

# Dealing with Heteroskedasticity

R-side Variance Modeling and different G matrices

2026-03-25

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# 1 Setup

```
library(nlme)
library(spida2)
library(lattice)
library(latticeExtra)
library(latex2exp)
```

## 2 Introduction

This is an example to illustrate methods to deal with heteroskedasticity. The methods can be used for ‘single-level’ and multi-level data.

To model single-level heteroskedasticity or multi-level within-group heteroskedasticity, you can use the ‘weights’ argument to the fitting functions in the **nlme** package, namely, ‘gls’, ‘nls’, ‘lme’ and ‘nlme’.

To model heteroskedasticity in the between-group model in multi-level models, i.e. heteroskedasticity in the  $G$  matrix, you can use the **pdBlocked** function together with other functions from the **pdClasses** family of functions.

For single-level models, use the ‘gls’ and ‘nls’ functions, and, for mixed models, the ‘lme’ and ‘nlme’ functions.

## 3 Generating a data set

Pay equity data set for a hypothetical university with two faculties: Medicine and Arts with a higher level and variance in Medicine vs Arts and a different gender gap:

```
{  
  # Starting data frame:  
  dd <- expand.grid(Faculty = c("Arts", "Med"), Sex = c("F", "M"), n = 1:400)  
  # sample(1000000,1)
```

```

set.seed(576530)
dd <- within(
  dd,
  {
    Age <- 45 + 5 * (Faculty == "Arts") + 5 * (Sex == "M") + 15 * rnorm(n)

    ..esal <- 100 + 20 * (Faculty == 'Med') +
      (4 + .3 * (Sex == "M") + .5 * (Faculty == "Med")) * (Age - 30)

    ..sdsal <- 10 + 15 * (Faculty == "Med") + (.2 + .2 * (Faculty == "Med")) * (Age - 30)

    Base <- ..esal + ..sdsal * rnorm(n)
    keep <- Age > 28 & Age < 80
    ..sdsal <- NULL
    ..esal <- NULL
  }
)
tab(dd, ~ Faculty + Sex + keep)

```

```
, , keep = FALSE
```

|         | Sex |    |       |
|---------|-----|----|-------|
| Faculty | F   | M  | Total |
| Arts    | 42  | 37 | 79    |
| Med     | 45  | 46 | 91    |
| Total   | 87  | 83 | 170   |

, , keep = TRUE

|         | Sex |     |       |
|---------|-----|-----|-------|
| Faculty | F   | M   | Total |
| Arts    | 358 | 363 | 721   |
| Med     | 355 | 354 | 709   |
| Total   | 713 | 717 | 1430  |

, , keep = Total

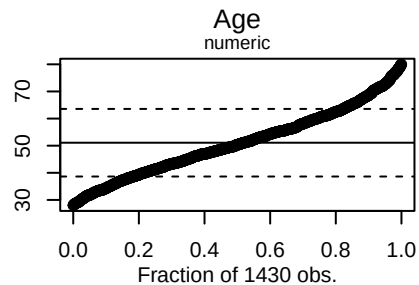
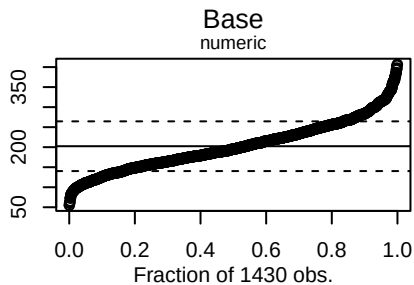
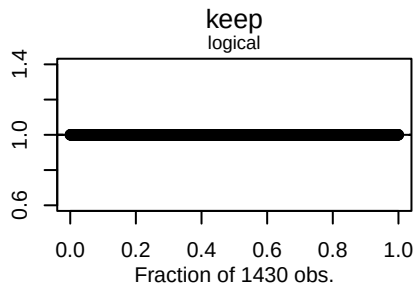
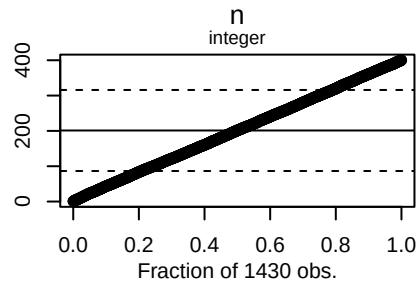
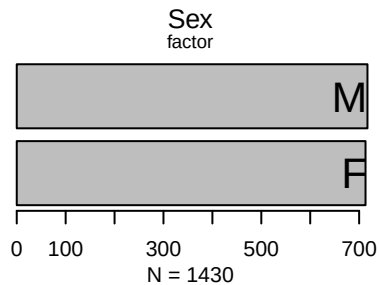
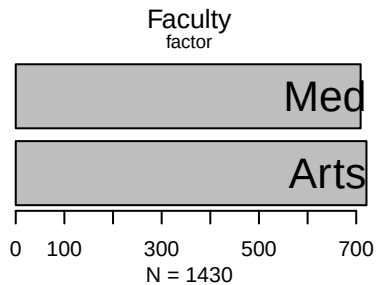
|         | Sex |     |       |
|---------|-----|-----|-------|
| Faculty | F   | M   | Total |
| Arts    | 400 | 400 | 800   |
| Med     | 400 | 400 | 800   |

```
Total 800 800 1600
```

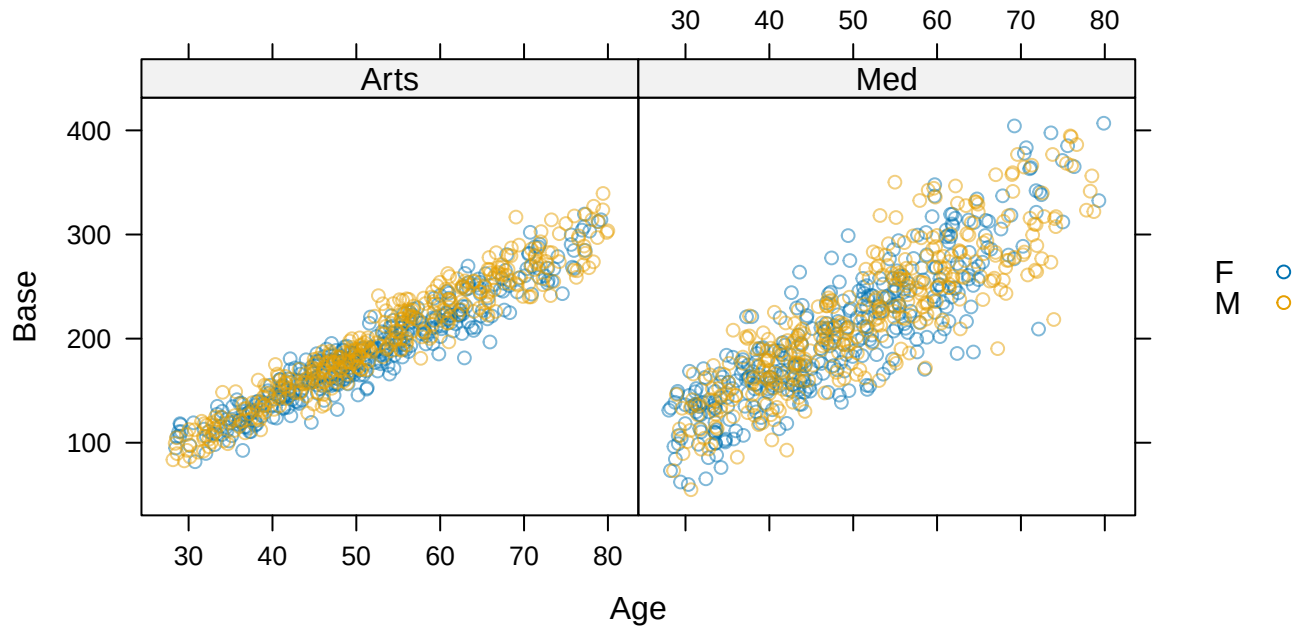
```
dd <- subset(dd, keep)
saveRDS(dd, file = 'salary.rds')
```

## 4 Analysis

```
dd <- readRDS('salary.rds')
xqplot(dd)
```



```
xyplot(Base ~ Age | Faculty, dd, groups = Sex, alpha = .5,  
       auto.key = T)
```



```
fit <- lm(Base ~ Age * Faculty * Sex, dd)
summary(fit)
```

Call:

```
lm(formula = Base ~ Age * Faculty * Sex, data = dd)
```

Residuals:

| Min     | 1Q     | Median | 3Q    | Max    |
|---------|--------|--------|-------|--------|
| -113.08 | -13.85 | 0.04   | 14.27 | 112.03 |

Coefficients:

|                     | Estimate | Std. Error | t value | Pr(> t ) |     |
|---------------------|----------|------------|---------|----------|-----|
| (Intercept)         | -19.7709 | 6.0681     | -3.258  | 0.00115  | **  |
| Age                 | 3.9913   | 0.1148     | 34.776  | < 2e-16  | *** |
| FacultyMed          | -14.8144 | 8.3213     | -1.780  | 0.07524  | .   |
| SexM                | -4.4577  | 8.3148     | -0.536  | 0.59196  |     |
| Age:FacultyMed      | 0.9579   | 0.1621     | 5.908   | 4.32e-09 | *** |
| Age:SexM            | 0.2011   | 0.1546     | 1.301   | 0.19361  |     |
| FacultyMed:SexM     | 17.8111  | 11.6461    | 1.529   | 0.12640  |     |
| Age:FacultyMed:SexM | -0.4304  | 0.2220     | -1.939  | 0.05276  | .   |

---

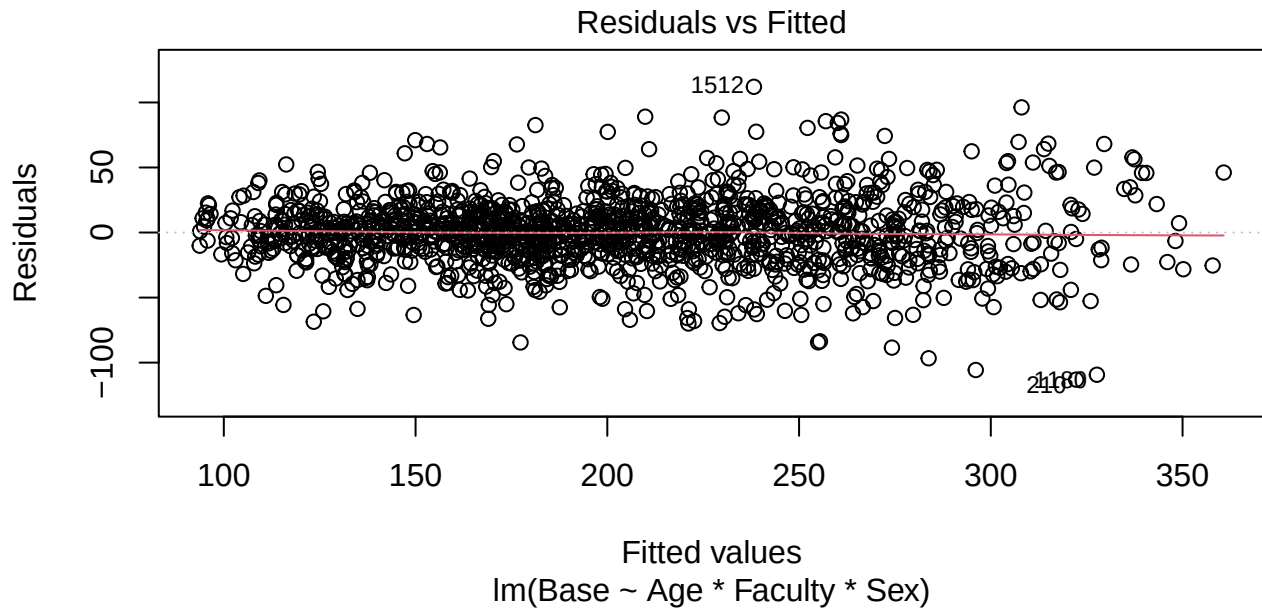
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

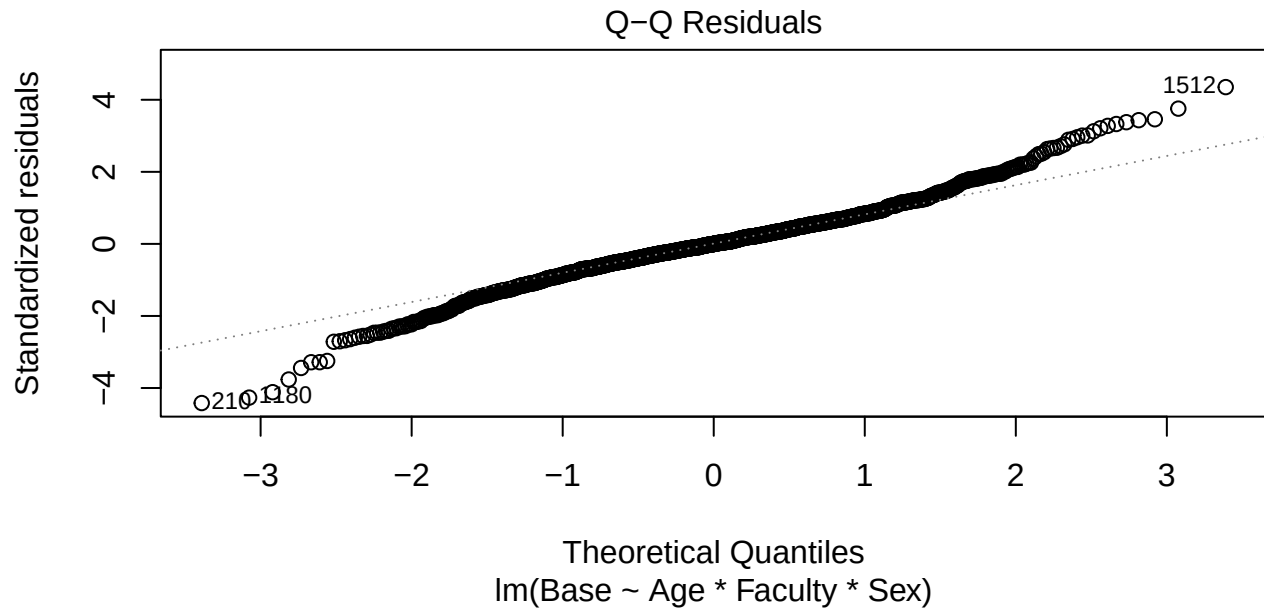
Residual standard error: 25.78 on 1422 degrees of freedom

