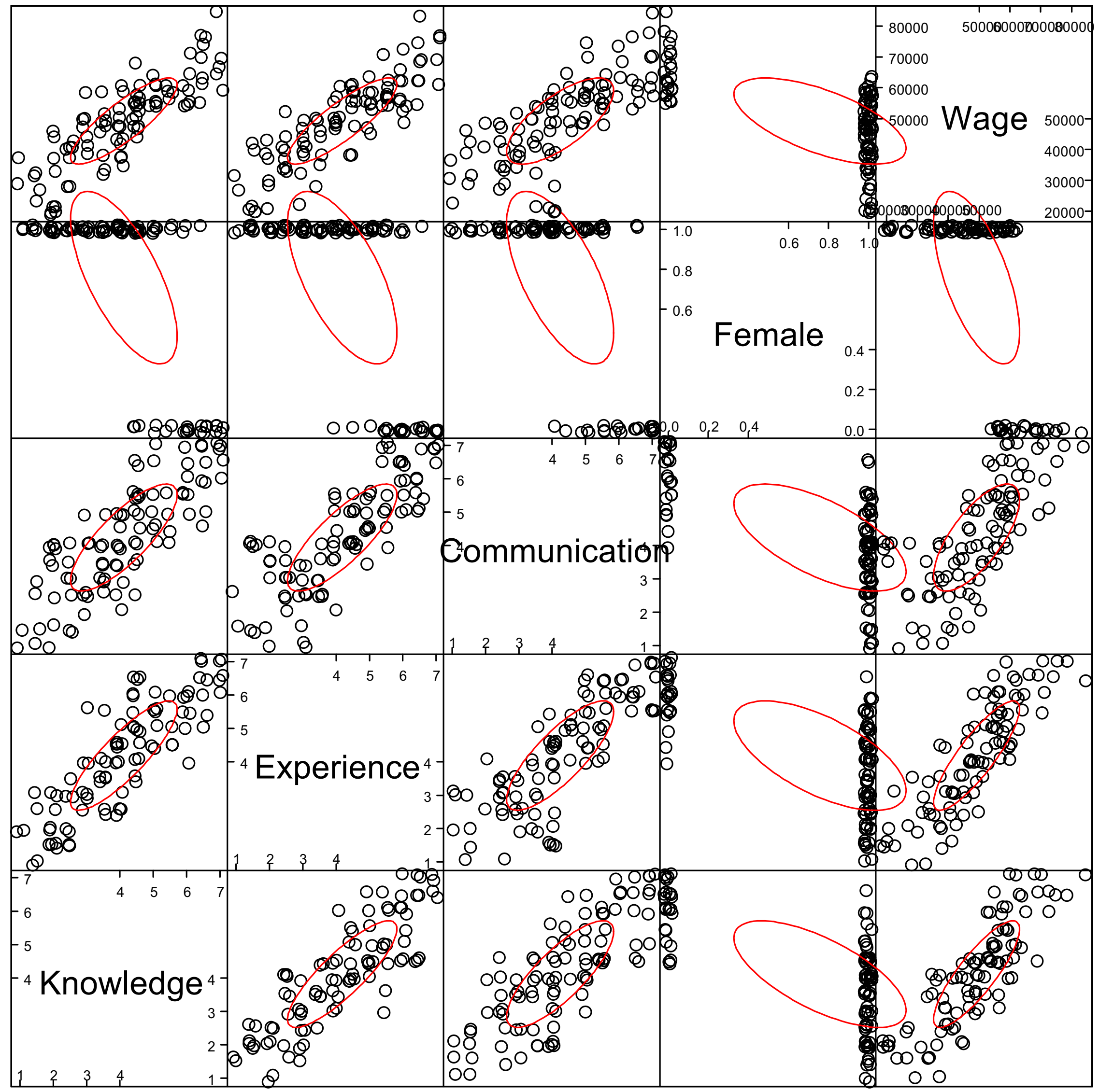
| | | | | | |
|---|---|---|---|---|---|
| 89 | Admin Asst | 5.0 | 5.5 | 7.0 | M |
| ... | 95 positions | | | | |