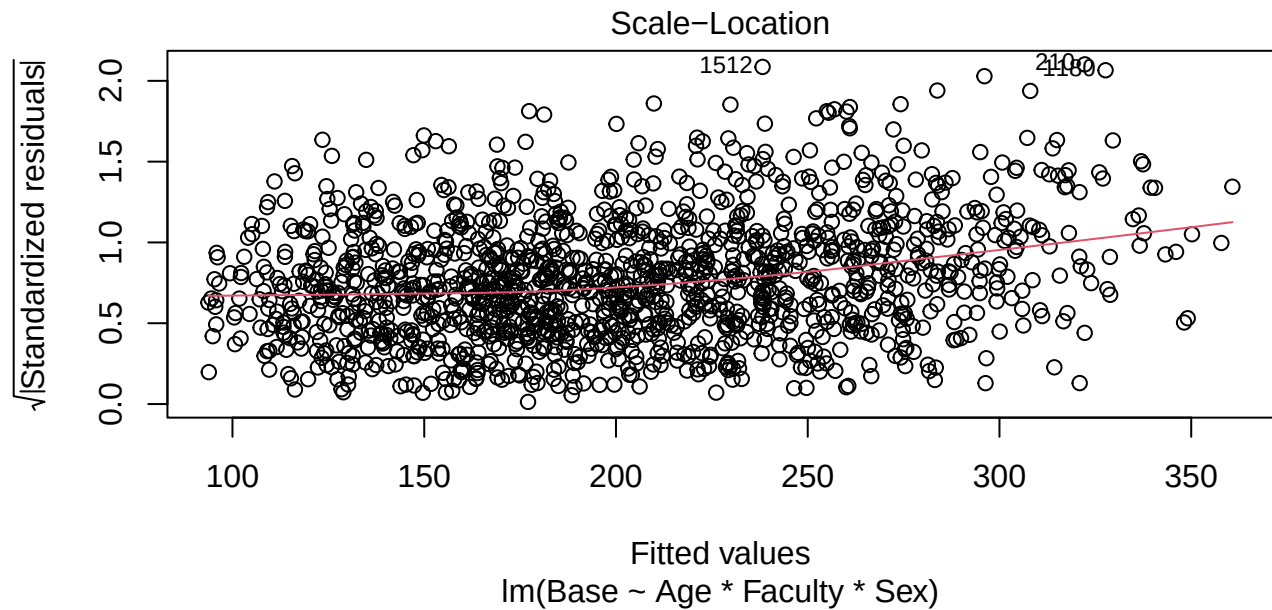
Multiple R-squared: 0.8283, Adjusted R-squared: 0.8275

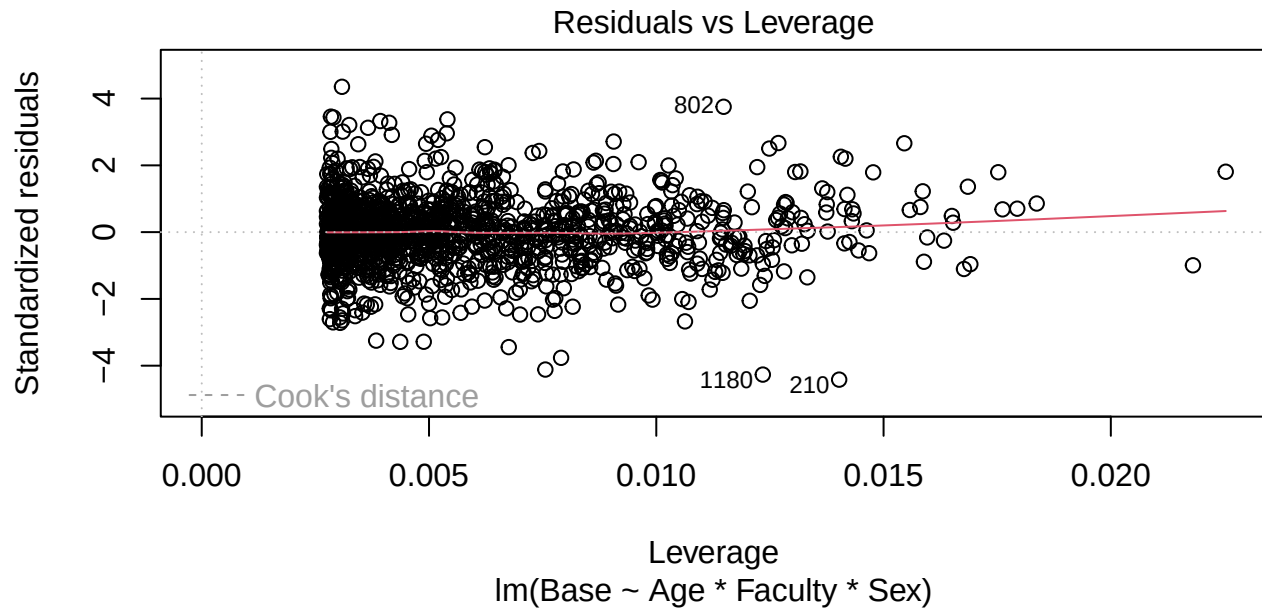
F-statistic: 980 on 7 and 1422 DF, p-value: < 2.2e-16

```
for(i in c(1,2,3,5)) {  
  plot(fit, which = i, add.smooth=T, mfcol = c(1,1))  
}
```

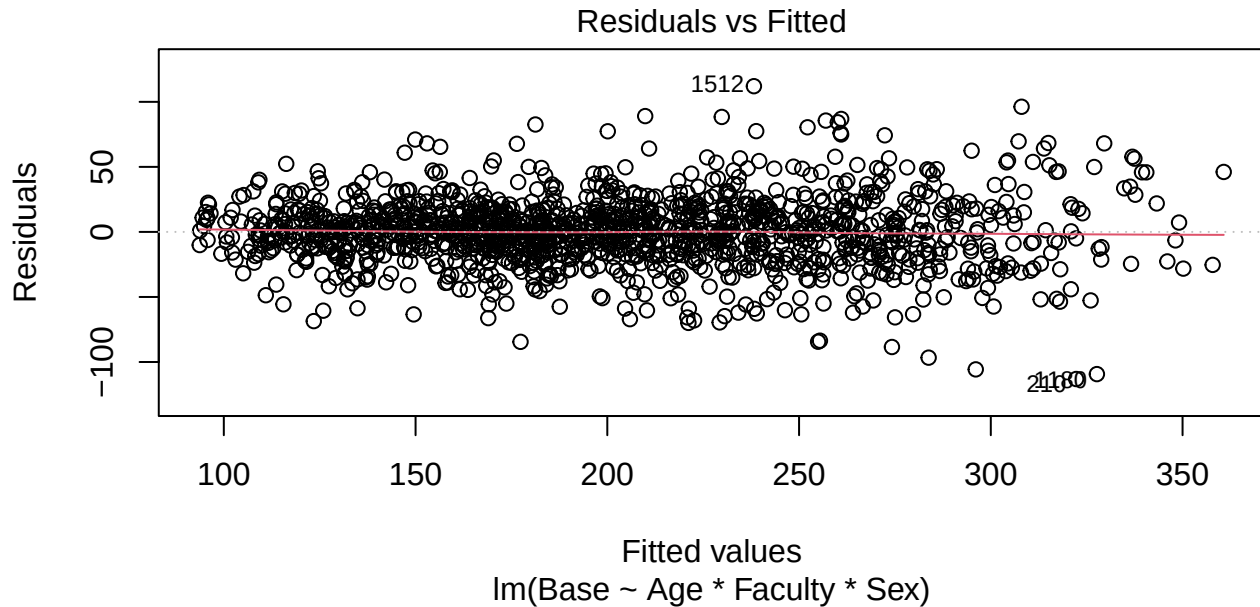




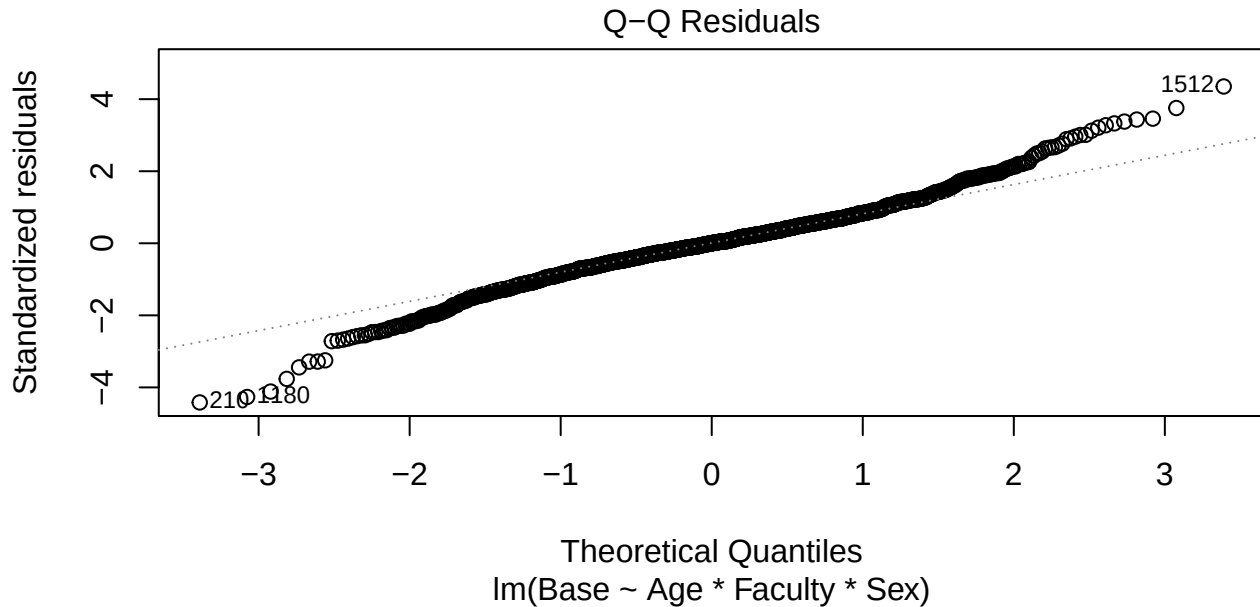




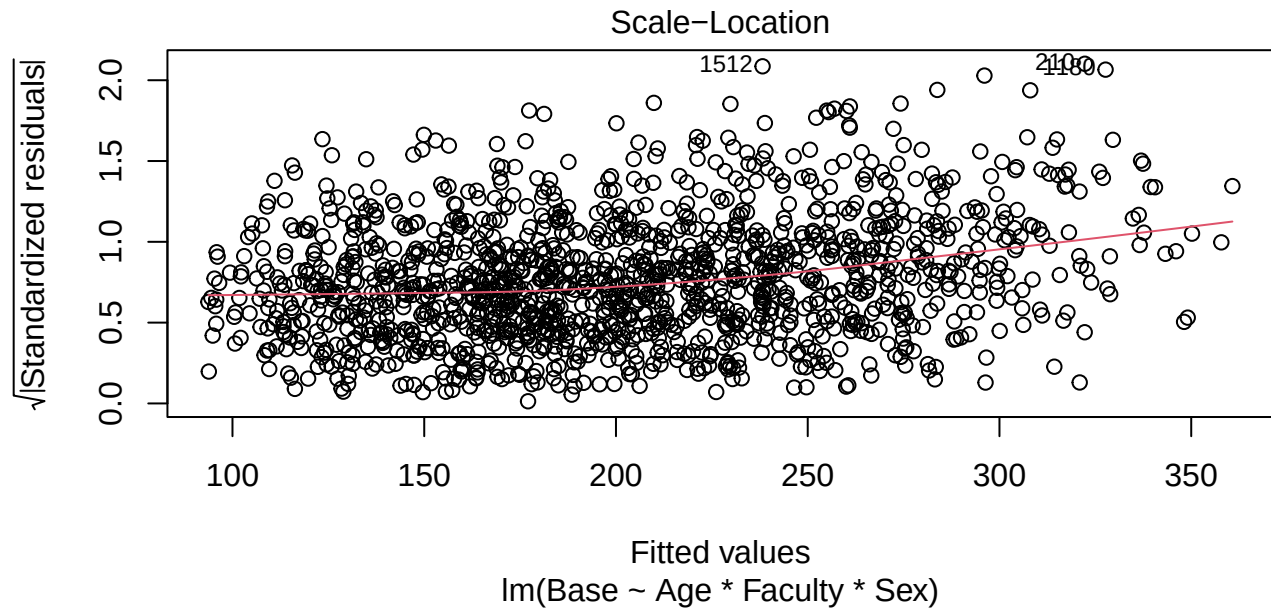
```
plot(fit, 1, add.smooth = T)
```



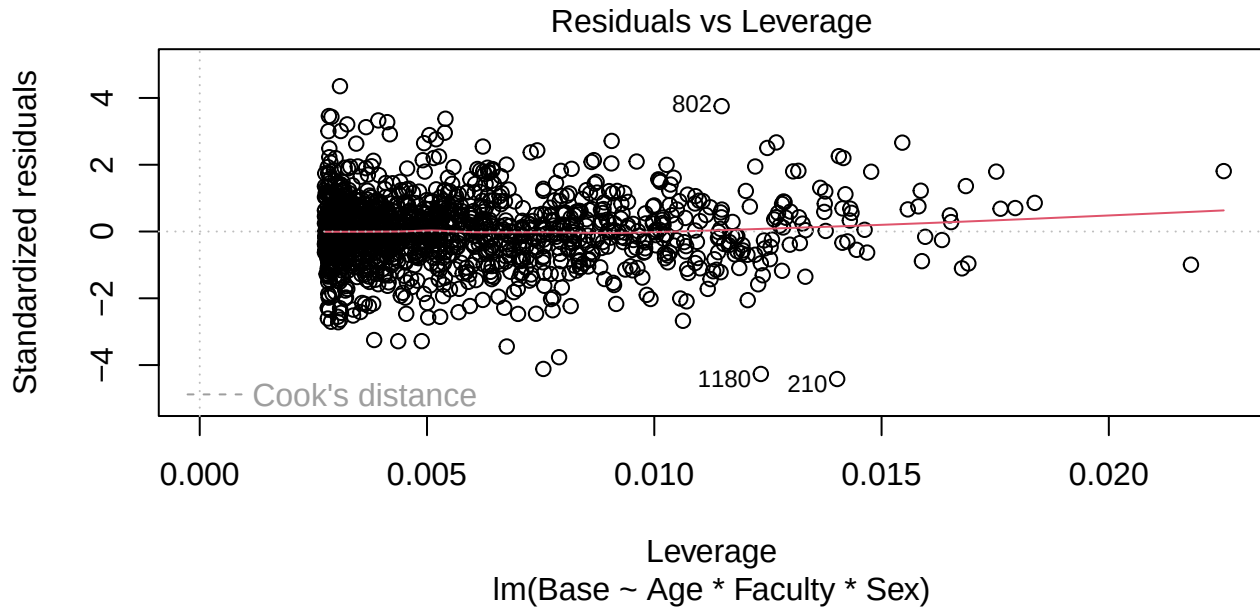
```
plot(fit, 2, add.smooth = T)
```



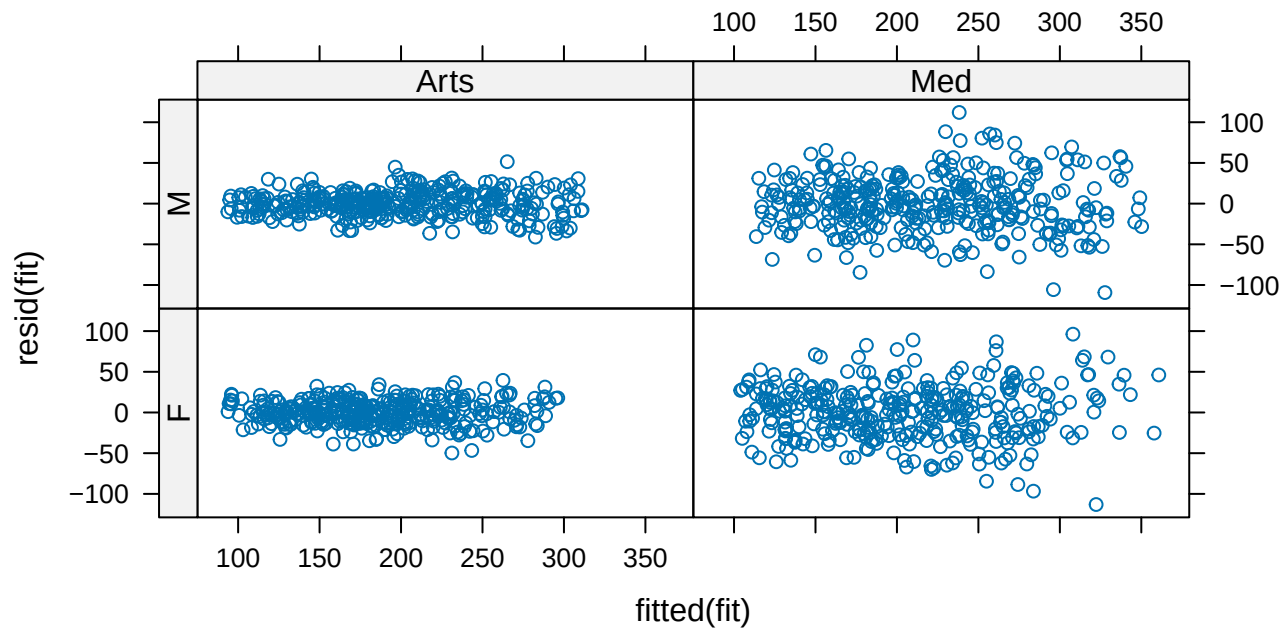
```
plot(fit, 3, add.smooth = T)
```



```
plot(fit, 5, add.smooth = T)
```



```
xyplot(resid(fit) ~ fitted(fit) | Faculty * Sex, dd) %>%  
  useOuterStrips
```