# Summary

```
> summary(Law)
        ID           Knowledge        Experience     Communication
    Min.: 1.0       Min.:1.000       Min.:1.000       Min.:1.000
 1st Qu.:24.5    1st Qu.:3.000    1st Qu.:3.000    1st Qu.:3.000
  Median:48.0     Median:4.000     Median:4.500     Median:4.000
    Mean:48.0       Mean:4.116       Mean:4.184       Mean:4.237
 3rd Qu.:71.5    3rd Qu.:5.000    3rd Qu.:5.500    3rd Qu.:5.500
    Max.:95.0       Max.:7.000       Max.:7.000       Max.:7.000

 Gender          Female                      Title            Wage
 F:72         Min.:0.0000    Secretary III:24       Min.:19350
 M:23      1st Qu.:1.0000     Secretary II:24    1st Qu.:39580
            Median:1.0000       Admin Asst:17     Median:50010
              Mean:0.7579          Clerk I:12       Mean:49180
           3rd Qu.:1.0000     Secretary IV: 6    3rd Qu.:58650
              Max.:1.0000      Secretary I: 5       Max.:84190
                                   (Other): 7
```


ID
Knowledge
Experience
Communication
Gender
Female
Title
Wage
Data
Fraction of 95 (NA: 0 )
M
F
Secretary IV
Secretary III
Secretary II
Secretary I
Receptionist
Manager
Clerk I
Admin Asst