## 5 Functions in nlme to deal with heteroskedasticity:

Overview:

`?varClasses`

- `varExp`: exponential of a covariate or yhat
- `varPower`: power of a covariate or yhat
- `varConstPower`: constant + power of a covariate or yhat
- `varConstProp`: constant + proportion of a covariate or yhat
- `varIdent`: different variance in different subgroups
- `varFixed`: fixed weights given by a covariate
- `varComb`: combination of variance functions
- You can also build your own but count on spending a few days figuring out how to do it

Using `nlme::gls` to do the same job as `lm`:

```
fit.lm <- lm(Base ~ Age * Faculty * Sex, dd)
fit <- gls(Base ~ Age * Faculty * Sex, dd) # same as fit with gls model
summary(fit.lm)
```

Call:

```
lm(formula = Base ~ Age * Faculty * Sex, data = dd)
```

Residuals:

| Min     | 1Q     | Median | 3Q    | Max    |
|---------|--------|--------|-------|--------|
| -113.08 | -13.85 | 0.04   | 14.27 | 112.03 |

Coefficients:

|                | Estimate | Std. Error | t value | Pr(> t ) |     |
|----------------|----------|------------|---------|----------|-----|
| (Intercept)    | -19.7709 | 6.0681     | -3.258  | 0.00115  | **  |
| Age            | 3.9913   | 0.1148     | 34.776  | < 2e-16  | *** |
| FacultyMed     | -14.8144 | 8.3213     | -1.780  | 0.07524  | .   |
| SexM           | -4.4577  | 8.3148     | -0.536  | 0.59196  |     |
| Age:FacultyMed | 0.9579   | 0.1621     | 5.908   | 4.32e-09 | *** |
| Age:SexM       | 0.2011   | 0.1546     | 1.301   | 0.19361  |     |

```

FacultyMed:SexM      17.8111      11.6461      1.529  0.12640
Age:FacultyMed:SexM  -0.4304       0.2220     -1.939  0.05276 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

```

Residual standard error: 25.78 on 1422 degrees of freedom
Multiple R-squared:  0.8283, Adjusted R-squared:  0.8275
F-statistic:  980 on 7 and 1422 DF,  p-value: < 2.2e-16

```

```
summary(fit)
```

```
Generalized least squares fit by REML
```

```
Model: Base ~ Age * Faculty * Sex
```

```
Data: dd
```

```

      AIC      BIC    logLik
13362.55 13409.89 -6672.276

```

```
Coefficients:
```

```

              Value Std.Error  t-value p-value
(Intercept)  -19.770856  6.068085 -3.25817  0.0011
Age           3.991303  0.114772 34.77603  0.0000
FacultyMed   -14.814393  8.321266 -1.78031  0.0752

```

|                     |           |           |          |        |
|---------------------|-----------|-----------|----------|--------|
| SexM                | -4.457711 | 8.314815  | -0.53612 | 0.5920 |
| Age:FacultyMed      | 0.957917  | 0.162128  | 5.90842  | 0.0000 |
| Age:SexM            | 0.201067  | 0.154597  | 1.30059  | 0.1936 |
| FacultyMed:SexM     | 17.811110 | 11.646092 | 1.52936  | 0.1264 |
| Age:FacultyMed:SexM | -0.430366 | 0.222007  | -1.93852 | 0.0528 |

Correlation:

|                     | (Intr) | Age    | FcltyM | SexM   | Ag:FcM | Ag:SxM | FcM:SM |
|---------------------|--------|--------|--------|--------|--------|--------|--------|
| Age                 | -0.974 |        |        |        |        |        |        |
| FacultyMed          | -0.729 | 0.711  |        |        |        |        |        |
| SexM                | -0.730 | 0.711  | 0.532  |        |        |        |        |
| Age:FacultyMed      | 0.690  | -0.708 | -0.972 | -0.503 |        |        |        |
| Age:SexM            | 0.723  | -0.742 | -0.528 | -0.973 | 0.526  |        |        |
| FacultyMed:SexM     | 0.521  | -0.508 | -0.715 | -0.714 | 0.695  | 0.695  |        |
| Age:FacultyMed:SexM | -0.504 | 0.517  | 0.710  | 0.677  | -0.730 | -0.696 | -0.972 |

Standardized residuals:

| Min          | Q1           | Med         | Q3          | Max         |
|--------------|--------------|-------------|-------------|-------------|
| -4.386111354 | -0.537358027 | 0.001570287 | 0.553507998 | 4.345247638 |

Residual standard error: 25.78107

Degrees of freedom: 1430 total; 1422 residual

```
pred <- with(dd, pred.grid(Faculty, Sex, Age = seq(30,75,1)))
ww <- as.data.frame(wald(fit, pred = pred))
head(ww)
```

|   | coef              | se                    | U2           | L2        | p-value          | t-value    | DF   | Faculty | Sex |
|---|-------------------|-----------------------|--------------|-----------|------------------|------------|------|---------|-----|
| 1 | 99.96824          | 2.820883              | 105.6100     | 94.32648  | 1.189622e-197    | 35.43863   | 1421 | Arts    | F   |
| 2 | 113.89137         | 2.499606              | 118.8906     | 108.89216 | 3.599115e-280    | 45.56373   | 1421 | Med     | F   |
| 3 | 101.54254         | 2.767341              | 107.0772     | 96.00786  | 6.440110e-208    | 36.69317   | 1421 | Arts    | M   |
| 4 | 120.36580         | 2.720735              | 125.8073     | 114.92433 | 1.704521e-269    | 44.24018   | 1421 | Med     | M   |
| 5 | 103.95955         | 2.720954              | 109.4015     | 98.51764  | 2.566811e-220    | 38.20703   | 1421 | Arts    | F   |
| 6 | 118.84059         | 2.404593              | 123.6498     | 114.03140 | 6.134238e-311    | 49.42233   | 1421 | Med     | F   |
|   | Age L.(Intercept) | L.Age                 | L.FacultyMed | L.SexM    | L.Age:FacultyMed | L.Age:SexM |      |         |     |
| 1 | 30                | 1                     | 30           | 0         | 0                | 0          | 0    |         |     |
| 2 | 30                | 1                     | 30           | 1         | 0                | 30         | 0    |         |     |
| 3 | 30                | 1                     | 30           | 0         | 1                | 0          | 30   |         |     |
| 4 | 30                | 1                     | 30           | 1         | 1                | 30         | 30   |         |     |
| 5 | 31                | 1                     | 31           | 0         | 0                | 0          | 0    |         |     |
| 6 | 31                | 1                     | 31           | 1         | 0                | 31         | 0    |         |     |
|   | L.FacultyMed:SexM | L.Age:FacultyMed:SexM |              |           |                  |            |      |         |     |
| 1 | 0                 | 0                     |              |           |                  |            |      |         |     |

|   |   |    |
|---|---|----|
| 2 | 0 | 0  |
| 3 | 0 | 0  |
| 4 | 1 | 30 |
| 5 | 0 | 0  |
| 6 | 0 | 0  |

```
caption <- "Predicted salary for men and women in each Faculty."
```

```
xyplot(coef ~ Age | Faculty, ww, groups = Sex,  
       type = 'l', auto.key = list(space='right'))
```

```
caption <- "Predicted salary for men and women in each Faculty with 95% error bands."
```

```
xyplot(coef ~ Age | Faculty, ww, groups = Sex,  
       type = 'l', auto.key = list(space='right'),  
       fit = ww$coef,  
       lower = ww$L2,  
       upper = ww$U2,  
       subscripts = TRUE) +  
glayer(panel.band(...))
```

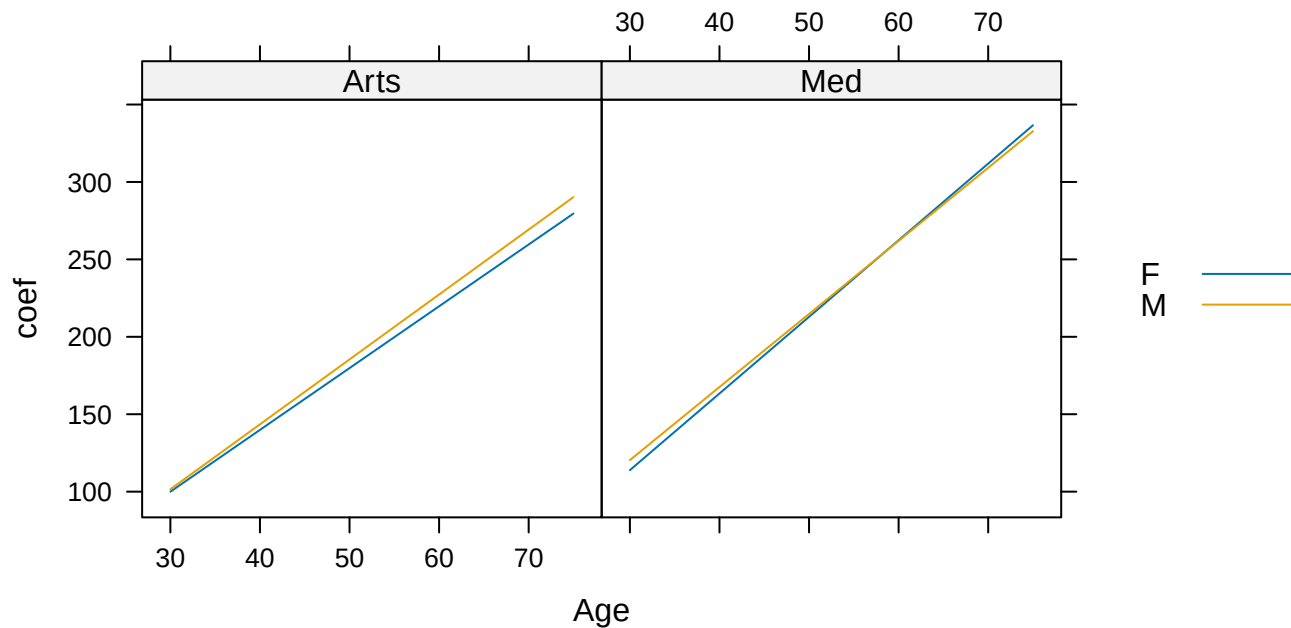


Figure 1: Predicted salary for men and women in each Faculty.

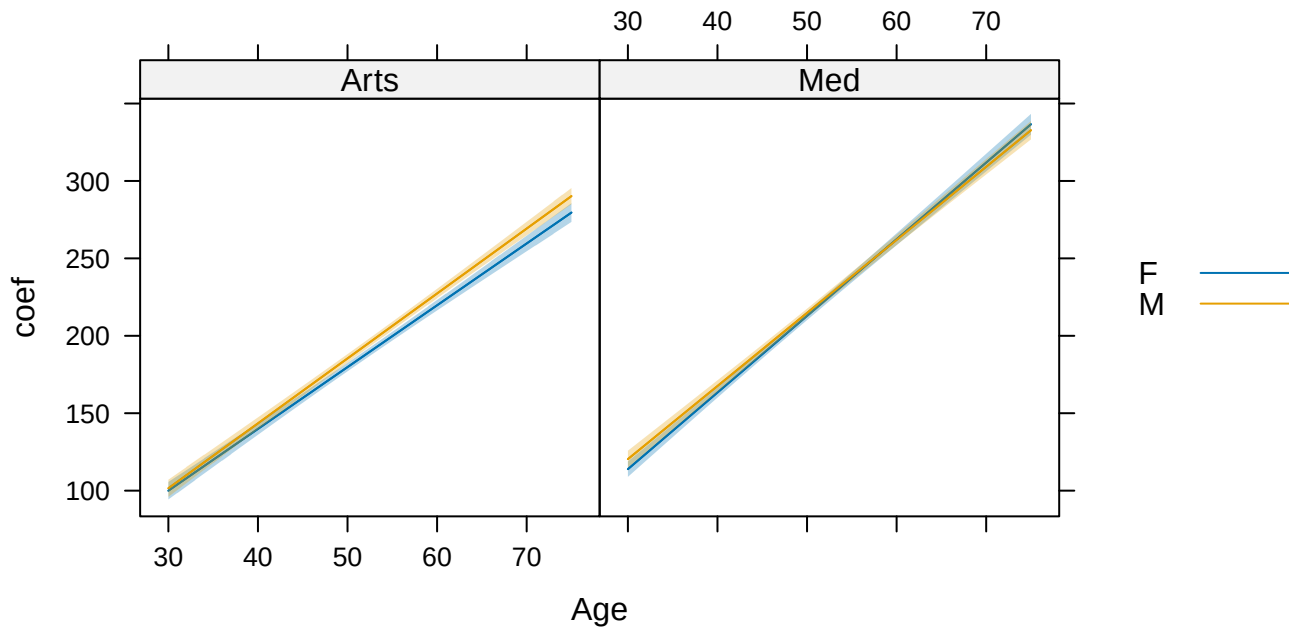


Figure 2: Predicted salary for men and women in each Faculty with 95% error bands.

## 6 Estimating the Gap

Estimating the gap, if it is potentially non-linear in  $X$ , requires estimating a many linear combinations of coefficients – a different linear combination for each value of ‘ $X$ ’ where the gap needs to be estimated.

`spida2::wald`, with `spida2::as.data.frame`, allows you to generate these combinations.

The **L** matrices that produce predicted values for each sex are extracted as follows:

```
Lmale <- subset(ww, Sex == 'M')$L
Lfemale <- subset(ww, Sex == 'F')$L
```

The **L** matrix that produces an estimate of the gap is then:

```
Lgap <- Lmale - Lfemale
```

which can then be used with the `wald` function to estimate the gap and its standard error at each value in the ‘pred’ data frame:

```
wgap <- wald(fit,
             Lgap,
             data = subset(ww, Sex == 'F', select = c(Faculty, Age)))
```

The ‘data’ parameter attaches the rows of its argument to the output of the function so the rows can be available for

further processing of the result when rendered as a data frame.

Note that the row of the **L** matrix producing each predicted value and values of predictors from the prediction data frame are available in each row together with the predicted values and their standard errors:

```
wgap <- as.data.frame(wgap)
head(wgap)
```

|                                                                     | coef     | se       | U2        | L2         | p-value    | t-value   | DF   | Faculty | Age |
|---------------------------------------------------------------------|----------|----------|-----------|------------|------------|-----------|------|---------|-----|
| 3                                                                   | 1.574296 | 3.951653 | 9.477602  | -6.3290094 | 0.69040306 | 0.3983893 | 1421 | Arts    | 30  |
| 4                                                                   | 6.474432 | 3.694648 | 13.863729 | -0.9148645 | 0.07992395 | 1.7523812 | 1421 | Med     | 30  |
| 7                                                                   | 1.775363 | 3.817388 | 9.410138  | -5.8594117 | 0.64195054 | 0.4650729 | 1421 | Arts    | 31  |
| 8                                                                   | 6.245134 | 3.560326 | 13.365786 | -0.8755194 | 0.07963061 | 1.7540901 | 1421 | Med     | 31  |
| 11                                                                  | 1.976430 | 3.684716 | 9.345863  | -5.3930025 | 0.59177576 | 0.5363860 | 1421 | Arts    | 32  |
| 12                                                                  | 6.015835 | 3.428148 | 12.872130 | -0.8404604 | 0.07950296 | 1.7548354 | 1421 | Med     | 32  |
| L.(Intercept) L.Age L.FacultyMed L.SexM L.Age:FacultyMed L.Age:SexM |          |          |           |            |            |           |      |         |     |
| 3                                                                   |          | 0        | 0         | 0          | 1          | 0         | 30   |         |     |
| 4                                                                   |          | 0        | 0         | 0          | 1          | 0         | 30   |         |     |
| 7                                                                   |          | 0        | 0         | 0          | 1          | 0         | 31   |         |     |
| 8                                                                   |          | 0        | 0         | 0          | 1          | 0         | 31   |         |     |
| 11                                                                  |          | 0        | 0         | 0          | 1          | 0         | 32   |         |     |
| 12                                                                  |          | 0        | 0         | 0          | 1          | 0         | 32   |         |     |

|    | L.FacultyMed:SexM | L.Age:FacultyMed:SexM |
|----|-------------------|-----------------------|
| 3  | 0                 | 0                     |
| 4  | 1                 | 30                    |
| 7  | 0                 | 0                     |
| 8  | 1                 | 31                    |
| 11 | 0                 | 0                     |
| 12 | 1                 | 32                    |

The result can then be plotted as follows. Note that the **spida2::panel.band** function is designed to use the values passed defined in **xyplot** as **fit**, **lower** and **upper**. It is necessary to add **subscripts = TRUE** so values are passed to **panel.band** in the correct positions.

```
caption <- paste('Estimated gaps, estimated without allowing for different',
                'variances in different Faculties.')
xyplot(coef ~ Age | Faculty, wgap,
       type = 'l', auto.key = list(space='right'))
```

```
caption <-
  paste("Estimated Male-Female wage gap with 95% error bands",
        "estimated without allowing for different variances in different Faculties.")
(
  p <- xyplot(coef ~ Age | Faculty, wgap,
```

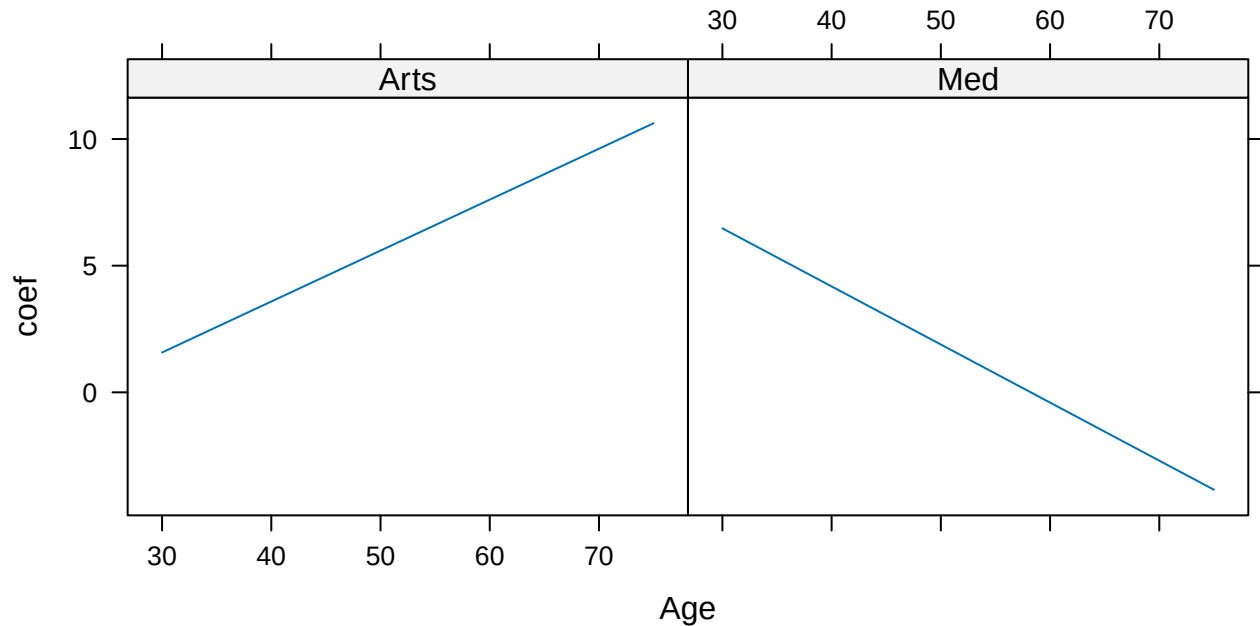


Figure 3: Estimated gaps, estimated without allowing for different variances in different Faculties.

```

    type = 'l', auto.key = list(space='right'),
    ylab = 'Wage gap (male - female)',
    ylim = c(-15, 20), xlim = c(30, 75),
    fit = wgap$coef,
    lower = wgap$L2,
    upper = wgap$U2,
    subscripts = TRUE) +
layer(panel.band(...)) +
layer_(panel.grid(h=-1,v=-1)) +
layer_(panel.abline(h=0))
)

```

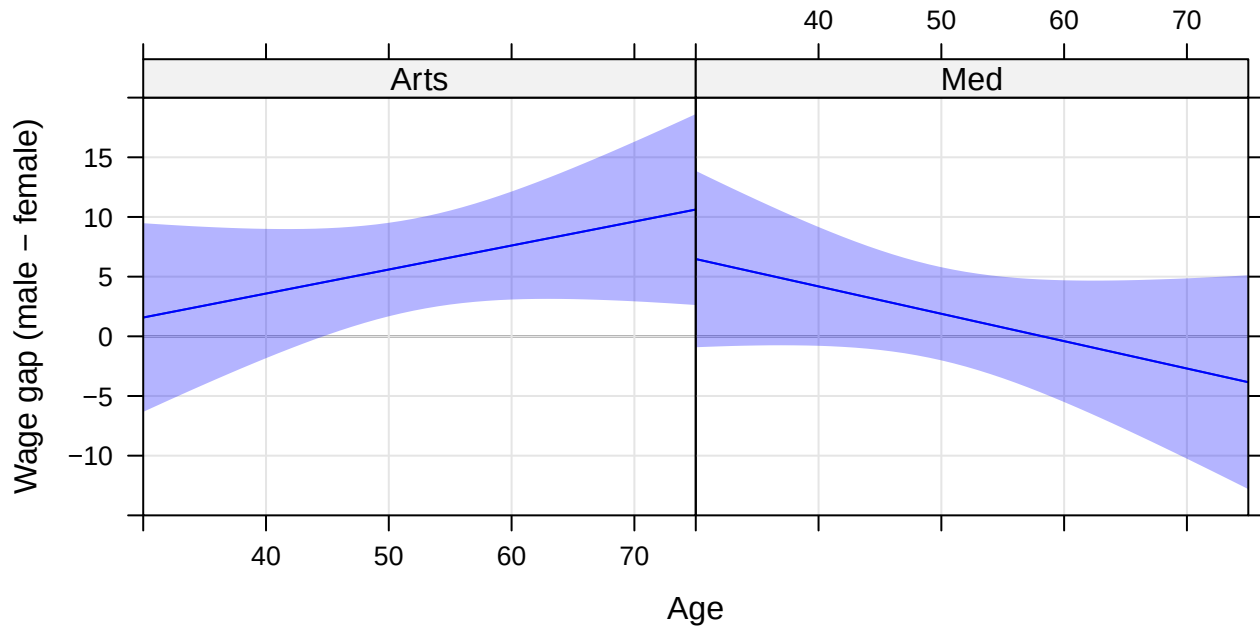


Figure 4: Estimated Male-Female wage gap with 95% error bands estimated without allowing for different variances in different Faculties.

## 7 Models with heteroskedasticity

We use the `weights` parameter of `glm` to fit linear models for prediction as well as a number of possible models for variance. The models use functions in the `nlme` package designed to work with the `weights` parameter.

The following illustrate a number of these functions.

Although these functions are called ‘variance functions’ they are in fact ‘standard deviation’ functions.

For example

```
weights = varConstPower(form = ~fitted(.)|Faculty)
```

models a multiplicative factor, which when multiplying the estimate of the residual standard error,  $\hat{\sigma}$ , yields the estimated standard deviation of epsilons.

The function `varConstPower` fits a model for the standard deviation of epsilons of the form:

$$\hat{\sigma} (\theta_1 + |v|^{\theta_2})$$

where  $\hat{\sigma}$  is the estimate of  $\sigma$ ,  $v$  is the ‘variance covariate’ supplied in a formula as the argument to the `form` parameter, and  $\theta_1$  and  $\theta_2$  are parameters to be estimated.

$v$  can be a variable in ‘data’ or one of two special variables: `resid(.)` or `fitted(.)`.

If the argument of `form` has a conditioning categorical variable, (following a `|`), then independent estimates of  $\theta_1$  and  $\theta_2$  are obtained for each level of the conditioning variable.

The following are three heteroskedastic models for base salary. See help files for descriptions of `varPower` and `varIdent`.

```
fitconpower <- gls(Base ~ Age * Faculty * Sex, dd,
                  weights = varConstPower(form = ~fitted(.)|Faculty))
summary(fitconpower)
```

Generalized least squares fit by REML

Model: Base ~ Age \* Faculty \* Sex

Data: dd

| AIC      | BIC      | logLik    |
|----------|----------|-----------|
| 12878.73 | 12947.11 | -6426.367 |

Variance function:

Structure: Constant plus power of variance covariate, different strata

Formula: ~fitted(.) | Faculty

Parameter estimates:

|       | Arts         | Med          |
|-------|--------------|--------------|
| const | 6.308813e+05 | 1.473026e+06 |

power 2.336016e+00 2.420192e+00

Coefficients:

|                     | Value      | Std.Error | t-value  | p-value |
|---------------------|------------|-----------|----------|---------|
| (Intercept)         | -17.889910 | 3.276807  | -5.45956 | 0.0000  |
| Age                 | 3.952852   | 0.065960  | 59.92837 | 0.0000  |
| FacultyMed          | -12.114040 | 7.692010  | -1.57489 | 0.1155  |
| SexM                | -8.661098  | 4.525437  | -1.91387 | 0.0558  |
| Age:FacultyMed      | 0.895920   | 0.163682  | 5.47355  | 0.0000  |
| Age:SexM            | 0.285801   | 0.090281  | 3.16568  | 0.0016  |
| FacultyMed:SexM     | 15.329922  | 11.072343 | 1.38452  | 0.1664  |
| Age:FacultyMed:SexM | -0.371087  | 0.230012  | -1.61334 | 0.1069  |

Correlation:

|                 | (Intr) | Age    | FcltyM | SexM   | Ag:FcM | Ag:SxM | FcM:SM |
|-----------------|--------|--------|--------|--------|--------|--------|--------|
| Age             | -0.974 |        |        |        |        |        |        |
| FacultyMed      | -0.426 | 0.415  |        |        |        |        |        |
| SexM            | -0.724 | 0.706  | 0.308  |        |        |        |        |
| Age:FacultyMed  | 0.393  | -0.403 | -0.971 | -0.284 |        |        |        |
| Age:SexM        | 0.712  | -0.731 | -0.303 | -0.972 | 0.294  |        |        |
| FacultyMed:SexM | 0.296  | -0.288 | -0.695 | -0.409 | 0.675  | 0.397  |        |

Age:FacultyMed:SexM -0.279 0.287 0.691 0.382 -0.712 -0.393 -0.971

Standardized residuals:

| Min           | Q1            | Med           | Q3           | Max          |
|---------------|---------------|---------------|--------------|--------------|
| -3.0357747190 | -0.6706411747 | -0.0007740241 | 0.6726131486 | 3.2485263975 |

Residual standard error: 1.685497e-05

Degrees of freedom: 1430 total; 1422 residual

```
fitpower <- update(fit, weights = varPower(form = ~fitted(.)|Faculty))
summary(fitpower)
```

Generalized least squares fit by REML

Model: Base ~ Age \* Faculty \* Sex

Data: dd

| AIC      | BIC      | logLik    |
|----------|----------|-----------|
| 12882.81 | 12940.67 | -6430.406 |

Variance function:

Structure: Power of variance covariate, different strata

Formula: ~fitted(.) | Faculty

Parameter estimates:

Arts            Med  
0.4459907 0.5896091

Coefficients:

|                     | Value      | Std.Error | t-value  | p-value |
|---------------------|------------|-----------|----------|---------|
| (Intercept)         | -18.013083 | 3.222974  | -5.58896 | 0.0000  |
| Age                 | 3.957298   | 0.064267  | 61.57558 | 0.0000  |
| FacultyMed          | -11.932641 | 7.338452  | -1.62604 | 0.1042  |
| SexM                | -7.897624  | 4.420878  | -1.78644 | 0.0742  |
| Age:FacultyMed      | 0.895227   | 0.157968  | 5.66713  | 0.0000  |
| Age:SexM            | 0.266513   | 0.087061  | 3.06121  | 0.0022  |
| FacultyMed:SexM     | 14.445347  | 10.604449 | 1.36220  | 0.1734  |
| Age:FacultyMed:SexM | -0.356574  | 0.221909  | -1.60685 | 0.1083  |

Correlation:

|                | (Intr) | Age    | FcltyM | SexM   | Ag:FcM | Ag:SxM | FcM:SM |
|----------------|--------|--------|--------|--------|--------|--------|--------|
| Age            | -0.973 |        |        |        |        |        |        |
| FacultyMed     | -0.439 | 0.427  |        |        |        |        |        |
| SexM           | -0.729 | 0.709  | 0.320  |        |        |        |        |
| Age:FacultyMed | 0.396  | -0.407 | -0.969 | -0.288 |        |        |        |
| Age:SexM       | 0.718  | -0.738 | -0.315 | -0.970 | 0.300  |        |        |

```
FacultyMed:SexM      0.304 -0.296 -0.692 -0.417  0.670  0.404
Age:FacultyMed:SexM -0.282  0.290  0.690  0.381 -0.712 -0.392 -0.968
```

Standardized residuals:

| Min          | Q1           | Med          | Q3          | Max         |
|--------------|--------------|--------------|-------------|-------------|
| -3.071825562 | -0.671447364 | -0.001009986 | 0.681397992 | 3.124627752 |

Residual standard error: 1.419601

Degrees of freedom: 1430 total; 1422 residual