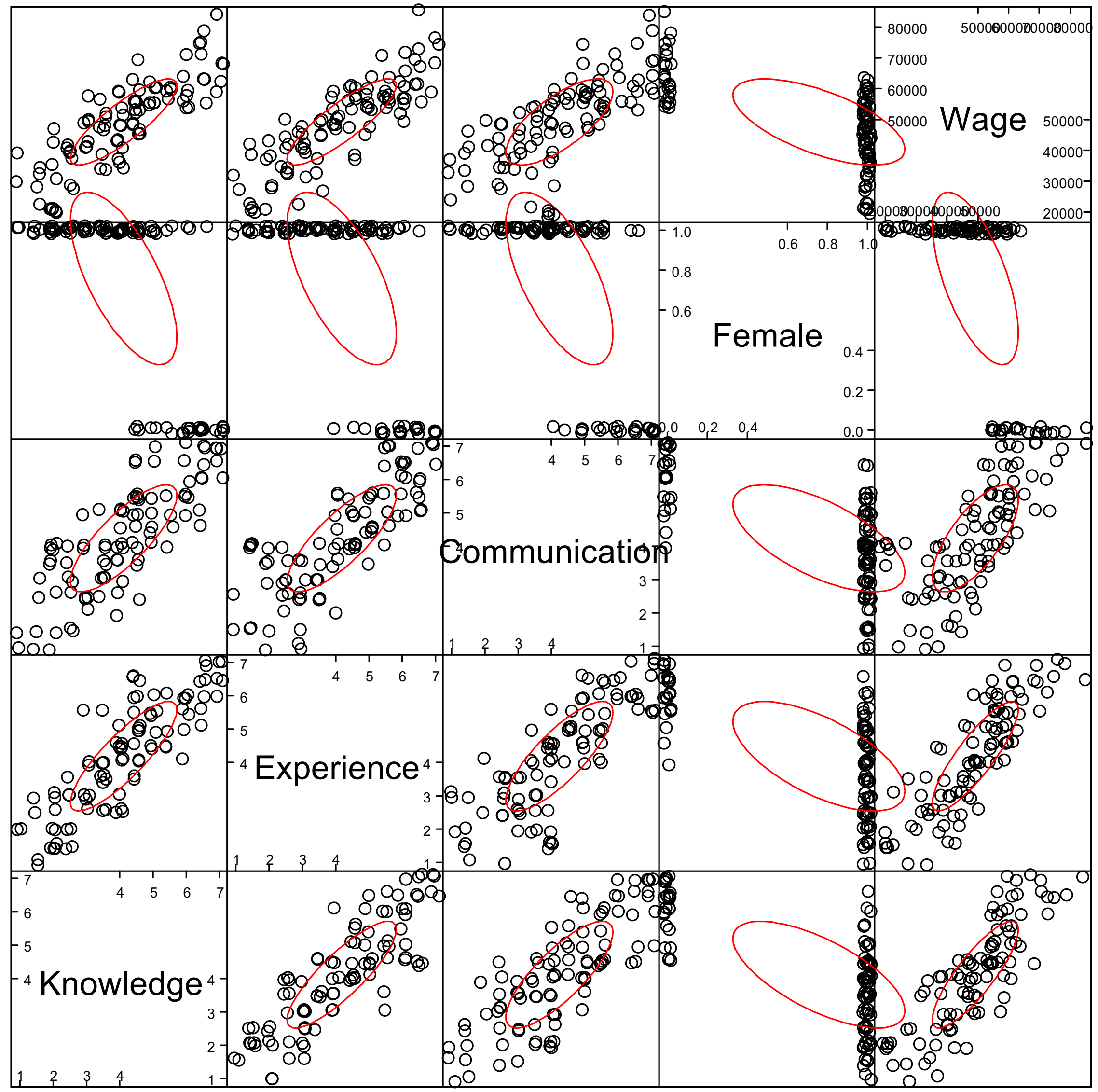

Wage
Female
Communication
Experience
Knowledge

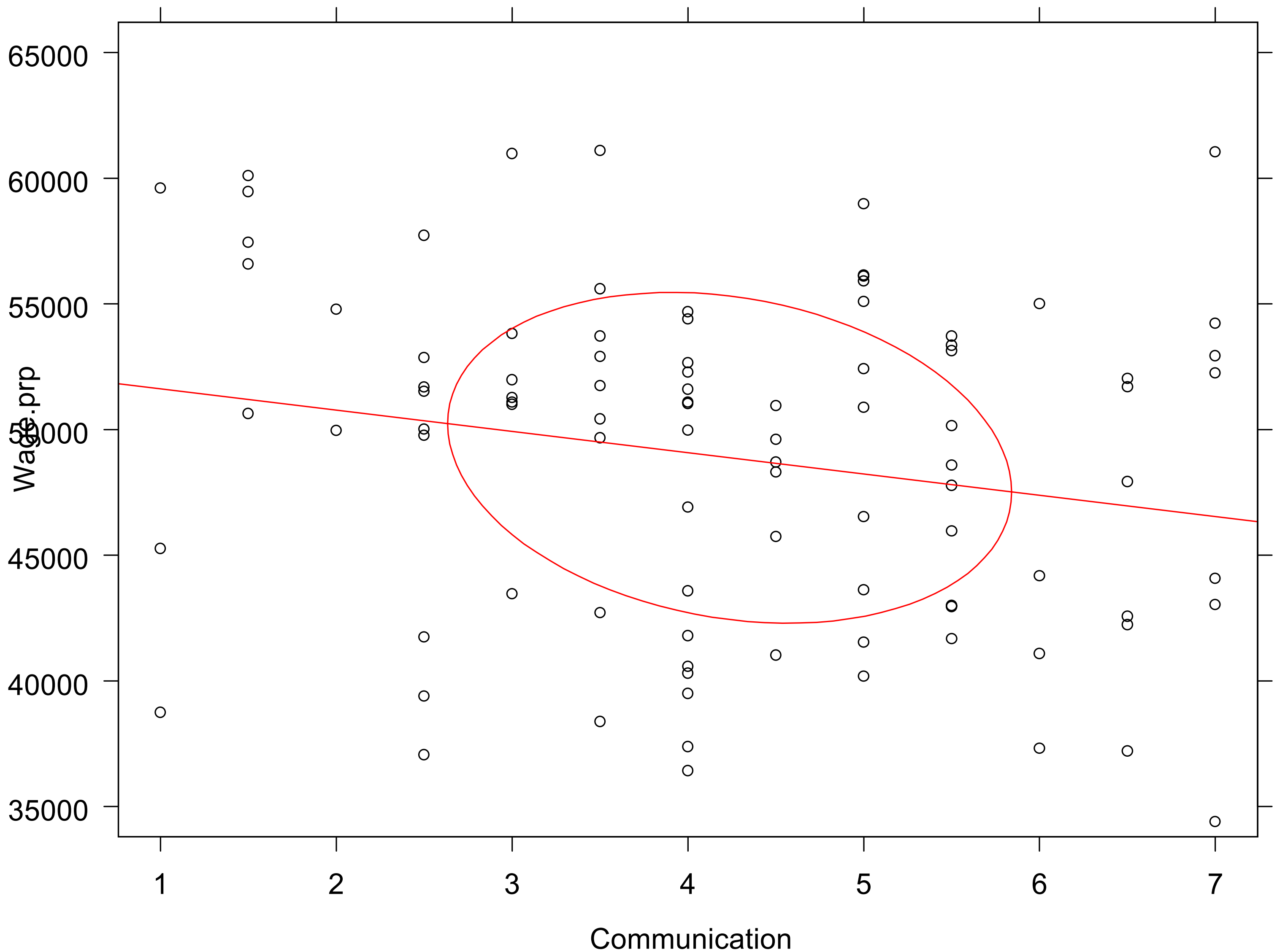

Wage
Female
Communication
Experience
Knowledge

# Regression output:

```
> fit <- lm(Wage ~ Knowledge + Experience
                + Communication + Female, data = Law)
> summary(fit)

Call: lm(formula = Wage ~ Knowledge + Experience
              + Communication + Female, data = Law)
Residuals:
    Min     1Q Median   3Q   Max
 -13279 -4824  972.8 5190 14498

Coefficients:
                    Value  Std. Error    t value    Pr(>|t|)
  (Intercept)  20153.7248   4031.6651     4.9989      0.0000
    Knowledge   3825.2482    899.8848     4.2508      0.0001
   Experience   4414.0819    854.7458     5.1642      0.0000
Communication   -846.8698    777.7750    -1.0888      0.2791
       Female  -2113.9833   2206.0361    -0.9583      0.3405

Residual standard error: 6577 on 90 degrees of freedom
```

```
Multiple R-Squared: 0.7864
F-statistic: 82.86 on 4 and 90 degrees of freedom,
the p-value is 0
```


Wage
Female
Communication
Experience
Knowledge


65000
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35000
Wage.prp
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2
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6
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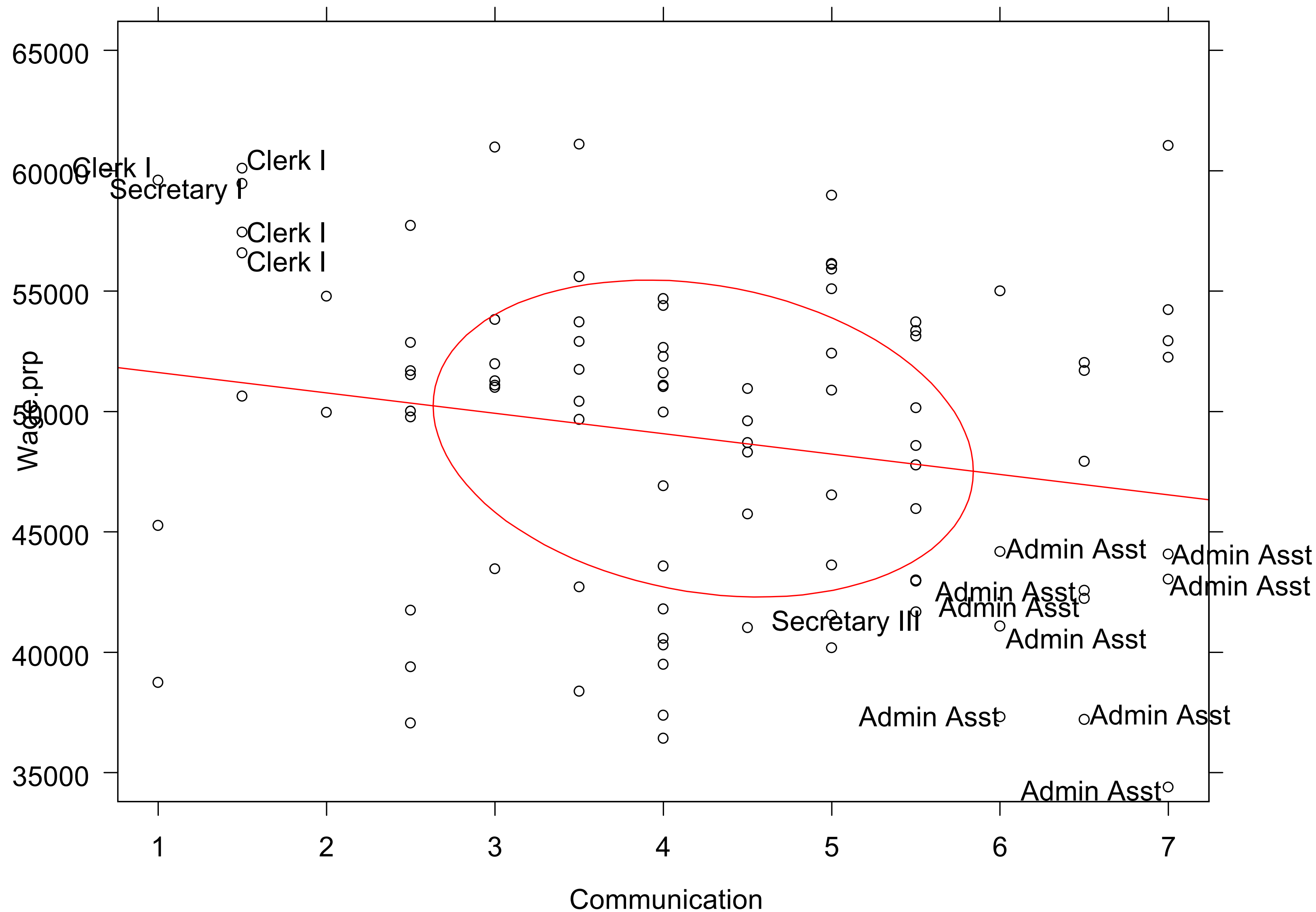
Communication