```
fitgroups <- update(fit, weights = varIdent(form = ~ 1 | Faculty))
summary(fitgroups)
```

Generalized least squares fit by REML

Model: Base ~ Age \* Faculty \* Sex

Data: dd

| AIC      | BIC      | logLik    |
|----------|----------|-----------|
| 12943.13 | 12995.73 | -6461.566 |

Variance function:

Structure: Different standard deviations per stratum

Formula: ~1 | Faculty

Parameter estimates:

|          |          |
|----------|----------|
| Arts     | Med      |
| 1.000000 | 2.240452 |

Coefficients:

|                     | Value      | Std.Error | t-value  | p-value |
|---------------------|------------|-----------|----------|---------|
| (Intercept)         | -19.770856 | 3.507592  | -5.63659 | 0.0000  |
| Age                 | 3.991303   | 0.066343  | 60.16206 | 0.0000  |
| FacultyMed          | -14.814393 | 8.165856  | -1.81419 | 0.0699  |
| SexM                | -4.457711  | 4.806290  | -0.92747 | 0.3538  |
| Age:FacultyMed      | 0.957917   | 0.162463  | 5.89622  | 0.0000  |
| Age:SexM            | 0.201067   | 0.089363  | 2.25000  | 0.0246  |
| FacultyMed:SexM     | 17.811110  | 11.602872 | 1.53506  | 0.1250  |
| Age:FacultyMed:SexM | -0.430366  | 0.224866  | -1.91388 | 0.0558  |

Correlation:

|                | (Intr) | Age    | FcltyM | SexM   | Ag:FcM | Ag:SxM | FcM:SM |
|----------------|--------|--------|--------|--------|--------|--------|--------|
| Age            | -0.974 |        |        |        |        |        |        |
| FacultyMed     | -0.430 | 0.419  |        |        |        |        |        |
| SexM           | -0.730 | 0.711  | 0.313  |        |        |        |        |
| Age:FacultyMed | 0.398  | -0.408 | -0.971 | -0.290 |        |        |        |

```

Age:SexM          0.723 -0.742 -0.311 -0.973  0.303
FacultyMed:SexM    0.302 -0.295 -0.704 -0.414  0.683  0.403
Age:FacultyMed:SexM -0.287  0.295  0.702  0.387 -0.722 -0.397 -0.971

```

Standardized residuals:

```

          Min          Q1          Med          Q3          Max
-3.386777242 -0.672809401  0.002716575  0.661769856  3.456258200

```

Residual standard error: 14.90247

Degrees of freedom: 1430 total; 1422 residual

Like `lme`, `gls` uses REML by default, so that different variance models can be assessed with likelihood methods (LRT, AIC, BIC), provided the linear model is the same. Difference linear models can be assessed with Wald tests of a containing model, or by comparisons if the fitted objects have been refit with `method = 'ML'`.

```
anova(fit , fitgroups, fitpower, fitconpower)
```

|             | Model | df | AIC      | BIC      | logLik    | Test   | L.Ratio  | p-value |
|-------------|-------|----|----------|----------|-----------|--------|----------|---------|
| fit         | 1     | 9  | 13362.55 | 13409.89 | -6672.276 |        |          |         |
| fitgroups   | 2     | 10 | 12943.13 | 12995.73 | -6461.566 | 1 vs 2 | 421.4192 | <.0001  |
| fitpower    | 3     | 11 | 12882.81 | 12940.67 | -6430.406 | 2 vs 3 | 62.3213  | <.0001  |
| fitconpower | 4     | 13 | 12878.73 | 12947.11 | -6426.367 | 3 vs 4 | 8.0773   | 0.0176  |

When interpreting the output of `anova` be very careful to interpret p-value only if the models compared are nested with respect to each other.

Avoid selecting models only on the basis of p-values, AIC or BIC. Don't drop a parameter just because it isn't significant if it plays a role in the interpretation of the model.

For example don't drop the parameter of a confounding variable needed to block a backdoor path. Don't drop a parameter if dropping it will affect the correct interpretation of another parameter of key interest. For example a curvature parameter may have a great deal of influence on the estimation of predicted values away from the points of intersection of a straight line and the curved line.

```
library(car)
compareCoefs(fit, fitgroups, fitpower, fitconpower, pvals = TRUE)
```

Calls:

```
1: gls(model = Base ~ Age * Faculty * Sex, data = dd)
2: gls(model = Base ~ Age * Faculty * Sex, data = dd, weights =
  varIdent(form = ~1 | Faculty))
3: gls(model = Base ~ Age * Faculty * Sex, data = dd, weights =
  varPower(form = ~fitted(.) | Faculty))
4: gls(model = Base ~ Age * Faculty * Sex, data = dd, weights =
  varConstPower(form = ~fitted(.) | Faculty))
```

|                | Model 1 | Model 2 | Model 3 | Model 4 |
|----------------|---------|---------|---------|---------|
| (Intercept)    | -19.77  | -19.77  | -18.01  | -17.89  |
| SE             | 6.07    | 3.51    | 3.22    | 3.28    |
| Pr(> z )       | 0.0011  | 1.7e-08 | 2.3e-08 | 4.8e-08 |
| Age            | 3.9913  | 3.9913  | 3.9573  | 3.9529  |
| SE             | 0.1148  | 0.0663  | 0.0643  | 0.0660  |
| Pr(> z )       | < 2e-16 | < 2e-16 | < 2e-16 | < 2e-16 |
| FacultyMed     | -14.81  | -14.81  | -11.93  | -12.11  |
| SE             | 8.32    | 8.17    | 7.34    | 7.69    |
| Pr(> z )       | 0.0750  | 0.0696  | 0.1039  | 0.1153  |
| SexM           | -4.46   | -4.46   | -7.90   | -8.66   |
| SE             | 8.31    | 4.81    | 4.42    | 4.53    |
| Pr(> z )       | 0.5919  | 0.3537  | 0.0740  | 0.0556  |
| Age:FacultyMed | 0.958   | 0.958   | 0.895   | 0.896   |
| SE             | 0.162   | 0.162   | 0.158   | 0.164   |
| Pr(> z )       | 3.5e-09 | 3.7e-09 | 1.5e-08 | 4.4e-08 |

|                     |        |        |        |        |
|---------------------|--------|--------|--------|--------|
| Age:SexM            | 0.2011 | 0.2011 | 0.2665 | 0.2858 |
| SE                  | 0.1546 | 0.0894 | 0.0871 | 0.0903 |
| Pr(> z )            | 0.1934 | 0.0244 | 0.0022 | 0.0015 |
| FacultyMed:SexM     | 17.8   | 17.8   | 14.4   | 15.3   |
| SE                  | 11.6   | 11.6   | 10.6   | 11.1   |
| Pr(> z )            | 0.1262 | 0.1248 | 0.1731 | 0.1662 |
| Age:FacultyMed:SexM | -0.430 | -0.430 | -0.357 | -0.371 |
| SE                  | 0.222  | 0.225  | 0.222  | 0.230  |
| Pr(> z )            | 0.0526 | 0.0556 | 0.1081 | 0.1067 |

## 7.1 Revisiting the gap with heteroskedasticity

```
wgap1 <- wald(fitgroups,
              Lgap,
              data = subset(ww, Sex == 'F', select = c(Faculty, Age)))

wgap2 <- wald(fitconpower,
              Lgap,
              data = subset(ww, Sex == 'F', select = c(Faculty, Age)))

opt <- options(max.print=30)
```

Model with different variances in different Faculties

```
print(wgap1)
```

|   | numDF    | denDF     | F-value  | p-value  |         |            |            |  |  |
|---|----------|-----------|----------|----------|---------|------------|------------|--|--|
| 1 | 4        | 1421      | 9.144169 | <.00001  |         |            |            |  |  |
|   | Estimate | Std.Error | DF       | t-value  | p-value | Lower 0.95 | Upper 0.95 |  |  |
| 3 | 1.574296 | 2.284210  | 1421     | 0.689208 | 0.49080 | -2.906490  | 6.055083   |  |  |
| 4 | 6.474432 | 4.784826  | 1421     | 1.353118 | 0.17623 | -2.911649  | 15.860514  |  |  |
| 7 | 1.775363 | 2.206600  | 1421     | 0.804570 | 0.42120 | -2.553180  | 6.103906   |  |  |
| 8 | 6.245134 | 4.610870  | 1421     | 1.354437 | 0.17581 | -2.799709  | 15.289976  |  |  |

```
[ reached 'max' / getOption("max.print") -- omitted 88 rows ]

wgap1 <- as.data.frame(wgap1)
head(wgap1)
```

|   | coef                                                                | se       | U2       | L2        | p-value   | t-value   | DF   | Faculty | Age |
|---|---------------------------------------------------------------------|----------|----------|-----------|-----------|-----------|------|---------|-----|
| 3 | 1.574296                                                            | 2.284210 | 6.142717 | -2.994125 | 0.4908048 | 0.6892081 | 1421 | Arts    | 30  |
|   | L.(Intercept) L.Age L.FacultyMed L.SexM L.Age:FacultyMed L.Age:SexM |          |          |           |           |           |      |         |     |
| 3 | 0                                                                   | 0        | 0        | 1         |           | 0         |      | 30      |     |
|   | L.FacultyMed:SexM L.Age:FacultyMed:SexM                             |          |          |           |           |           |      |         |     |
| 3 | 0                                                                   |          |          | 0         |           |           |      |         |     |

```
[ reached 'max' / getOption("max.print") -- omitted 2 rows ]
[ reached 'max' / getOption("max.print") -- omitted 3 rows ]
```

Model with different variances by age in different Faculties

```
print(wgap2)
```

|   | numDF     | denDF     | F-value  | p-value   |         |            |            |  |  |
|---|-----------|-----------|----------|-----------|---------|------------|------------|--|--|
| 1 | 4         | 1421      | 9.077471 | <.00001   |         |            |            |  |  |
|   | Estimate  | Std.Error | DF       | t-value   | p-value | Lower 0.95 | Upper 0.95 |  |  |
| 3 | -0.087053 | 1.994770  | 1421     | -0.043641 | 0.96520 | -4.000064  | 3.825958   |  |  |
| 4 | 4.110264  | 4.224149  | 1421     | 0.973040  | 0.33070 | -4.175973  | 12.396501  |  |  |

```

7    0.198748 1.918793 1421 0.103580 0.91752 -3.565222 3.962719
8    4.024979 4.052370 1421 0.993241 0.32076 -3.924292 11.974249
[ reached 'max' / getOption("max.print") -- omitted 88 rows ]

```

```

wgap2 <- as.data.frame(wgap2)
head(wgap2)

```

```

      coef      se      U2      L2  p-value    t-value    DF Faculty
3 -0.08705332 1.994770 3.902487 -4.076594 0.9651969 -0.04364077 1421    Arts
  Age L.(Intercept) L.Age L.FacultyMed L.SexM L.Age:FacultyMed L.Age:SexM
3 30              0      0              0      1              0          30
  L.FacultyMed:SexM L.Age:FacultyMed:SexM
3              0              0
[ reached 'max' / getOption("max.print") -- omitted 2 rows ]
[ reached 'max' / getOption("max.print") -- omitted 3 rows ]

```

```

options(opt)      # reset to default values

```

Combining results for plotting

```

wgap$model <- 'none'
wgap1$model <- 'by Faculty'
wgap2$model <- 'by Age and Faculty'

```

```

wg <- rbind(wgap, wgap1, wgap2)

levels <- c('none','by Faculty','by Age and Faculty')
wg$model <- factor(wg$model, levels = levels)

caption <- paste("Estimated Male-Female wage gap with 95% error bands estimated",
                 "with different variance models")

xyplot(coef ~ Age | model * Faculty, wg,
       type = 'l', auto.key = list(space='right'),
       ylab = 'Wage gap (male - female)',
       ylim = c(-15,20),
       xlim = c(30,75),
       fit = wg$coef,
       lower = wg$L2,
       upper = wg$U2,
       subscripts = TRUE) +
  layer(panel.fit(..., alpha = .2)) +
  layer_(panel.abline(h=0)) +
  layer_(panel.grid(h=-1,v=-1)) -> plhet
useOuterStrips(plhet)

```

**Question:** Where are the bands wider and where are they narrower when incorporating heteroskedasticity in the model? Do the patterns you see make sense?

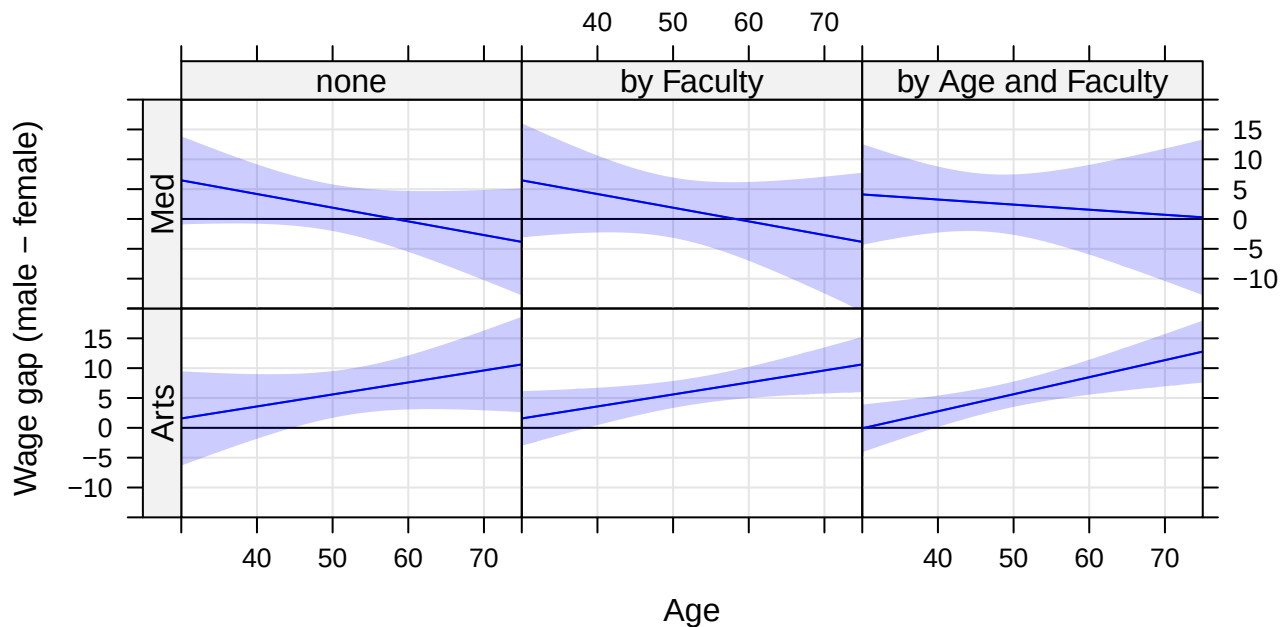


Figure 5: Estimated Male-Female wage gap with 95% error bands estimated with different variance models

## 8 G-side modeling and heteroskedasticity

- Different patterns for  $G$  matrices
- How different subgroups have different  $G$  matrices

## 9 Summary

The **nlme** package provides tools to build different forms of  $G$  matrices using different ‘pd’ (for ‘positive-definite’) functions from the ‘pdClasses’ set of functions.

See `?pdClasses`

- `pdSymm`: general positive-definite matrix, without constraints
- `pdLogChol`: general positive-definite matrix, without constraints, using a log-Choleski parametrization, i.e. a Choleski lower-triangular factor  $G = AA'$  with diagonal elements  $= \exp(\eta_{ii})$ , to ensure that diagonal elements are positive.
- `pdDiag`: diagonal matrix
- `pdIdent`: multiple of an identity so different components are uncorrelated
- `pdCompSymm`: compound symmetry structure: constant diagonal and constant off-diagonal elements
- `pdBlocked`: block diagonal with diagonal blocks consisting of matrices above

Example using high school data allowing different  $G$  matrices for Public and Catholic schools