Clerk I
Clerk I
Secretary I
Clerk I
Clerk I
Admin Asst
Admin Asst
Admin Asst
Admin Asst
Admin Asst
Secretary III
Admin Asst
Admin Asst
Admin Asst
Admin Asst
65000
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Wage.prp
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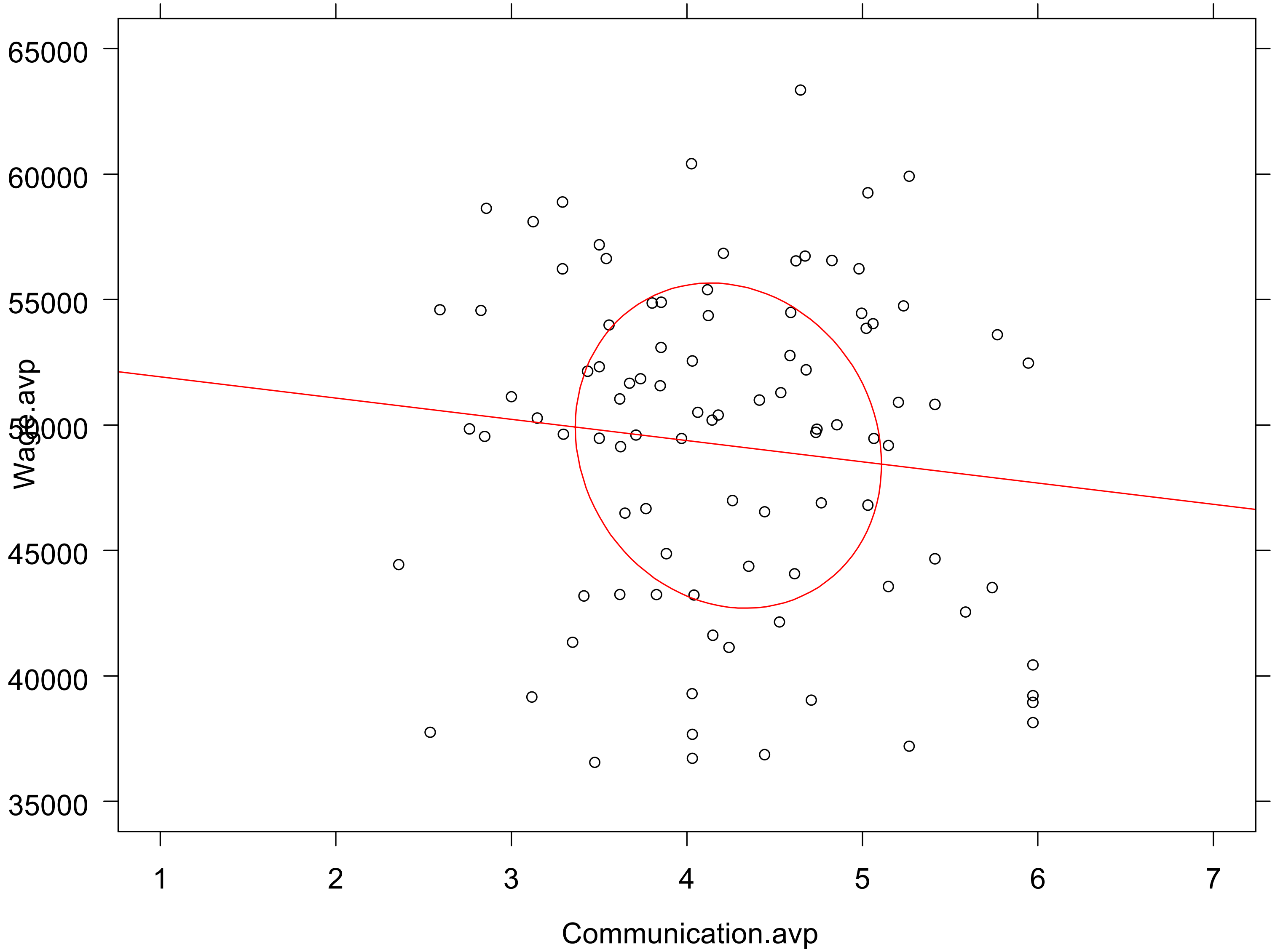
Communication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Wage.avp
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Communication.avp

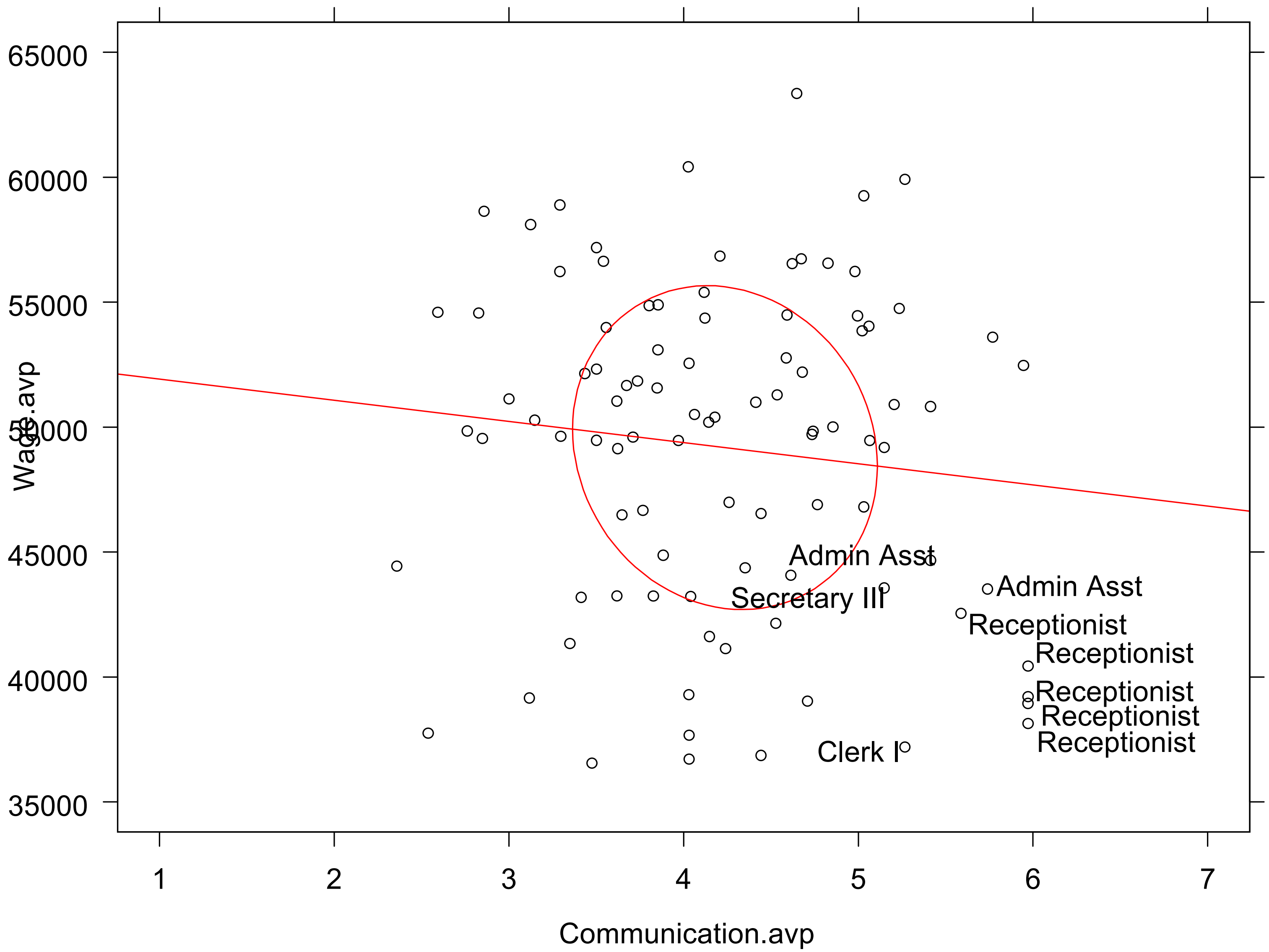
Note:

Horizontal width of ellipse in PRP $= s_X$

Horizontal width of ellipse in AVP $= s_{X\cdot(\mathrm{other\,Xs})}$

$$\sqrt{VIF} = \frac{s_X}{s_{X\cdot(\mathrm{other\,Xs})}}$$

So showing the PRP and the AVP side-by-side on the same horizontal scale can be used to convey information on multicollinearity.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40000
35000
Wage.avp
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Communication.avp
Admin Asst
Admin Asst
Secretary III
Receptionist
Receptionist
Receptionist
Receptionist
Receptionist
Clerk I