```
library(spida2)
data(hs)
head(hs)
```

|   | school | mathach | ses    | Sex    | Minority | Size | Sector   | PRACAD | DISCLIM |
|---|--------|---------|--------|--------|----------|------|----------|--------|---------|
| 1 | 1317   | 12.862  | 0.882  | Female | No       | 455  | Catholic | 0.95   | -1.694  |
| 2 | 1317   | 8.961   | 0.932  | Female | Yes      | 455  | Catholic | 0.95   | -1.694  |
| 3 | 1317   | 4.756   | -0.158 | Female | Yes      | 455  | Catholic | 0.95   | -1.694  |
| 4 | 1317   | 21.405  | 0.362  | Female | Yes      | 455  | Catholic | 0.95   | -1.694  |
| 5 | 1317   | 20.748  | 1.372  | Female | No       | 455  | Catholic | 0.95   | -1.694  |
| 6 | 1317   | 18.362  | 0.132  | Female | Yes      | 455  | Catholic | 0.95   | -1.694  |

```
fit <- lme(mathach ~ ses*Sex*Sector, data = hs,
  random = ~ 1 + ses + Sex | school,
  control = list(msMaxIter = 1000, msVerbose = TRUE, returnObject = TRUE))
```

|    |            |         |         |         |          |          |          |
|----|------------|---------|---------|---------|----------|----------|----------|
| 0: | 11084.607: | 1.43628 | 2.35617 | 1.73578 | -3.90855 | -1.18740 | -1.57954 |
| 1: | 11084.530: | 1.46063 | 2.56643 | 1.89696 | -3.91496 | -1.20276 | -1.58647 |
| 2: | 11084.466: | 1.38214 | 2.57225 | 1.86942 | -3.91840 | -1.21270 | -1.59070 |
| 3: | 11084.428: | 1.40825 | 2.71251 | 1.81607 | -3.94397 | -1.28029 | -1.63523 |
| 4: | 11084.401: | 1.42100 | 2.83179 | 1.89025 | -4.01856 | -1.47903 | -1.77994 |
| 5: | 11084.337: | 1.52199 | 2.86685 | 1.81499 | -4.21718 | -2.02122 | -2.17911 |

|     |            |         |         |         |          |          |          |
|-----|------------|---------|---------|---------|----------|----------|----------|
| 6:  | 11084.267: | 1.54383 | 2.93374 | 1.84357 | -4.39393 | -2.52427 | -2.54863 |
| 7:  | 11084.146: | 1.60233 | 3.00899 | 1.96521 | -4.89077 | -3.90330 | -3.59920 |
| 8:  | 11084.086: | 1.66618 | 3.00893 | 2.03162 | -5.35200 | -5.15301 | -4.58148 |
| 9:  | 11084.061: | 1.73117 | 3.05659 | 2.12221 | -5.81484 | -6.40002 | -5.56328 |
| 10: | 11084.036: | 1.79221 | 3.09112 | 2.15713 | -6.28226 | -7.63810 | -6.55846 |
| 11: | 11084.022: | 1.85174 | 3.18426 | 2.19331 | -6.74986 | -8.87491 | -7.55145 |
| 12: | 11084.007: | 1.93212 | 3.32540 | 2.21578 | -7.39832 | -10.5733 | -8.93294 |
| 13: | 11084.005: | 2.01562 | 3.44210 | 2.25614 | -8.04463 | -12.2762 | -10.3115 |
| 14: | 11083.999: | 2.03109 | 3.46292 | 2.24462 | -8.17696 | -12.6109 | -10.6000 |
| 15: | 11083.995: | 2.09594 | 3.54437 | 2.25539 | -8.70123 | -13.9749 | -11.7275 |
| 16: | 11083.992: | 2.18464 | 3.64722 | 2.27276 | -9.44605 | -15.9090 | -13.3334 |
| 17: | 11083.990: | 2.21315 | 3.67094 | 2.27357 | -9.72329 | -16.6182 | -13.9398 |
| 18: | 11083.988: | 2.28535 | 3.75139 | 2.28778 | -10.4133 | -18.3955 | -15.4392 |
| 19: | 11083.987: | 2.31926 | 3.79245 | 2.29776 | -10.8267 | -19.4495 | -16.3469 |
| 20: | 11083.985: | 2.37483 | 3.86501 | 2.31619 | -11.5083 | -21.1885 | -17.8427 |
| 21: | 11083.984: | 2.39742 | 3.90433 | 2.33417 | -11.9991 | -22.4212 | -18.9376 |
| 22: | 11083.984: | 2.43685 | 3.95317 | 2.34641 | -12.4889 | -23.6690 | -20.0149 |
| 23: | 11083.984: | 2.45230 | 3.96672 | 2.34841 | -12.6639 | -24.1142 | -20.4015 |
| 24: | 11083.983: | 2.49116 | 3.99301 | 2.34651 | -13.0553 | -25.1095 | -21.2693 |
| 25: | 11083.983: | 2.50925 | 4.01268 | 2.34442 | -13.2473 | -25.5960 | -21.6967 |
| 26: | 11083.983: | 2.56213 | 4.08253 | 2.34415 | -13.8683 | -27.1691 | -23.0781 |

|     |            |         |         |         |          |          |          |
|-----|------------|---------|---------|---------|----------|----------|----------|
| 27: | 11083.982: | 2.59214 | 4.13462 | 2.33688 | -14.2267 | -28.0711 | -23.8850 |
| 28: | 11083.982: | 2.61988 | 4.16960 | 2.33966 | -14.5859 | -28.9793 | -24.6856 |
| 29: | 11083.982: | 2.62629 | 4.17389 | 2.34390 | -14.7120 | -29.2960 | -24.9696 |
| 30: | 11083.982: | 2.64352 | 4.18542 | 2.35464 | -15.0643 | -30.1785 | -25.7661 |
| 31: | 11083.982: | 2.66749 | 4.21323 | 2.35747 | -15.4164 | -31.0639 | -26.5588 |
| 32: | 11083.982: | 2.68938 | 4.23711 | 2.36042 | -15.7681 | -31.9455 | -27.3562 |
| 33: | 11083.981: | 2.73976 | 4.29829 | 2.36073 | -16.4862 | -33.7491 | -28.9799 |
| 34: | 11083.981: | 2.76792 | 4.33383 | 2.35698 | -16.8957 | -34.7735 | -29.9153 |
| 35: | 11083.981: | 2.79480 | 4.36493 | 2.35784 | -17.3059 | -35.8014 | -30.8468 |
| 36: | 11083.981: | 2.81965 | 4.39222 | 2.36005 | -17.7159 | -36.8271 | -31.7810 |
| 37: | 11083.981: | 2.83248 | 4.40408 | 2.36344 | -17.9760 | -37.4753 | -32.3782 |
| 38: | 11083.981: | 2.85470 | 4.42908 | 2.36610 | -18.3901 | -38.5077 | -33.3288 |
| 39: | 11083.981: | 2.89759 | 4.47965 | 2.36793 | -19.1320 | -40.3592 | -35.0295 |
| 40: | 11083.981: | 2.92530 | 4.51417 | 2.36753 | -19.6519 | -41.6524 | -36.2324 |
| 41: | 11083.981: | 2.95437 | 4.54790 | 2.36822 | -20.1725 | -42.9497 | -37.4307 |
| 42: | 11083.981: | 2.96944 | 4.56347 | 2.36848 | -20.4547 | -43.6519 | -38.0836 |
| 43: | 11083.981: | 2.99376 | 4.58745 | 2.36906 | -20.9366 | -44.8488 | -39.2030 |
| 44: | 11083.981: | 3.01875 | 4.61523 | 2.36989 | -21.4189 | -46.0481 | -40.3196 |
| 45: | 11083.981: | 3.03645 | 4.63673 | 2.37068 | -21.7846 | -46.9560 | -41.1692 |
| 46: | 11083.981: | 3.05897 | 4.66691 | 2.37206 | -22.2916 | -48.2124 | -42.3524 |
| 47: | 11083.981: | 3.08326 | 4.69482 | 2.37298 | -22.7988 | -49.4712 | -43.5331 |

|     |            |         |         |         |          |          |          |
|-----|------------|---------|---------|---------|----------|----------|----------|
| 48: | 11083.981: | 3.09252 | 4.70377 | 2.37311 | -23.0022 | -49.9748 | -44.0096 |
| 49: | 11083.981: | 3.11768 | 4.73055 | 2.37358 | -23.5433 | -51.3161 | -45.2743 |
| 50: | 11083.980: | 3.14030 | 4.75349 | 2.37346 | -24.0452 | -52.5588 | -46.4525 |
| 51: | 11083.980: | 3.16286 | 4.77878 | 2.37388 | -24.5475 | -53.8029 | -47.6290 |
| 52: | 11083.980: | 3.17410 | 4.79300 | 2.37423 | -24.8233 | -54.4846 | -48.2784 |
| 53: | 11083.980: | 3.20117 | 4.82500 | 2.37512 | -25.4718 | -56.0885 | -49.8034 |
| 54: | 11083.980: | 3.21791 | 4.84575 | 2.37613 | -25.9208 | -57.1967 | -50.8655 |
| 55: | 11083.980: | 3.23628 | 4.86664 | 2.37668 | -26.3702 | -58.3076 | -51.9246 |
| 56: | 11083.980: | 3.25469 | 4.88624 | 2.37694 | -26.8194 | -59.4178 | -52.9846 |
| 57: | 11083.980: | 3.25813 | 4.88762 | 2.37648 | -26.8970 | -59.6093 | -53.1694 |
| 58: | 11083.980: | 3.26897 | 4.89871 | 2.37639 | -27.1668 | -60.2755 | -53.8080 |
| 59: | 11083.980: | 3.29330 | 4.92565 | 2.37653 | -27.7916 | -61.8179 | -55.2880 |
| 60: | 11083.980: | 3.31642 | 4.95260 | 2.37683 | -28.4159 | -63.3578 | -56.7707 |
| 61: | 11083.980: | 3.32780 | 4.96559 | 2.37710 | -28.7213 | -64.1113 | -57.4954 |
| 62: | 11083.980: | 3.35559 | 4.99794 | 2.37818 | -29.5057 | -66.0448 | -59.3611 |
| 63: | 11083.980: | 3.35892 | 5.00226 | 2.37871 | -29.6353 | -66.3631 | -59.6733 |
| 64: | 11083.980: | 3.37747 | 5.02277 | 2.37908 | -30.1555 | -67.6452 | -60.9120 |
| 65: | 11083.980: | 3.38577 | 5.03179 | 2.37916 | -30.3890 | -68.2207 | -61.4686 |
| 66: | 11083.980: | 3.39419 | 5.04037 | 2.37897 | -30.6224 | -68.7958 | -62.0256 |
| 67: | 11083.980: | 3.40889 | 5.05512 | 2.37848 | -31.0337 | -69.8090 | -63.0087 |
| 68: | 11083.980: | 3.42289 | 5.07055 | 2.37855 | -31.4451 | -70.8220 | -63.9921 |

|     |            |         |         |         |          |          |          |
|-----|------------|---------|---------|---------|----------|----------|----------|
| 69: | 11083.980: | 3.44819 | 5.09868 | 2.37870 | -32.2136 | -72.7135 | -65.8333 |
| 70: | 11083.980: | 3.46014 | 5.11235 | 2.37903 | -32.5840 | -73.6250 | -66.7206 |
| 71: | 11083.980: | 3.48526 | 5.14190 | 2.38014 | -33.4045 | -75.6428 | -68.6903 |
| 72: | 11083.980: | 3.48623 | 5.14344 | 2.38043 | -33.4670 | -75.7954 | -68.8437 |
| 73: | 11083.980: | 3.48834 | 5.14563 | 2.38051 | -33.5299 | -75.9502 | -68.9947 |
| 74: | 11083.980: | 3.49038 | 5.14781 | 2.38056 | -33.5928 | -76.1048 | -69.1458 |
| 75: | 11083.980: | 3.50636 | 5.16519 | 2.38063 | -34.0958 | -77.3420 | -70.3547 |
| 76: | 11083.980: | 3.51365 | 5.17270 | 2.38032 | -34.3202 | -77.8939 | -70.8940 |
| 77: | 11083.980: | 3.52441 | 5.18355 | 2.37968 | -34.6564 | -78.7211 | -71.7024 |
| 78: | 11083.980: | 3.53477 | 5.19488 | 2.37969 | -34.9928 | -79.5481 | -72.5109 |
| 79: | 11083.980: | 3.54476 | 5.20607 | 2.37989 | -35.3291 | -80.3750 | -73.3196 |
| 80: | 11083.980: | 3.55417 | 5.21705 | 2.38043 | -35.6654 | -81.2017 | -74.1286 |
| 81: | 11083.980: | 3.56276 | 5.22720 | 2.38109 | -35.9889 | -81.9966 | -74.9071 |
| 82: | 11083.980: | 3.57213 | 5.23755 | 2.38127 | -36.3124 | -82.7919 | -75.6852 |
| 83: | 11083.980: | 3.58150 | 5.24771 | 2.38132 | -36.6359 | -83.5872 | -76.4634 |
| 84: | 11083.980: | 3.59089 | 5.25763 | 2.38121 | -36.9593 | -84.3824 | -77.2417 |
| 85: | 11083.980: | 3.60509 | 5.27257 | 2.38093 | -37.4537 | -85.5981 | -78.4316 |
| 86: | 11083.980: | 3.61904 | 5.28782 | 2.38099 | -37.9482 | -86.8136 | -79.6215 |
| 87: | 11083.980: | 3.63005 | 5.29995 | 2.38106 | -38.3559 | -87.8159 | -80.6033 |
| 88: | 11083.980: | 3.64102 | 5.31209 | 2.38124 | -38.7637 | -88.8182 | -81.5852 |
| 89: | 11083.980: | 3.66334 | 5.33711 | 2.38192 | -39.6286 | -90.9438 | -83.6687 |

```
90:      11083.980:  3.66861  5.34283  2.38193 -39.8266 -91.4305 -84.1456
```

```
Warning in lme.formula(mathach ~ ses * Sex * Sector, data = hs, random = ~1 + : nlminb problem, convergence = singular convergence (7)
```

```
summary(fit)
```

```
Linear mixed-effects model fit by REML
```

```
Data: hs
```

|  | AIC      | BIC     | logLik    |
|--|----------|---------|-----------|
|  | 12850.32 | 12934.1 | -6410.161 |

```
Random effects:
```

```
Formula: ~1 + ses + Sex | school
```

```
Structure: General positive-definite, Log-Cholesky parametrization
```

|             | StdDev    | Corr        |
|-------------|-----------|-------------|
| (Intercept) | 1.5536822 | (Intr) ses  |
| ses         | 0.2287419 | 0.989       |
| SexMale     | 0.5637833 | 0.995 0.992 |
| Residual    | 6.1034068 |             |