# Model with adjustment for Receptionists:

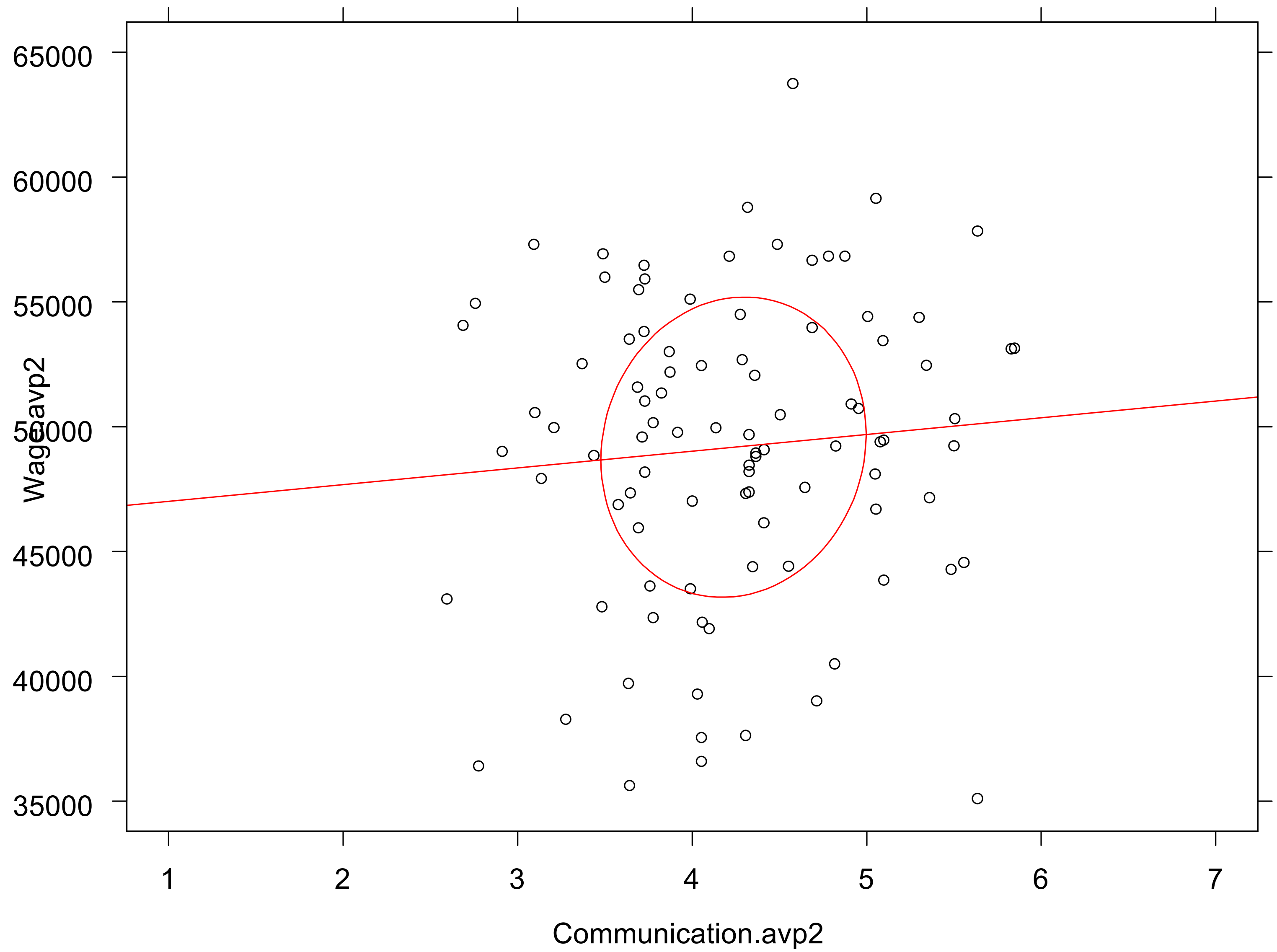
```
> fit <- lm( Wage ~Knowledge + Experience + Communication +
                 Female + Recept, Law)

> summary(fit)

Call: lm(formula = Wage ~Knowledge + Experience + Communication +
                                     Female + Recept, data = Law)
Residuals:
    Min     1Q Median   3Q   Max
 -15008 -3101  441.7 4261 14321

Coefficients:
                   Value  Std. Error     t value    Pr(>|t|)
  (Intercept)  23317.3282   3869.9155     6.0253      0.0000
    Knowledge   3234.6101    857.4269     3.7725      0.0003
   Experience   2955.3080    892.0224     3.3130      0.0013
Communication    671.6964    835.8108     0.8036      0.4237
       Female  -2606.6538   2069.4971    -1.2596      0.2111
       Recept -13098.1539   3539.6990    -3.7004      0.0004
```

```
Residual standard error: 6158 on 89 degrees of freedom
Multiple R-Squared: 0.8149
F-statistic: 78.37 on 5 and 89 degrees of freedom, the p-value is
```


65000
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Wage.avp2
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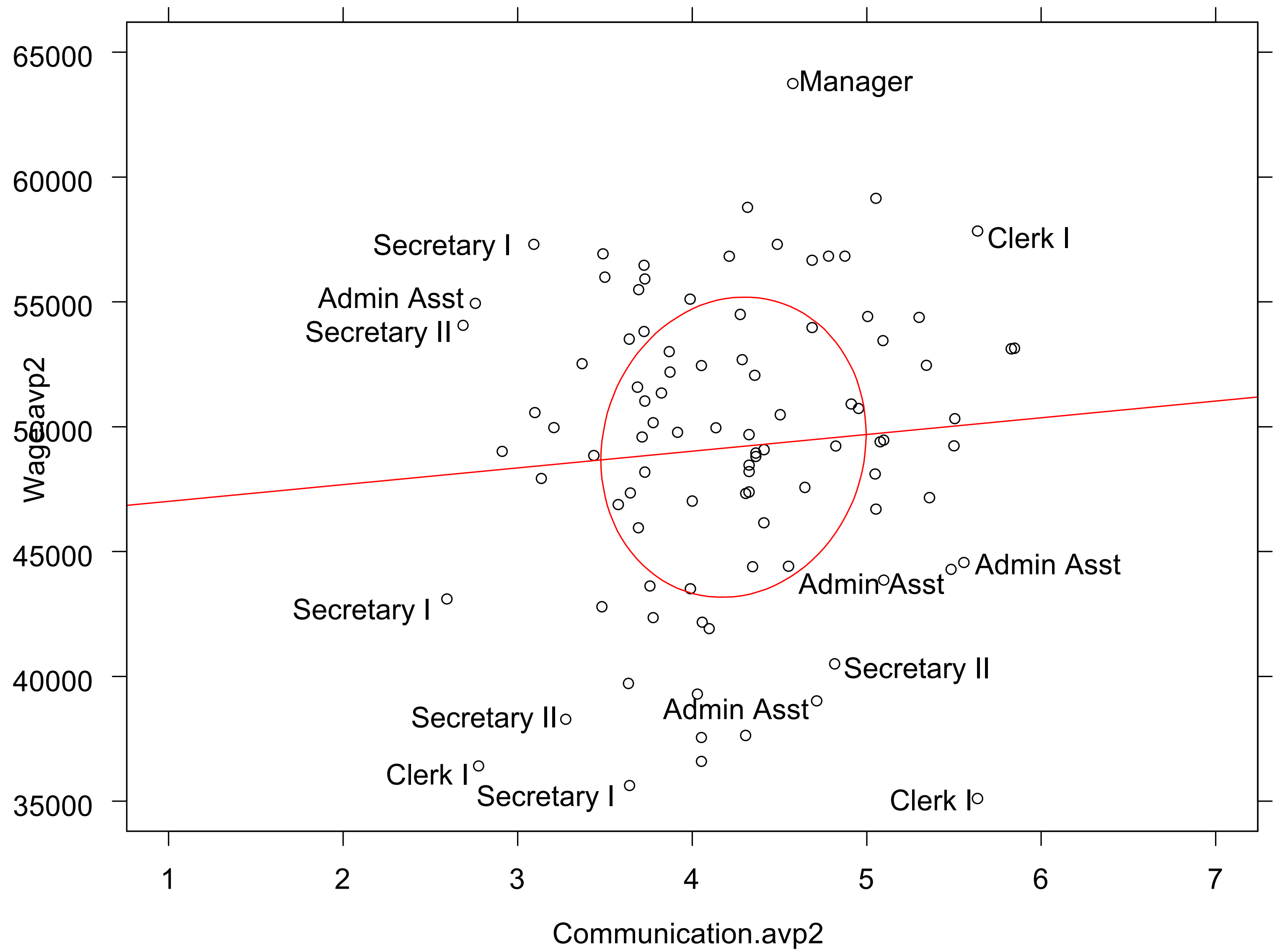
6
7
Communication.avp2


Manager
Clerk I
Secretary I
Admin Asst
Secretary II
Admin Asst
Admin Asst
Secretary I
Secretary II
Admin Asst
Secretary II
Clerk I
Secretary I
Clerk I
65000
60000
55000
50000
45000
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35000
Wage.avp2
1
2
3
4
5
6
7
Communication.avp2

# 17 References:


- Berk K.N. (1998) “Regression diagnostic plots in 3-D,” *Technometrics*, 40 (1): pp. 39-47.
- Fox, John (1997) *Applied Regression Analysis, Linear Models, and Related Methods.* Sage.
- Freedman, Pisani and Purves, (1997), *Statistics*, (3rd ed.) Norton.
- Monette, G. (1990). “The Geometry of Multiple Regression and Interactive 3D Graphics,” *Modern Methods of Data Analysis*, (J. Fox and J.S. Long, eds.), Sage, Newbury Park, Ca., pp. 209-256.
- Stone, Mervyn (1987), “Coordinate-free multivariable statistics: An illustrated geometric progression from Halmos to Gauss and Bayes”, Oxford University Press (Oxford)