```
Fixed effects: mathach ~ ses * Sex * Sector
```

|                          | Value     | Std.Error | DF   | t-value   | p-value |
|--------------------------|-----------|-----------|------|-----------|---------|
| (Intercept)              | 12.858142 | 0.4360005 | 1931 | 29.491118 | 0.0000  |
| ses                      | 2.116848  | 0.3682502 | 1931 | 5.748397  | 0.0000  |
| SexMale                  | 1.416743  | 0.5281693 | 1931 | 2.682365  | 0.0074  |
| SectorPublic             | -1.448005 | 0.6430916 | 38   | -2.251631 | 0.0302  |
| ses:SexMale              | -0.659243 | 0.5408905 | 1931 | -1.218811 | 0.2231  |
| ses:SectorPublic         | 1.343742  | 0.5522021 | 1931 | 2.433424  | 0.0150  |
| SexMale:SectorPublic     | -0.452527 | 0.6967158 | 1931 | -0.649514 | 0.5161  |
| ses:SexMale:SectorPublic | 0.037923  | 0.7746682 | 1931 | 0.048954  | 0.9610  |

Standardized Within-Group Residuals:

| Min         | Q1          | Med        | Q3         | Max        |
|-------------|-------------|------------|------------|------------|
| -3.08765052 | -0.74262146 | 0.05512576 | 0.78985595 | 2.63308994 |

Number of Observations: 1977

Number of Groups: 40

```
sapply(hs, class)
```

| school    | mathach   | ses       | Sex      | Minority | Size      | Sector   | PRACAD    |
|-----------|-----------|-----------|----------|----------|-----------|----------|-----------|
| "integer" | "numeric" | "numeric" | "factor" | "factor" | "integer" | "factor" | "numeric" |
| DISCLIM   |           |           |          |          |           |          |           |

```
"numeric"
```

```
hs$Sex <- factor(hs$Sex)
contrasts(hs$Sex)
```

```
      Male
Female    0
Male      1
```

```
contrasts(hs$Sex) <- rbind(Female = -1/2, Male = 1/2)
contrasts(hs$Sex)
```

```
      [,1]
Female -0.5
Male    0.5
```

```
# What does this do?
```

```
# The 0 for 'Sex' is the average of the Male and Female levels.
```

```
fit <- lme(mathach ~ ses*Sex*Sector, data = hs,
  random = ~ 1 + ses + Sex | school,
  control = list(msMaxIter = 1000, msVerbose = TRUE, returnObject = TRUE))
```

|     |            |         |         |         |          |          |          |
|-----|------------|---------|---------|---------|----------|----------|----------|
| 0:  | 11084.622: | 1.43255 | 2.35496 | 1.72612 | -3.96969 | -3.18307 | -1.56569 |
| 1:  | 11084.520: | 1.42275 | 2.65095 | 1.96164 | -3.97852 | -3.20604 | -1.57569 |
| 2:  | 11084.469: | 1.37131 | 2.65056 | 1.94082 | -3.98062 | -3.21224 | -1.57862 |
| 3:  | 11084.451: | 1.39555 | 2.64179 | 1.89375 | -3.98503 | -3.22432 | -1.58791 |
| 4:  | 11084.429: | 1.40470 | 2.69342 | 1.83765 | -4.00783 | -3.28735 | -1.63411 |
| 5:  | 11084.389: | 1.44387 | 2.88695 | 1.81721 | -4.11945 | -3.59254 | -1.86659 |
| 6:  | 11084.343: | 1.46846 | 2.90440 | 1.78869 | -4.23856 | -3.93090 | -2.12860 |
| 7:  | 11084.247: | 1.51454 | 2.95223 | 1.83804 | -4.52587 | -4.72041 | -2.75502 |
| 8:  | 11084.129: | 1.59940 | 2.98465 | 1.98404 | -5.10575 | -6.27698 | -4.01908 |
| 9:  | 11084.080: | 1.65694 | 2.96287 | 2.05474 | -5.57787 | -7.51154 | -5.04477 |
| 10: | 11084.053: | 1.70968 | 3.07153 | 2.12287 | -6.05111 | -8.74611 | -6.06482 |
| 11: | 11084.030: | 1.75800 | 3.15481 | 2.15685 | -6.53172 | -9.97036 | -7.09810 |
| 12: | 11084.016: | 1.83886 | 3.29436 | 2.21387 | -7.23586 | -11.7787 | -8.61936 |
| 13: | 11084.004: | 1.89169 | 3.38203 | 2.21930 | -7.71711 | -12.9877 | -9.65970 |
| 14: | 11084.000: | 1.94663 | 3.46125 | 2.23701 | -8.19596 | -14.2006 | -10.6969 |
| 15: | 11083.994: | 2.03037 | 3.57257 | 2.24703 | -8.93373 | -16.0537 | -12.2987 |
| 16: | 11083.991: | 2.07363 | 3.62833 | 2.24739 | -9.33840 | -17.0610 | -13.1798 |
| 17: | 11083.990: | 2.11587 | 3.68632 | 2.25613 | -9.74218 | -18.0715 | -14.0575 |
| 18: | 11083.989: | 2.15415 | 3.73865 | 2.26295 | -10.1475 | -19.0769 | -14.9410 |
| 19: | 11083.987: | 2.19776 | 3.79757 | 2.27415 | -10.6808 | -20.3893 | -16.1069 |
| 20: | 11083.986: | 2.24173 | 3.85588 | 2.28839 | -11.2136 | -21.7036 | -17.2710 |

|     |            |         |         |         |          |          |          |
|-----|------------|---------|---------|---------|----------|----------|----------|
| 21: | 11083.984: | 2.27696 | 3.89943 | 2.30580 | -11.7487 | -23.0089 | -18.4450 |
| 22: | 11083.983: | 2.29908 | 3.92479 | 2.31836 | -12.1214 | -23.9136 | -19.2653 |
| 23: | 11083.983: | 2.32558 | 3.95786 | 2.32837 | -12.4931 | -24.8223 | -20.0813 |
| 24: | 11083.983: | 2.34433 | 3.98018 | 2.33517 | -12.7913 | -25.5451 | -20.7395 |
| 25: | 11083.983: | 2.36513 | 4.00639 | 2.33934 | -13.0890 | -26.2686 | -21.3969 |
| 26: | 11083.982: | 2.42101 | 4.07814 | 2.34493 | -13.8516 | -28.1220 | -23.0826 |
| 27: | 11083.982: | 2.47879 | 4.15353 | 2.33978 | -14.6138 | -29.9685 | -24.7759 |
| 28: | 11083.981: | 2.50175 | 4.18267 | 2.34058 | -14.9208 | -30.7137 | -25.4559 |
| 29: | 11083.981: | 2.52151 | 4.20719 | 2.33970 | -15.1927 | -31.3714 | -26.0606 |
| 30: | 11083.981: | 2.54971 | 4.24125 | 2.33876 | -15.6018 | -32.3587 | -26.9726 |
| 31: | 11083.981: | 2.57632 | 4.27347 | 2.34007 | -16.0111 | -33.3455 | -27.8851 |
| 32: | 11083.981: | 2.62438 | 4.33029 | 2.34493 | -16.8182 | -35.2860 | -29.6896 |
| 33: | 11083.981: | 2.64343 | 4.35201 | 2.35001 | -17.2180 | -36.2411 | -30.5884 |
| 34: | 11083.981: | 2.66568 | 4.37867 | 2.35357 | -17.6175 | -37.1999 | -31.4831 |
| 35: | 11083.981: | 2.68731 | 4.40503 | 2.35703 | -18.0171 | -38.1576 | -32.3789 |
| 36: | 11083.981: | 2.68818 | 4.40706 | 2.35747 | -18.0479 | -38.2305 | -32.4492 |
| 37: | 11083.981: | 2.70730 | 4.43214 | 2.36036 | -18.4296 | -39.1432 | -33.3087 |
| 38: | 11083.981: | 2.72750 | 4.45685 | 2.36206 | -18.8112 | -40.0563 | -34.1680 |
| 39: | 11083.980: | 2.77180 | 4.50942 | 2.36338 | -19.6343 | -42.0251 | -36.0241 |
| 40: | 11083.980: | 2.78589 | 4.52459 | 2.36075 | -19.8859 | -42.6257 | -36.5962 |
| 41: | 11083.980: | 2.79942 | 4.54029 | 2.36076 | -20.1380 | -43.2285 | -37.1658 |

|     |            |         |         |         |          |          |          |
|-----|------------|---------|---------|---------|----------|----------|----------|
| 42: | 11083.980: | 2.81271 | 4.55574 | 2.36085 | -20.3902 | -43.8310 | -37.7358 |
| 43: | 11083.980: | 2.82553 | 4.57060 | 2.36128 | -20.6423 | -44.4329 | -38.3063 |
| 44: | 11083.980: | 2.83254 | 4.57865 | 2.36208 | -20.7942 | -44.7950 | -38.6513 |
| 45: | 11083.980: | 2.85287 | 4.60220 | 2.36544 | -21.2640 | -45.9129 | -39.7198 |
| 46: | 11083.980: | 2.87468 | 4.62800 | 2.36710 | -21.7333 | -47.0311 | -40.7865 |
| 47: | 11083.980: | 2.88809 | 4.64417 | 2.36830 | -22.0473 | -47.7775 | -41.5037 |
| 48: | 11083.980: | 2.90263 | 4.66136 | 2.36877 | -22.3613 | -48.5252 | -42.2194 |
| 49: | 11083.980: | 2.93478 | 4.69919 | 2.36818 | -23.0449 | -50.1525 | -43.7795 |
| 50: | 11083.980: | 2.93506 | 4.69939 | 2.36628 | -23.0375 | -50.1348 | -43.7658 |
| 51: | 11083.980: | 2.93544 | 4.69975 | 2.36640 | -23.0444 | -50.1511 | -43.7814 |
| 52: | 11083.980: | 2.94434 | 4.71009 | 2.36684 | -23.2479 | -50.6350 | -44.2470 |
| 53: | 11083.980: | 2.95929 | 4.72746 | 2.36764 | -23.5938 | -51.4572 | -45.0381 |
| 54: | 11083.980: | 2.97331 | 4.74364 | 2.36927 | -23.9397 | -52.2789 | -45.8296 |
| 55: | 11083.980: | 2.97491 | 4.74526 | 2.37065 | -24.0097 | -52.4445 | -45.9904 |
| 56: | 11083.980: | 2.98675 | 4.75883 | 2.37095 | -24.2894 | -53.1095 | -46.6306 |
| 57: | 11083.980: | 3.02006 | 4.79710 | 2.37087 | -25.0670 | -54.9584 | -48.4106 |
| 58: | 11083.980: | 3.01267 | 4.78849 | 2.36961 | -24.8844 | -54.5247 | -47.9930 |
| 59: | 11083.980: | 3.01989 | 4.79681 | 2.37025 | -25.0669 | -54.9582 | -48.4109 |
| 60: | 11083.980: | 3.02992 | 4.80835 | 2.37076 | -25.3158 | -55.5497 | -48.9810 |
| 61: | 11083.980: | 3.03959 | 4.81949 | 2.37154 | -25.5647 | -56.1410 | -49.5514 |
| 62: | 11083.980: | 3.04340 | 4.82391 | 2.37216 | -25.6736 | -56.3994 | -49.8013 |

|     |            |         |         |         |          |          |          |
|-----|------------|---------|---------|---------|----------|----------|----------|
| 63: | 11083.980: | 3.05412 | 4.83626 | 2.37253 | -25.9485 | -57.0526 | -50.4318 |
| 64: | 11083.980: | 3.07160 | 4.85625 | 2.37255 | -26.3914 | -58.1051 | -51.4479 |
| 65: | 11083.980: | 3.08886 | 4.87587 | 2.37276 | -26.8342 | -59.1574 | -52.4642 |
| 66: | 11083.980: | 3.10571 | 4.89488 | 2.37265 | -27.2769 | -60.2092 | -53.4812 |
| 67: | 11083.980: | 3.11360 | 4.90382 | 2.37287 | -27.4863 | -60.7067 | -53.9623 |
| 68: | 11083.980: | 3.12112 | 4.91230 | 2.37334 | -27.6957 | -61.2040 | -54.4436 |
| 69: | 11083.980: | 3.13363 | 4.92638 | 2.37439 | -28.0568 | -62.0612 | -55.2742 |
| 70: | 11083.980: | 3.14632 | 4.94077 | 2.37473 | -28.4074 | -62.8939 | -56.0807 |
| 71: | 11083.980: | 3.15871 | 4.95481 | 2.37498 | -28.7578 | -63.7263 | -56.8877 |
| 72: | 11083.980: | 3.17437 | 4.97251 | 2.37504 | -29.1926 | -64.7589 | -57.8885 |
| 73: | 11083.980: | 3.18530 | 4.98481 | 2.37457 | -29.4988 | -65.4866 | -58.5950 |
| 74: | 11083.980: | 3.19603 | 4.99688 | 2.37474 | -29.8053 | -66.2144 | -59.3011 |
| 75: | 11083.980: | 3.20008 | 5.00137 | 2.37522 | -29.9338 | -66.5193 | -59.5976 |

```
summary(fit)
```

Linear mixed-effects model fit by REML

Data: hs

|  | AIC      | BIC     | logLik    |
|--|----------|---------|-----------|
|  | 12850.32 | 12934.1 | -6410.161 |

Random effects:

```

Formula: ~1 + ses + Sex | school
Structure: General positive-definite, Log-Cholesky parametrization
          StdDev    Corr
(Intercept) 1.8341705 (Intr) ses
ses          0.2312824 0.979
Sex1         0.5675749 0.990 0.984
Residual     6.1033216

```

```

Fixed effects: mathach ~ ses * Sex * Sector
              Value Std.Error   DF   t-value p-value
(Intercept)  13.566539 0.4436315 1931  30.580645 0.0000
ses          1.787165 0.2847212 1931   6.276894 0.0000
Sex1         1.416321 0.5284630 1931   2.680076 0.0074
SectorPublic -1.674441 0.6507528   38 -2.573083 0.0141
ses:Sex1     -0.658959 0.5410327 1931 -1.217966 0.2234
ses:SectorPublic 1.362985 0.4255229 1931   3.203083 0.0014
Sex1:SectorPublic -0.451953 0.6971068 1931 -0.648327 0.5169
ses:Sex1:SectorPublic 0.037107 0.7748415 1931   0.047890 0.9618

```

Standardized Within-Group Residuals:

| Min | Q1 | Med | Q3 | Max |
|-----|----|-----|----|-----|
|-----|----|-----|----|-----|

```
-3.08764759 -0.74258107 0.05495886 0.78983031 2.63306300
```

```
Number of Observations: 1977
```

```
Number of Groups: 40
```

```
getG(fit)
```

```
Random effects variance covariance matrix
```

```
          (Intercept)          ses          Sex1
```

```
(Intercept)      3.36420 0.415510 1.03100
```

```
ses              0.41551 0.053492 0.12918
```

```
Sex1             1.03100 0.129180 0.32214
```

```
Standard Deviations: 1.8342 0.23128 0.56757
```

```
#
```

```
# create 2 indicator variables
```

```
#
```

```
hs <- within(
```

```
  hs,
```

```
{
```

```
  public <- 1*(Sector == 'Public')
```

```
  catholic <- 1 - public
```

```
}
)
head(hs)
```

|        | school | mathach | ses    | Sex    | Minority | Size | Sector   | PRACAD | DISCLIM | catholic |
|--------|--------|---------|--------|--------|----------|------|----------|--------|---------|----------|
| 1      | 1317   | 12.862  | 0.882  | Female | No       | 455  | Catholic | 0.95   | -1.694  | 1        |
| 2      | 1317   | 8.961   | 0.932  | Female | Yes      | 455  | Catholic | 0.95   | -1.694  | 1        |
| 3      | 1317   | 4.756   | -0.158 | Female | Yes      | 455  | Catholic | 0.95   | -1.694  | 1        |
| 4      | 1317   | 21.405  | 0.362  | Female | Yes      | 455  | Catholic | 0.95   | -1.694  | 1        |
| 5      | 1317   | 20.748  | 1.372  | Female | No       | 455  | Catholic | 0.95   | -1.694  | 1        |
| 6      | 1317   | 18.362  | 0.132  | Female | Yes      | 455  | Catholic | 0.95   | -1.694  | 1        |
| public |        |         |        |        |          |      |          |        |         |          |
| 1      | 0      |         |        |        |          |      |          |        |         |          |
| 2      | 0      |         |        |        |          |      |          |        |         |          |
| 3      | 0      |         |        |        |          |      |          |        |         |          |
| 4      | 0      |         |        |        |          |      |          |        |         |          |
| 5      | 0      |         |        |        |          |      |          |        |         |          |
| 6      | 0      |         |        |        |          |      |          |        |         |          |

```
fit2 <- lme(mathach ~ ses*Sex*Sector, data = hs,
            random = list(
```

```

school = pdBlocked(
  list(
    pdSymm(~ 0 + public + public:(ses + Sex)),
    pdSymm(~ 0 + catholic + catholic:(ses + Sex))
  )
),
control = list(msMaxIter = 1000, msVerbose = TRUE, returnObject = TRUE))

```

|     |            |         |           |         |          |           |         |          |           |       |
|-----|------------|---------|-----------|---------|----------|-----------|---------|----------|-----------|-------|
| 0:  | 11084.390: | 1.05737 | -0.341129 | 1.97316 | 0.456040 | -0.182753 | 1.38388 | 0.184004 | -0.199094 | -0.62 |
| 1:  | 11083.394: | 1.07953 | -0.459759 | 2.41469 | 0.564682 | -0.254473 | 1.61011 | 0.121957 | -0.220327 | -0.96 |
| 2:  | 11083.264: | 1.11767 | -0.441501 | 2.42084 | 0.631018 | -0.259650 | 1.63959 | 0.213505 | -0.226344 | -0.90 |
| 3:  | 11082.880: | 1.10423 | -0.448680 | 2.54210 | 0.679252 | -0.278264 | 1.70560 | 0.198827 | -0.236458 | -0.91 |
| 4:  | 11082.810: | 1.07952 | -0.458722 | 2.55147 | 0.654541 | -0.281441 | 1.69865 | 0.158494 | -0.235397 | -0.93 |
| 5:  | 11082.794: | 1.08323 | -0.466684 | 2.57836 | 0.643830 | -0.293643 | 1.69072 | 0.172237 | -0.225284 | -0.94 |
| 6:  | 11082.773: | 1.08389 | -0.478147 | 2.60643 | 0.650579 | -0.299989 | 1.69912 | 0.165299 | -0.230789 | -0.95 |
| 7:  | 11082.755: | 1.08760 | -0.487083 | 2.63611 | 0.663103 | -0.306415 | 1.70860 | 0.165085 | -0.234958 | -0.95 |
| 8:  | 11082.689: | 1.11138 | -0.514471 | 2.86693 | 0.735892 | -0.347230 | 1.76522 | 0.144469 | -0.222310 | -0.99 |
| 9:  | 11082.671: | 1.14827 | -0.623937 | 3.04875 | 0.759823 | -0.475698 | 1.76868 | 0.155582 | -0.300724 | -1.06 |
| 10: | 11082.649: | 1.14689 | -0.663899 | 3.16724 | 0.815635 | -0.479267 | 1.70894 | 0.123353 | -0.246715 | -1.01 |
| 11: | 11082.639: | 1.13225 | -0.666323 | 3.21041 | 0.821427 | -0.480438 | 1.70436 | 0.113789 | -0.222937 | -0.99 |

|     |            |         |           |         |          |           |         |          |           |         |
|-----|------------|---------|-----------|---------|----------|-----------|---------|----------|-----------|---------|
| 12: | 11082.631: | 1.11617 | -0.668714 | 3.23924 | 0.811546 | -0.493014 | 1.74981 | 0.125312 | -0.233608 | -0.99   |
| 13: | 11082.629: | 1.11333 | -0.674051 | 3.28516 | 0.804764 | -0.512634 | 1.76637 | 0.134362 | -0.231259 | -1.00   |
| 14: | 11082.625: | 1.11003 | -0.695887 | 3.32847 | 0.818432 | -0.532338 | 1.78006 | 0.130960 | -0.244090 | -1.00   |
| 15: | 11082.622: | 1.12200 | -0.714874 | 3.37519 | 0.827725 | -0.550703 | 1.78163 | 0.131015 | -0.249632 | -1.01   |
| 16: | 11082.619: | 1.13628 | -0.765680 | 3.51468 | 0.867137 | -0.601682 | 1.78037 | 0.120582 | -0.258293 | -1.01   |
| 17: | 11082.616: | 1.14822 | -0.822053 | 3.65899 | 0.885670 | -0.636956 | 1.81378 | 0.126484 | -0.264053 | -1.03   |
| 18: | 11082.614: | 1.15336 | -0.856713 | 3.80605 | 0.905217 | -0.691558 | 1.84148 | 0.128970 | -0.271409 | -1.04   |
| 19: | 11082.612: | 1.16201 | -0.911564 | 3.95087 | 0.926419 | -0.733852 | 1.86489 | 0.130859 | -0.273472 | -1.04   |
| 20: | 11082.611: | 1.17055 | -0.963204 | 4.09654 | 0.951030 | -0.780390 | 1.88487 | 0.130547 | -0.275993 | -1.05   |
| 21: | 11082.611: | 1.18565 | -1.01397  | 4.24102 | 0.974697 | -0.827678 | 1.90659 | 0.131334 | -0.280684 | -1.066  |
| 22: | 11082.611: | 1.19659 | -1.06559  | 4.38510 | 0.997687 | -0.876244 | 1.93086 | 0.132581 | -0.285456 | -1.074  |
| 23: | 11082.610: | 1.21977 | -1.14739  | 4.61410 | 1.03426  | -0.954370 | 1.97015 | 0.134557 | -0.293915 | -1.090  |
| 24: | 11082.610: | 1.23935 | -1.22881  | 4.84318 | 1.07161  | -1.03217  | 2.00978 | 0.136323 | -0.302211 | -1.1061 |
| 25: | 11082.610: | 1.24534 | -1.24891  | 4.89648 | 1.07985  | -1.05076  | 2.01997 | 0.136903 | -0.303597 | -1.1097 |
| 26: | 11082.610: | 1.26142 | -1.30334  | 5.04731 | 1.10417  | -1.10256  | 2.04671 | 0.137920 | -0.308373 | -1.1207 |
| 27: | 11082.610: | 1.27709 | -1.35783  | 5.19829 | 1.12844  | -1.15434  | 2.07340 | 0.138900 | -0.312695 | -1.1315 |
| 28: | 11082.610: | 1.29197 | -1.41258  | 5.34922 | 1.15277  | -1.20613  | 2.10035 | 0.139994 | -0.316746 | -1.1421 |
| 29: | 11082.610: | 1.30810 | -1.46722  | 5.50012 | 1.17678  | -1.25821  | 2.12765 | 0.140945 | -0.320304 | -1.1531 |
| 30: | 11082.610: | 1.32402 | -1.52206  | 5.65090 | 1.20100  | -1.31056  | 2.15526 | 0.141792 | -0.323365 | -1.1642 |
| 31: | 11082.610: | 1.34035 | -1.57676  | 5.80161 | 1.22494  | -1.36301  | 2.18303 | 0.142639 | -0.326476 | -1.1753 |
| 32: | 11082.610: | 1.35619 | -1.62904  | 5.94511 | 1.24788  | -1.41335  | 2.20994 | 0.143281 | -0.328929 | -1.1862 |

|     |            |         |          |         |         |          |         |          |           |          |
|-----|------------|---------|----------|---------|---------|----------|---------|----------|-----------|----------|
| 33: | 11082.610: | 1.37161 | -1.68132 | 6.08863 | 1.27106 | -1.46355 | 2.23665 | 0.143959 | -0.331608 | -1.19711 |
| 34: | 11082.610: | 1.38388 | -1.72135 | 6.19788 | 1.28862 | -1.50232 | 2.25774 | 0.144342 | -0.333051 | -1.20570 |
| 35: | 11082.610: | 1.39619 | -1.76128 | 6.30714 | 1.30629 | -1.54107 | 2.27873 | 0.144677 | -0.334574 | -1.21433 |
| 36: | 11082.610: | 1.42468 | -1.85356 | 6.55898 | 1.34711 | -1.63090 | 2.32767 | 0.145369 | -0.337610 | -1.23433 |
| 37: | 11082.610: | 1.44765 | -1.92602 | 6.75561 | 1.37933 | -1.70228 | 2.36725 | 0.145538 | -0.338773 | -1.25070 |
| 38: | 11082.610: | 1.47012 | -1.99842 | 6.95266 | 1.41138 | -1.77303 | 2.40605 | 0.145987 | -0.340691 | -1.26666 |
| 39: | 11082.610: | 1.50490 | -2.10875 | 7.25242 | 1.46023 | -1.88152 | 2.46596 | 0.146462 | -0.342940 | -1.29111 |
| 40: | 11082.610: | 1.49062 | -2.06284 | 7.12679 | 1.43989 | -1.83691 | 2.44174 | 0.146124 | -0.341244 | -1.28108 |
| 41: | 11082.610: | 1.50518 | -2.10902 | 7.25172 | 1.46067 | -1.88268 | 2.46717 | 0.146161 | -0.341698 | -1.29166 |
| 42: | 11082.610: | 1.52004 | -2.15516 | 7.37654 | 1.48139 | -1.92863 | 2.49282 | 0.146128 | -0.342099 | -1.30222 |
| 43: | 11082.610: | 1.53488 | -2.20144 | 7.50127 | 1.50195 | -1.97468 | 2.51869 | 0.146156 | -0.342371 | -1.31274 |

```
summary(fit2)
```

Linear mixed-effects model fit by REML

Data: hs

|  | AIC      | BIC      | logLik    |
|--|----------|----------|-----------|
|  | 12875.58 | 13037.55 | -6408.791 |

Random effects:

Composite Structure: Blocked

Block 1: public, public:ses, public:SexFemale, public:SexMale

Formula: ~0 + public + public:(ses + Sex) | school

Structure: General positive-definite

|                  | StdDev    | Corr   |                      |
|------------------|-----------|--------|----------------------|
| public           | 4.2378236 |        | public pblc:s pbl:SF |
| public:ses       | 0.7112644 | 0.328  |                      |
| public:SexFemale | 3.9631951 | -0.890 | 0.140                |
| public:SexMale   | 3.2923978 | -0.779 | 0.327 0.975          |

Block 2: catholic, catholic:ses, catholic:SexFemale, catholic:SexMale

Formula: ~0 + catholic + catholic:(ses + Sex) | school

Structure: General positive-definite

|                    | StdDev     | Corr   |                      |
|--------------------|------------|--------|----------------------|
| catholic           | 10.0815330 |        | cathlc cthlc: cth:SF |
| catholic:ses       | 0.2797655  | 0.967  |                      |
| catholic:SexFemale | 10.8285768 | -0.993 | -0.945               |
| catholic:SexMale   | 10.8708306 | -0.987 | -0.934 0.999         |
| Residual           | 6.0896180  |        |                      |

Fixed effects: mathach ~ ses \* Sex \* Sector

| Value | Std.Error | DF | t-value | p-value |
|-------|-----------|----|---------|---------|
|-------|-----------|----|---------|---------|

|                       |           |           |      |          |        |
|-----------------------|-----------|-----------|------|----------|--------|
| (Intercept)           | 13.607292 | 0.4085240 | 1931 | 33.30843 | 0.0000 |
| ses                   | 1.817810  | 0.2863644 | 1931 | 6.34789  | 0.0000 |
| Sex1                  | 1.372260  | 0.5186425 | 1931 | 2.64587  | 0.0082 |
| SectorPublic          | -1.864650 | 0.7002876 | 38   | -2.66269 | 0.0113 |
| ses:Sex1              | -0.572227 | 0.5553368 | 1931 | -1.03041 | 0.3029 |
| ses:SectorPublic      | 1.282661  | 0.4556227 | 1931 | 2.81518  | 0.0049 |
| Sex1:SectorPublic     | -0.410088 | 0.7199490 | 1931 | -0.56961 | 0.5690 |
| ses:Sex1:SectorPublic | -0.141571 | 0.7923163 | 1931 | -0.17868 | 0.8582 |

Standardized Within-Group Residuals:

|  | Min         | Q1          | Med        | Q3         | Max        |
|--|-------------|-------------|------------|------------|------------|
|  | -3.08236335 | -0.73872867 | 0.05331624 | 0.79468179 | 2.63721946 |

Number of Observations: 1977

Number of Groups: 40

`getG(fit2)`

Random effects variance covariance matrix

|            | public   | public:ses | public:SexFemale | public:SexMale |
|------------|----------|------------|------------------|----------------|
| public     | 17.95900 | 0.98902    | -14.94100        | -10.86500      |
| public:ses | 0.98902  | 0.50590    | 0.39329          | 0.76625        |

|                      |           |              |                    |                  |
|----------------------|-----------|--------------|--------------------|------------------|
| public:SexFemale     | -14.94100 | 0.39329      | 15.70700           | 12.72100         |
| public:SexMale       | -10.86500 | 0.76625      | 12.72100           | 10.84000         |
| catholic             | 0.00000   | 0.00000      | 0.00000            | 0.00000          |
| catholic:ses         | 0.00000   | 0.00000      | 0.00000            | 0.00000          |
| catholic:SexFemale   | 0.00000   | 0.00000      | 0.00000            | 0.00000          |
| catholic:SexMale     | 0.00000   | 0.00000      | 0.00000            | 0.00000          |
|                      | catholic  | catholic:ses | catholic:SexFemale | catholic:SexMale |
| public               | 0.0000    | 0.000000     | 0.000              | 0.0000           |
| public:ses           | 0.0000    | 0.000000     | 0.000              | 0.0000           |
| public:SexFemale     | 0.0000    | 0.000000     | 0.000              | 0.0000           |
| public:SexMale       | 0.0000    | 0.000000     | 0.000              | 0.0000           |
| catholic             | 101.6400  | 2.726300     | -108.370           | -108.2000        |
| catholic:ses         | 2.7263    | 0.078269     | -2.862             | -2.8401          |
| catholic:SexFemale   | -108.3700 | -2.862000    | 117.260            | 117.6300         |
| catholic:SexMale     | -108.2000 | -2.840100    | 117.630            | 118.1700         |
| Standard Deviations: | 4.2378    | 0.71126      | 3.9632             | 3.2924           |
|                      | 10.082    | 0.27977      | 10.829             | 10.871           |